4 Homework for Introduction to Category Theory (week 4)

4.1 Final coalgebra as greatest fixed point

Consider the category $\mathbb{Set}$ of sets and functions. Fix a set Σ as our alphabet and consider the functor $F\colon \mathbb{Set} \to \mathbb{Set}$, defined by $F(X) = 2 \times X^{\Sigma}$. As a reminder: X^{Σ} is shorthand for $\mathrm{Hom}_{\mathbb{Set}}(\Sigma, X)$. The coalgebras for this functor are deterministic automata.

Let 1 be the final object in $\mathbb{Set}$ (in other words, a singleton set). There is a unique map

$$F(1) \xrightarrow{\;!\;} 1.$$

We can apply F to the map ! to obtain

$$F(F(1)) \xrightarrow{F(!)} F(1).$$

We may repeat this process and compose all those map. Hence we obtain

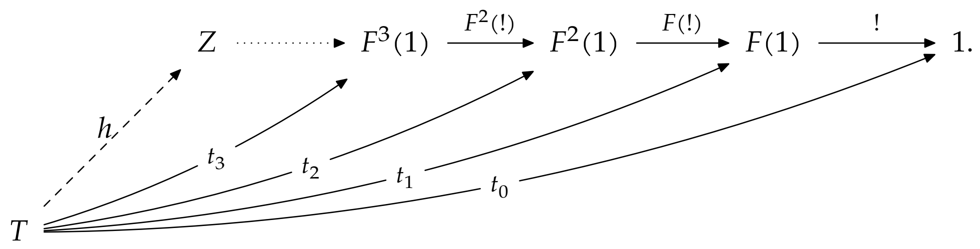
$$\cdots\cdots \to F^3(1) \xrightarrow{F^2(!)} F^2(1) \xrightarrow{F(!)} F(1) \xrightarrow{\;!\;} 1.$$

1. Try to find a explicit description of the object $F^n(1)$ for each $n \in \mathbb{N}$. What does the map $F^n(!)\colon F^{n+1}(1) \to F^n(1)$ do?

2. Consider the set $Z = \{\mathcal{L} \mid \mathcal{L} \subseteq \Sigma^*\}$ of all formal languages. Define maps $z_n\colon Z \to F^n(1)$ such that $F^n(!) \circ z_{n+1} = z_n$ for each $n \in \mathbb{N}$.

3. Let T be any set and let $t_n\colon Z \to F^n(1)$ be maps such that $F^n(!) \circ t_{n+1} = t_n$ for each n. Define a map $h\colon T \to Z$ such that $z_n \circ h = t_n$. (If you want: show that h is unique with this property.)

The three exercises above can be summarised in the following diagram. (I have omitted the individual maps z_n.) This also shows that Z is a *limit* of the above diagram. Do you recognise the universal mapping property?



4. Take $T = F(Z)$ and define the maps $t_n = F^n(!) \circ F(z_n)$ for each n. Show that $F^n(!) \circ t_{n+1} = t_n$ for all n. (Hint: this should follow abstractly, by functoriality of F.)

5. By point (3) we obtain a map $h\colon F(Z) \to Z$. Show that this map is bijective.

These last two points show that Z is a fixpoint of F, in the sense that $F(Z) \cong Z$. It is often (but not always) the case that we can find a fixpoint like this by iterating the functor F. This is reminiscent of the *Kleene fixpoint theorem.* In this case, since we started with 1, we have obtained the *greatest fixpoint of F*.

The last exercise also shows that we have both a coalgebra structure, $h^{-1}\colon Z \to F(Z)$, *and* an algebra structure, $h\colon F(Z) \to Z$. The universal property of Z makes it easy to map into it, this can be used to easily prove that (Z, h^{-1}) is the *final coalgebra.*