3 Homework for Introduction to Category Theory (week 3)

3.1 Elements of a set

Consider the category $\mathbf{Set}$ of sets and functions. Recall that a singleton set $\{*\}$ is a final object in $\mathbf{Set}$, so we might as well call this object 1.

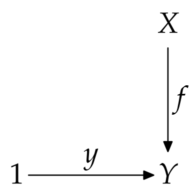
1. Show that there is a natural transformation $\phi\colon \mathrm{Hom}_{\mathbf{Set}}(1,-) \implies \mathrm{id}_{\mathbf{Set}}$. (That is, give an arrow $\phi_X\colon \mathrm{Hom}_{\mathbf{Set}}(1,X) \to X$ for every object X in $\mathbf{Set}$ and show that the square for naturality commutes.)

2. Show that ϕ is a natural isomorphism.

Another way to put this result is: Elements of X are in a (natural) one-to-one correspondence with functions $\{*\} \to X$.

3.2 Pullbacks

Let $\mathbb{C}$ be a category with a final object 1. Let $f\colon X \to Y$ and $y\colon 1 \to Y$ be arrows and consider the following diagram.



We will consider pullbacks of this diagram in three categories.

1. In $\mathbf{Set}$, the arrow $y\colon 1 \to Y$ can be considered as an element $y \in Y$ by the previous exercise. Show that the preimage $f^{-1}(y) = \{x \mid f(x) = y\}$ is the pullback of the above diagram. This object is sometimes called a *fiber.*

2. In $\mathbf{Vect}_{\mathbb{R}}$, the initial object is given by $1 = \{0\}$ (this also happens to be the initial object). Show that the kernel $\ker(f) = \{x \mid f(x) = 0\}$ is the pullback of the above diagram.

3. We can consider a poset $(P, \leq)$ as a category. (Recall: its objects are the elements of P, and there is a single arrow $g\colon A \to B$ iff $A \leq B$.) What is the pullback of above diagram in this category?