5 Homework for Introduction to Category Theory (week 5)

5.1 Composition of adjoint pairs

Consider three categories $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and functors F_1, F_2, G_1, G_2 as such:

$$\mathcal{C}_1 \underset{G_1}{\overset{F_1}{\rightleftarrows}} \mathcal{C}_2 \underset{G_2}{\overset{F_2}{\rightleftarrows}} \mathcal{C}_3$$

1. Show that if $F_1 \dashv G_1$ and $F_2 \dashv G_2$, then $F_2 \circ F_1 \dashv G_1 \circ G_2$. (In words: left adjoints are preserved under composition.)

5.2 RAPL

Recall the following theorem (which we unfortunately did not prove in the lectures). Consider categories $\mathcal{C}$ and $\mathcal{D}$ and functor between them.

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

Theorem (RAPL). If $F \dashv G$, then the functor F preserves all colimits and the functor G preserves all limits.

In this exercise we will see what this means in a bunch of examples. The colimits we will consider are simply coproducts and initial objects. So the theorem states the natural maps $F(X) + F(Y) \to F(X + Y)$ and $0 \to F(0)$ are isomorphisms. (And for limits we consider products and final objects, meaning $G(X \times Y) \to G(X) \times G(Y)$ and $G(1) \to 1$ are isomorphisms.) Note that the meaning of $+, \times, 0, 1$ depends on the category.

In each case, first try to define the given functors on arrows. Then try to convince yourself that the claimed adjunction actually holds, by considering the natural one-to-one correspondence $\{FX \to Y \text{ in } \mathcal{D}\} \cong \{X \to GY \text{ in } \mathcal{C}\}$.

1. Consider $\mathcal{C} = \mathsf{Set}$ and $\mathcal{D} = \mathsf{Set}_*$ (the category of sets with a specified "null value" and maps preserving these). Define $F(X) = (X + 1, \iota_2(*))$, which extends a set X by adding a new point, and $G(Y, y_0) = Y$, which simply "forgets" about the point.

   What does the fact that F preserves $+$ tell you about the coproducts in Set_*?

2. $\mathcal{C} = \mathbb{Set}$ and $\mathcal{D} = \mathbb{Set}$. Fix a set A and take $F(X) = X \times A$ and $G(Y) = \mathrm{Hom}_{\mathbb{Set}}(A, Y)$. This is adjunction is known as *Currying*.

   What does it mean that F preserves 0 and +? Slightly harder, what does it mean that G preserves 1 and ×?

3. Consider two sets A and B and a function $f: A \to B$. Take the powerset of both sets, and consider the posets as categories: $\mathcal{C} = (\mathcal{P}(A), \subseteq)$ and $\mathcal{D} = (\mathcal{P}(B), \subseteq)$. Now we may take $F(U) = \{f(u) \mid u \in U\}$ (direct image) and $G(V) = \{a \in A \mid f(a) \in V\}$ (inverse image).

   Relate the categorical $+, \times, 0, 1$ to usual set-theoretic constructions (unions, intersections, ...). What does it mean that $G = f^{-1}$ preserves products?

4. Consider $\mathcal{C}$ = the set of first-order formulas with (at most the) free variables $y_1, ..., y_n$ and entailment ($\models$) as arrows. Similarly, take $\mathcal{D}$ = the set of first-order formulas with free variables $x, y_1, ..., y_n$. Define $F(\phi) = \phi$ and $G(\psi) = \forall x.\psi$.

   The fact that G preserves products should be a familiar fact of first-order logic. What are the products? Write the fact that G preserves products as a logical equivalence.

5. Let $\mathcal{C} = \mathbb{Grp}$ the category of groups and group homomorphisms, and $\mathcal{D} = \mathbb{CRing}$ the category of commutative rings and ring homomorphisms. Define $G(R) = R^\times = \{r \in R \mid r \text{ is a unit}\}$, the group of units (i.e. invertible elements) of R. Now G has a left adjoint F, which is a bit harder to describe.[1]

   The product in both categories is given by Cartesian product (and pointwise operations). Using RAPL, finish the sentence: "$(r, s) \in R \times S$ is a unit if and only if ..."

6. This may be a confusing example. $\mathcal{C} = \mathbb{Set}$ and $\mathcal{D} = \mathbb{Set} \times \mathbb{Set}$. Take $F(X) = (X, X)$, the *diagonal* functor, and $G(Y_1, Y_2) = Y_1 \times Y_2$. The fact that this forms an adjunction is exactly because of the UMP of the product.

   By RAPL, the functor G (i.e., forming products) preserves products. What does that mean?

[1] If you insist: $F(G) = \mathbb{Z}[G]$, the *group ring* of G. It is like a polynomial ring, but the multiplication of the elements of G is given by the group multiplication.