2 Homework for Introduction to Category Theory (week 2)

2.1 The opposite category

In the lecture, we have defined the *opposite category*, $\mathcal{C}^{\text{op}}$. This is perhaps a bit confusing, because we have reused the same collection of objects and arrows, only changing the (co)domains and composition of arrows. There is another way to define it, which is a bit more explicit. Given a category $\mathcal{C}$, we define $\mathcal{C}^{\text{op}}$ as follows.

- For each object X of $\mathcal{C}$, there is an object $X^{\leftarrow}$ in $\mathcal{C}^{\text{op}}$.
- For each arrow $f\colon X \to Y$ in $\mathcal{C}$ there is an arrow $f^{\leftarrow}\colon Y^{\leftarrow} \to X^{\leftarrow}$ in $\mathcal{C}^{\text{op}}$. (Note that this notation already defines the domain an codomain for $f^{\leftarrow}$.)
- The identity arrow for an object $X^{\leftarrow}$ is $\text{id}_X^{\leftarrow}$.
- Given two maps $f^{\leftarrow}\colon X^{\leftarrow} \to Y^{\leftarrow}, g^{\leftarrow}\colon Y^{\leftarrow} \to Z^{\leftarrow}$, their composition is defined by $g^{\leftarrow} \circ f^{\leftarrow} = (f \circ g)^{\leftarrow}$.

1. Show $\mathbb{Rel}^{\text{op}} \cong \mathbb{Rel}$. (That is, show that there is a functor $F\colon \mathbb{Rel}^{\text{op}} \to \mathbb{Rel}$ which has an inverse functor.)

2. Given a (real) vector space V, we can consider the set $\text{Hom}_{\mathbb{Vect}_{\mathbb{R}}}(V, \mathbb{R})$. This set itself is a vector space by using pointwise addition, pointwise scalar multiplication, and the constant zero function. This is called the *dual vector space of* V, denoted V^*.

   Show that this defines a functor $(-)^*\colon \mathbb{Vect}_{\mathbb{R}}^{\text{op}} \to \mathbb{Vect}_{\mathbb{R}}$. Pay attention to the direction of the arrows!

3. Is $(-)^*\colon \mathbb{Vect}_{\mathbb{R}}^{\text{op}} \to \mathbb{Vect}_{\mathbb{R}}$ an isomorphism?

The dual vector space is an important tool in linear algebra. What the functor does on maps, $f \mapsto f^*$, corresponds to matrix transposition (in the finite dimensional case). Some facts on matrices, such as $(AB)^{\text{T}} = B^{\text{T}}A^{\text{T}}$, follow directly from functoriality.

2.2 Products as final objects

In the exercises of last week, you have seen the concept of an *initial object*. The dual concept is called a *final object* (or *terminal object*). Concretely: an object X is called *final* if for all objects Y, there is a unique map $h\colon Y \to X$.

1. In $\mathbb{Set}$, show that a singleton set is final.

Given an arbitrary category $\mathcal{C}$ and two objects X, Y in $\mathcal{C}$, we define a new category $\mathbb{Cone}_{\mathcal{C}}(X, Y)$. Its objects are triples $(Z, f\colon Z \to X, g\colon Z \to Y)$, called *cones.* Given two cones $(Z, f, g), (Z', f', g')$, the arrows from the first to the second are given by arrows $h\colon Z \to Z'$ in $\mathcal{C}$ such that $f = f' \circ h$ and $g = g' \circ h$.

2. Get familiar with this category by drawing a diagram showing all the structure of a single arrow h between two cones.

3. Define the identity arrows and composition for this category.

4. Show that an object P with maps $\pi_1\colon P \to X$ and $\pi_2\colon P \to Y$ is a product of X and Y in the category $\mathcal{C}$ if and only if the cone (P, π_1, π_2) is a final object in $\mathbb{Cone}_{\mathcal{C}}(X, Y)$.

This shows that there are often multiple ways to define the same concept. Having different perspectives can be handy: some proofs are easier with one definition than with another definition.