1 Homework for Introduction to Category Theory (week 1)

1.1 Uniqueness of identities and inverses

In the lecture we skipped the proof of the fact that *identity arrows are unique.* Using only the axioms of a category, prove the following:

1. If $\mathrm{id}'_X \colon X \to X$ is an arrow such that $\mathrm{id}'_x \circ f = f$ and $g \circ \mathrm{id}'_X = g$ for all arrows $f \colon Y \to X$ and $g \colon X \to Z$, then $\mathrm{id}'_X = \mathrm{id}_X$.
2. If $f \colon X \to Y$ is an isomorphism, then the arrow g such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$ is unique.

Both results justify that we talk about *the* identity arrows and *the* inverse of an isomorphism. The inverse of an isomorphism f is typically written as f^{-1}.

1.2 The empty set

In many cases, one can describe a property of an object by referencing just the arrows around it. In this exercise, we will see that we can characterise the *empty set* in the category $\mathbb{Set}$ by only using arrows.

1. Before we do that, recall the first-order definition of an empty set. In the first-order theory of sets, we can only talk about membership ($\in$), and a set X is *empty* if it satisfies the following formula:

   $$\phi(X) = \forall Y. \neg(Y \in X)$$

   Show that the empty set is unique, i.e., if $\phi(X)$ and $\phi(X')$ hold, then $X = X'$. Hint: You may need to use the "axiom of extensionality" from set theory.

2. In category theory, an object X is called *initial* if for each object Y, there is a unique arrow $f \colon X \to Y$. (Put differently: the hom-sets $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ are singleton sets for each Y.)

   Show that an initial object (if it exists) is unique, i.e., if X and X' are initial, then there is an isomorphism $f \colon X \to X'$.

3. Show that, in the category $\mathbb{Set}$, a set X is empty if and only if X is initial. (You may use any fact from set theory, doing this purely from axioms of set theory would be a pain.)

4. What is the initial object in the category $\mathbb{Vect}_{\mathbb{R}}$?