On Certificates, Expected Runtimes, and Termination in Probabilistic Pushdown Automata

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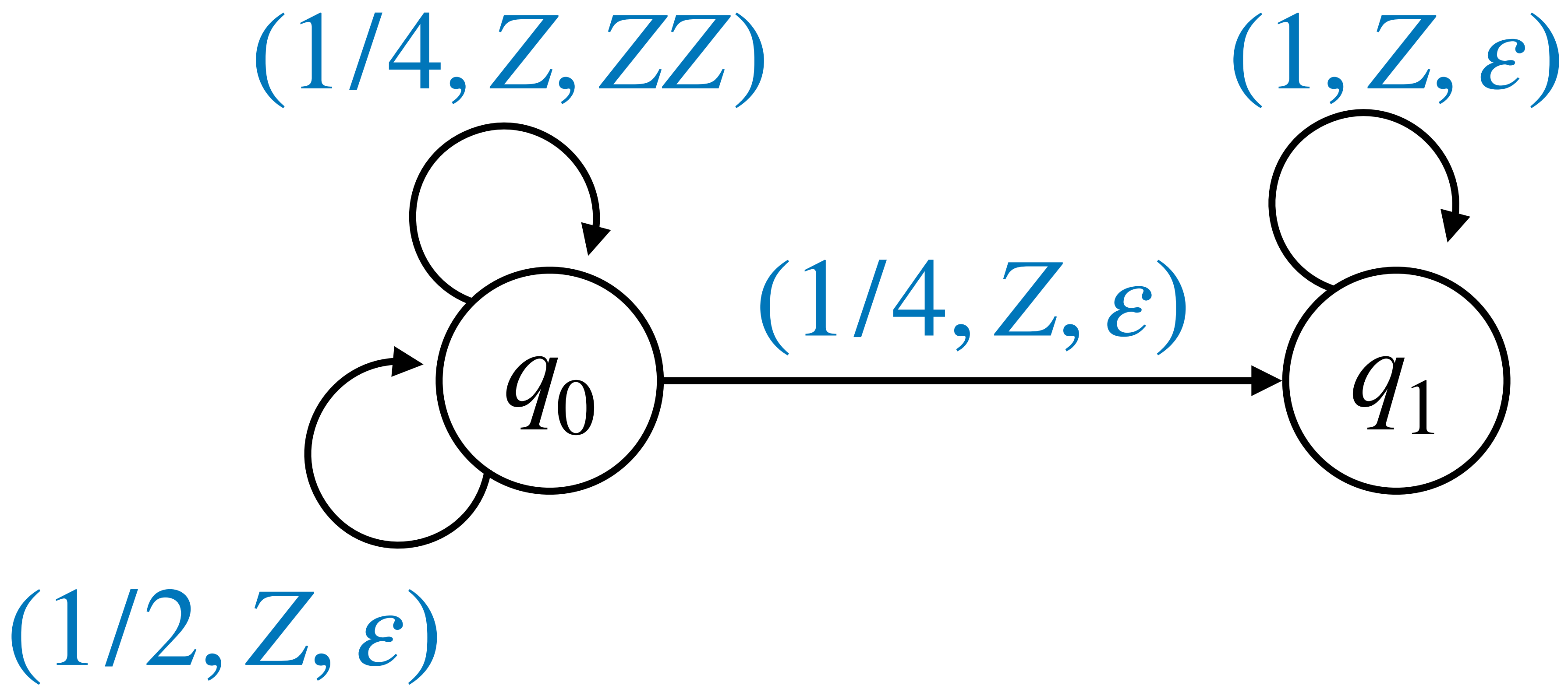
DFG Research Training Group 2236

European Research Council

LICS 2023

Probabilistic Pushdown Automata (pPDA)

[Esparza, Kucera, Mayr LICS '04, Etessami & Yannakakis STACS '05]



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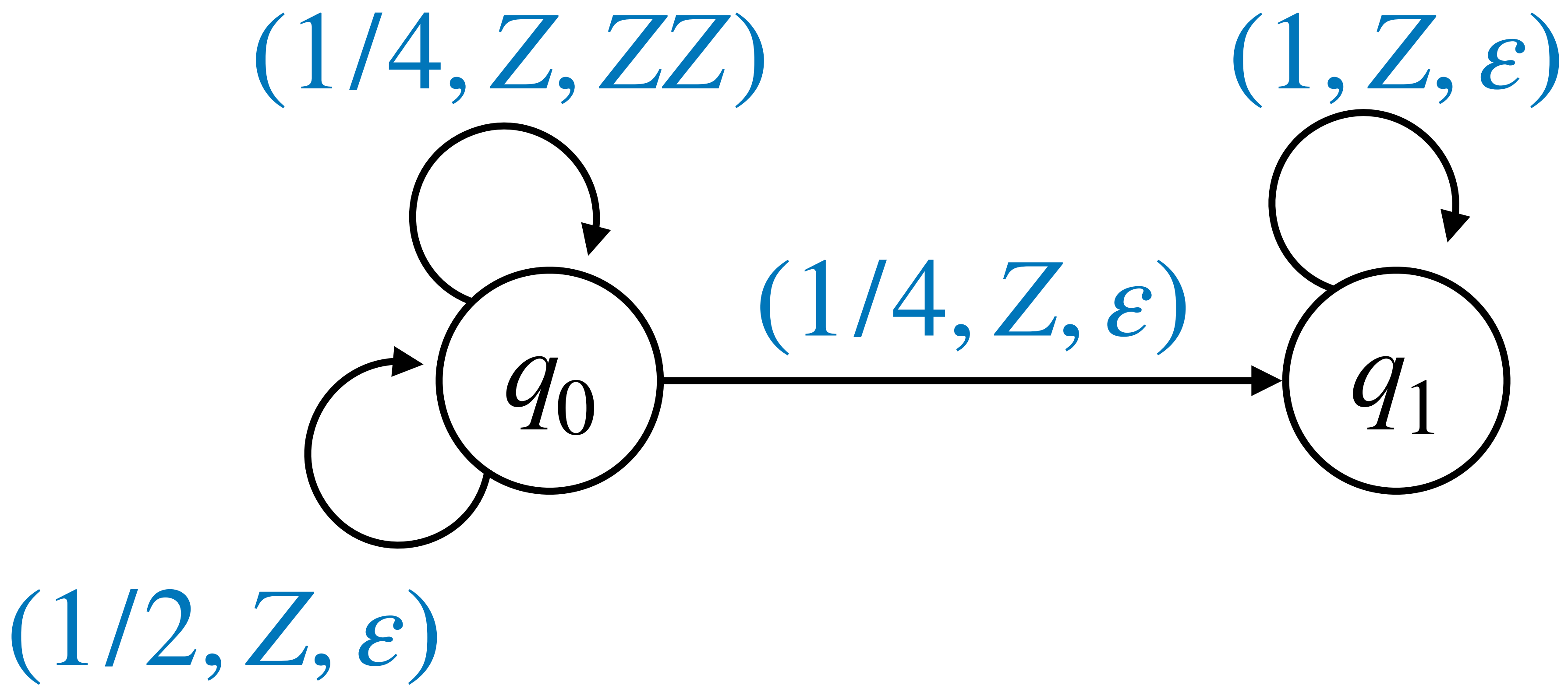

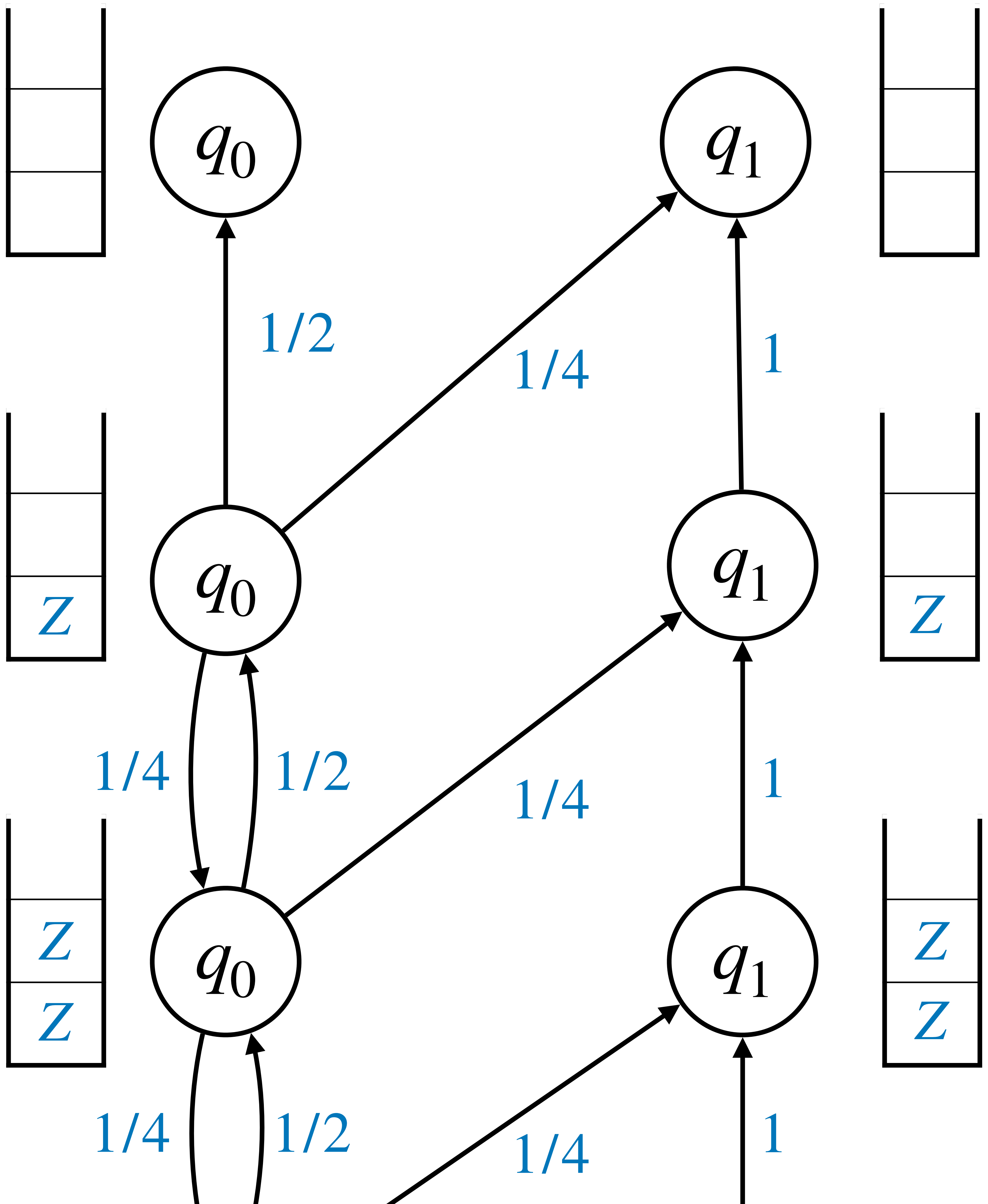

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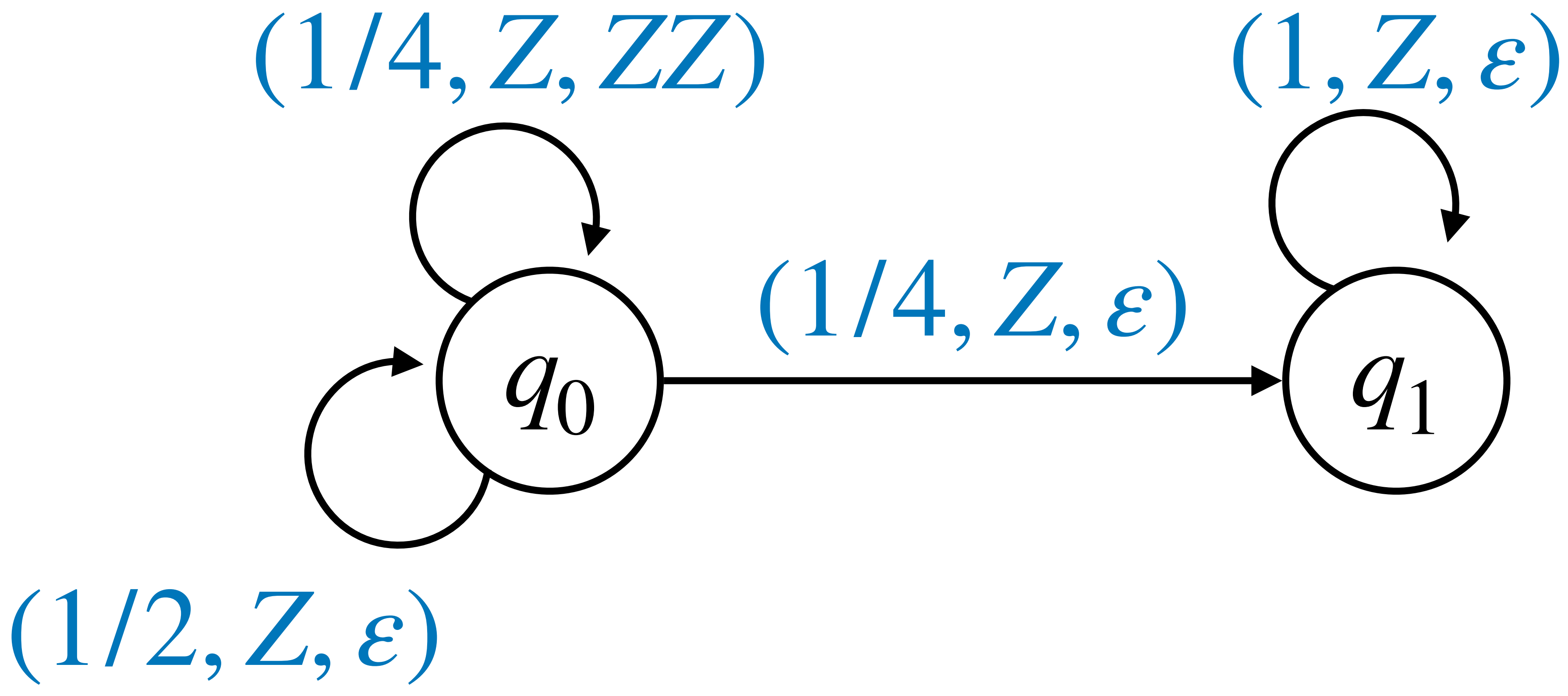


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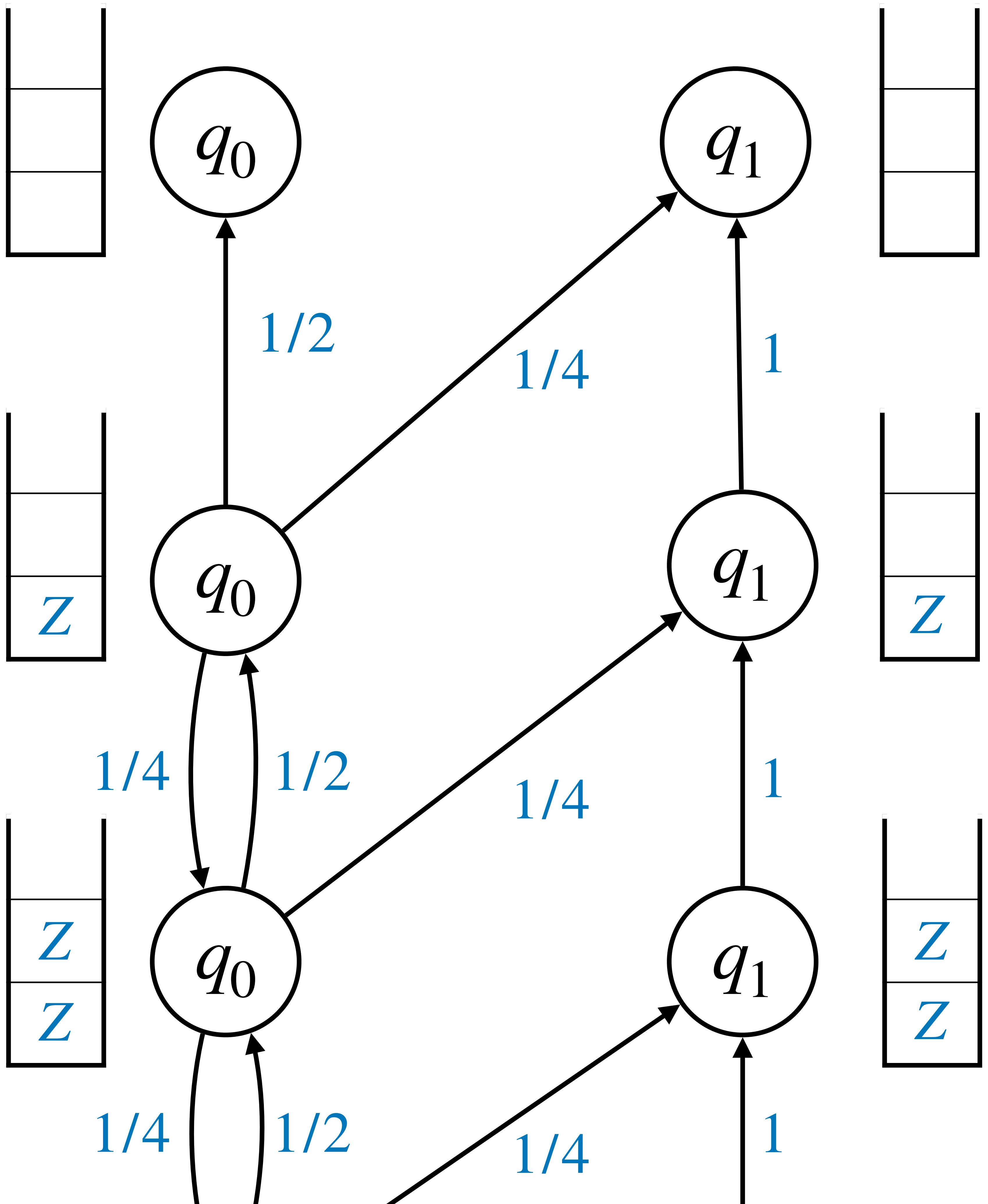
boolean f(boolean x) {
  if(x) {
    pchoice {
      0.5: return true;
      0.25: return false;
      0.25: return f(f(x));
    }
  } else {
    return false;
  }
}
```



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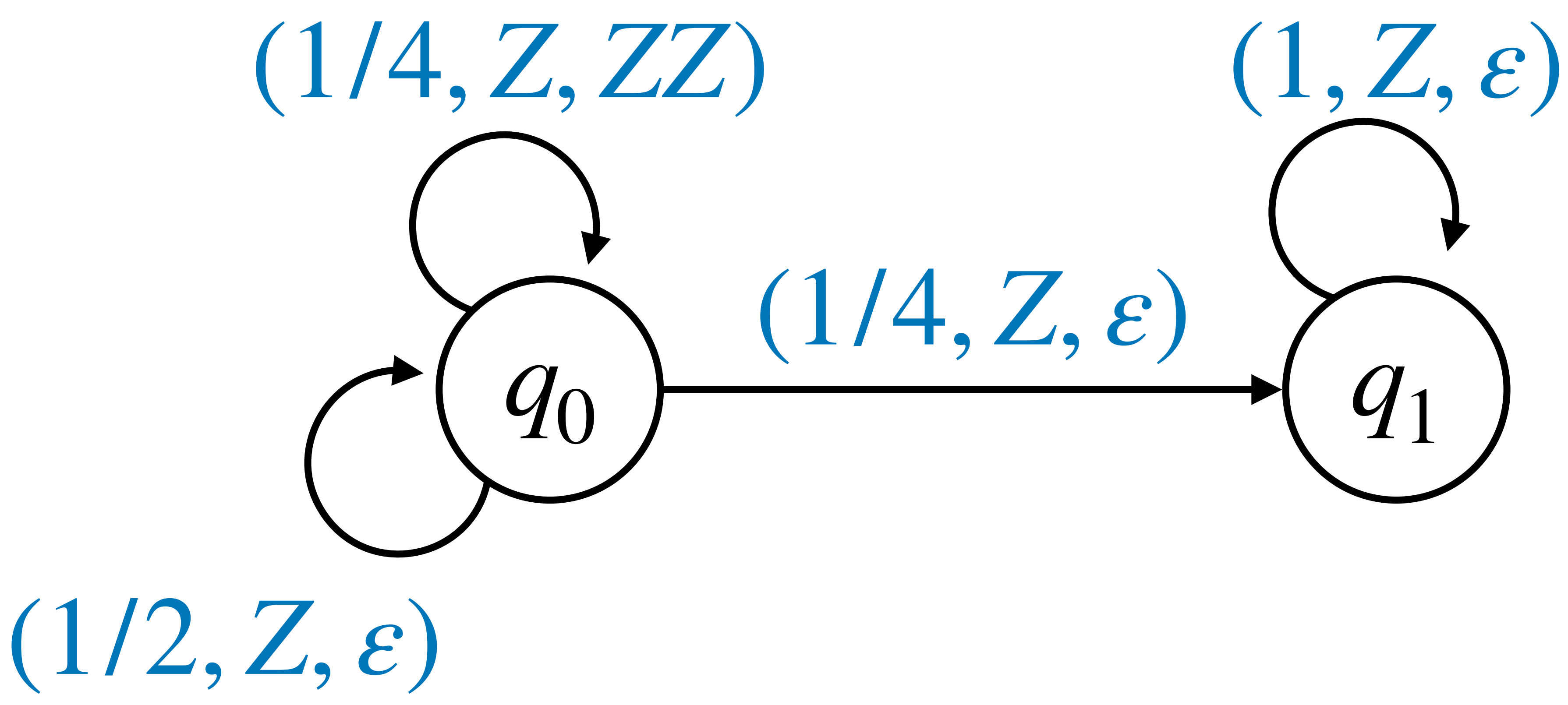


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```
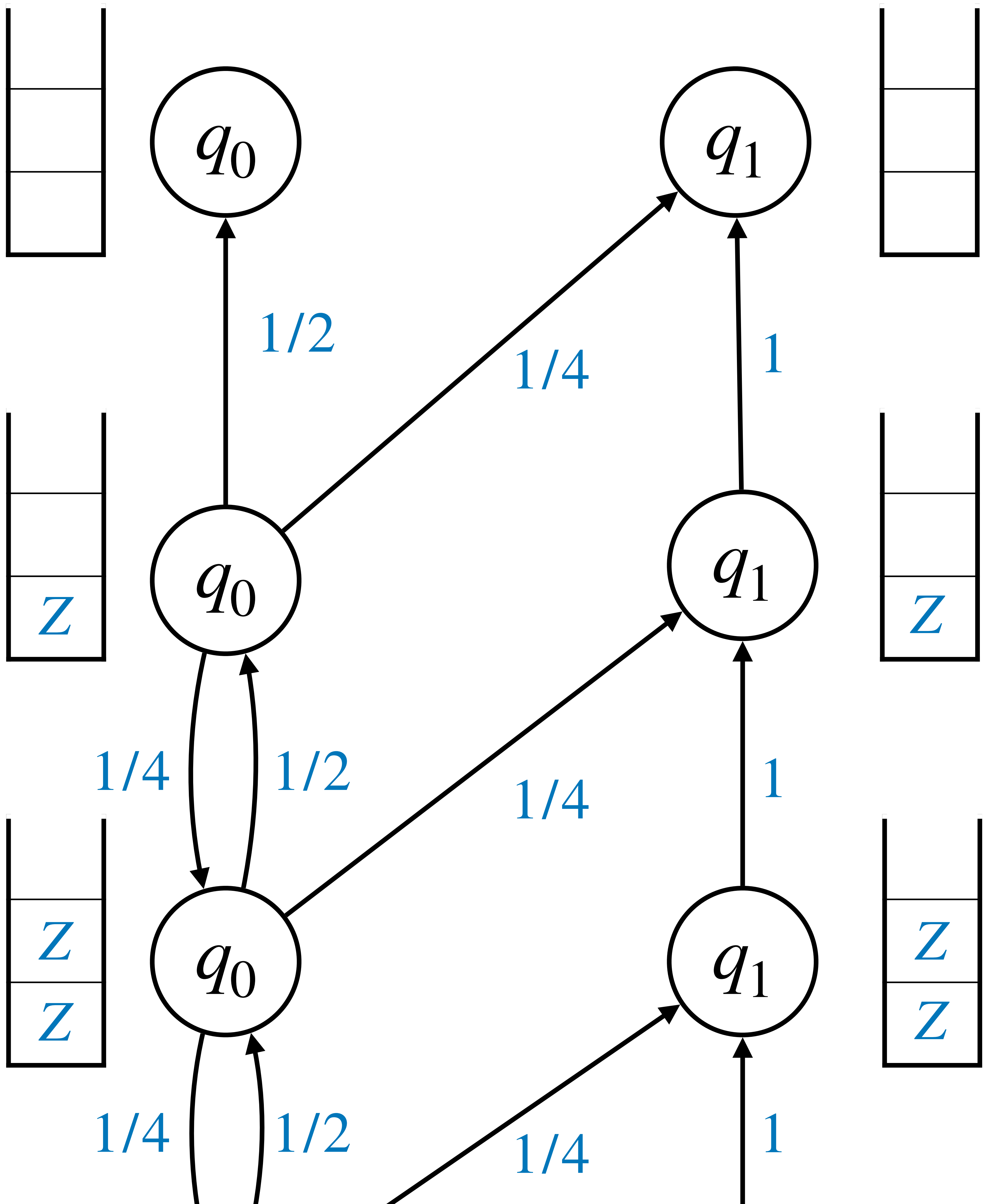

`f(true)` returns `true` with probability $2 - \sqrt{2}$



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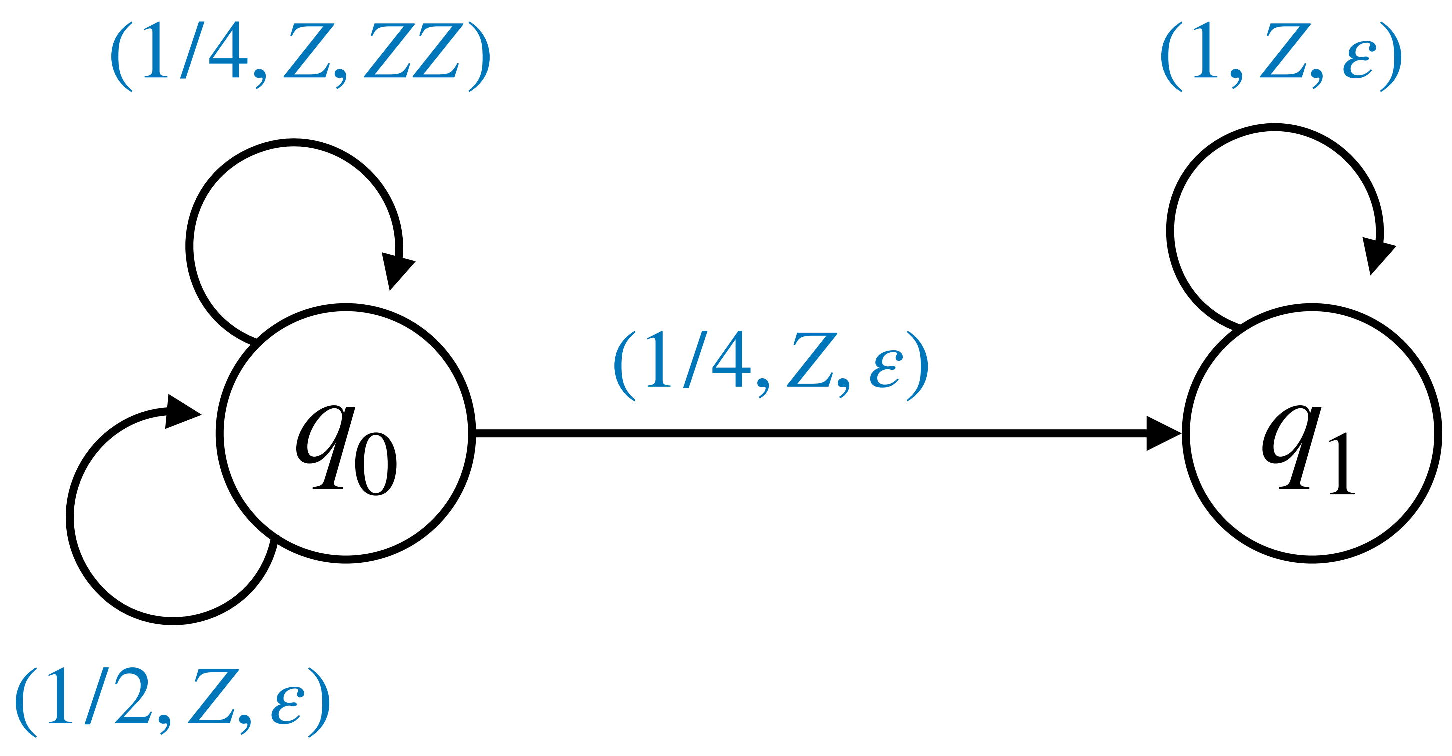

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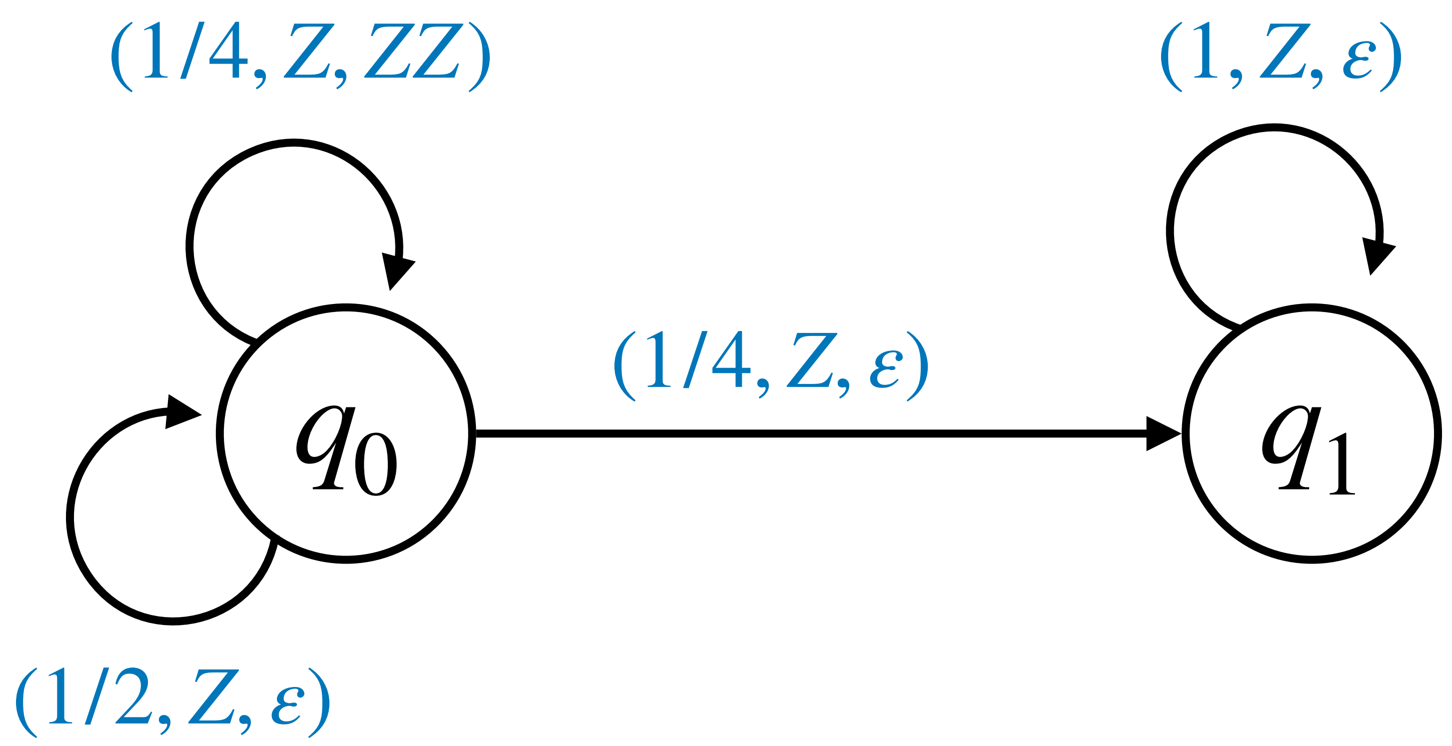

AST and PAST

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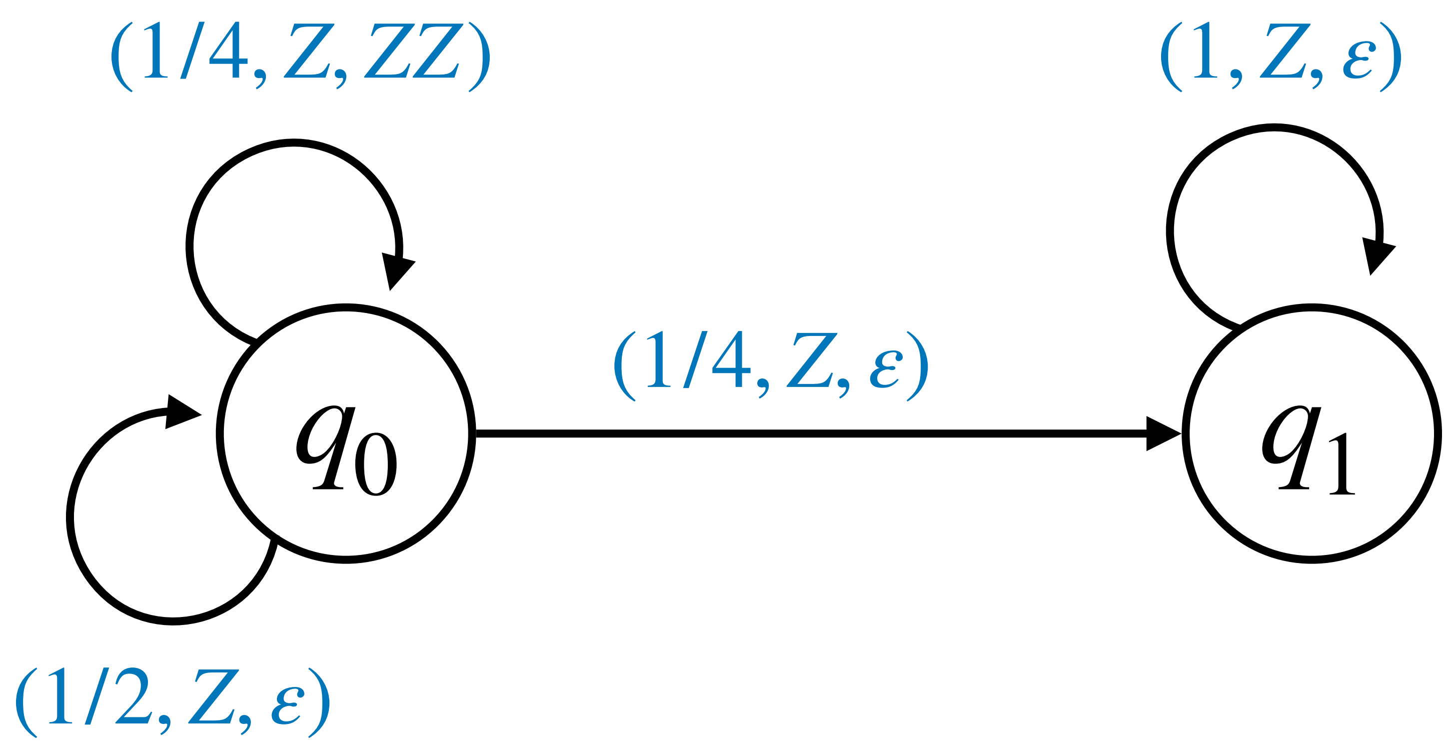

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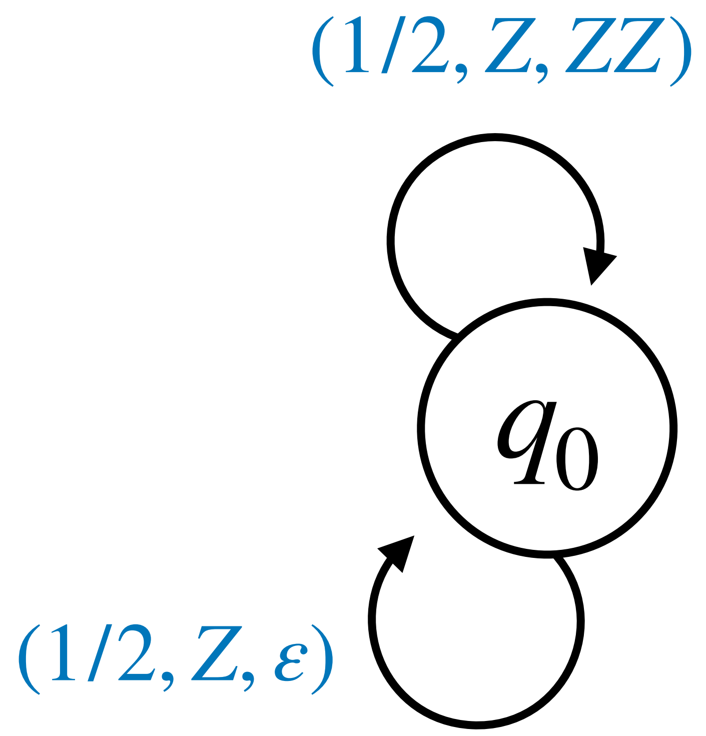

AST, but not PAST

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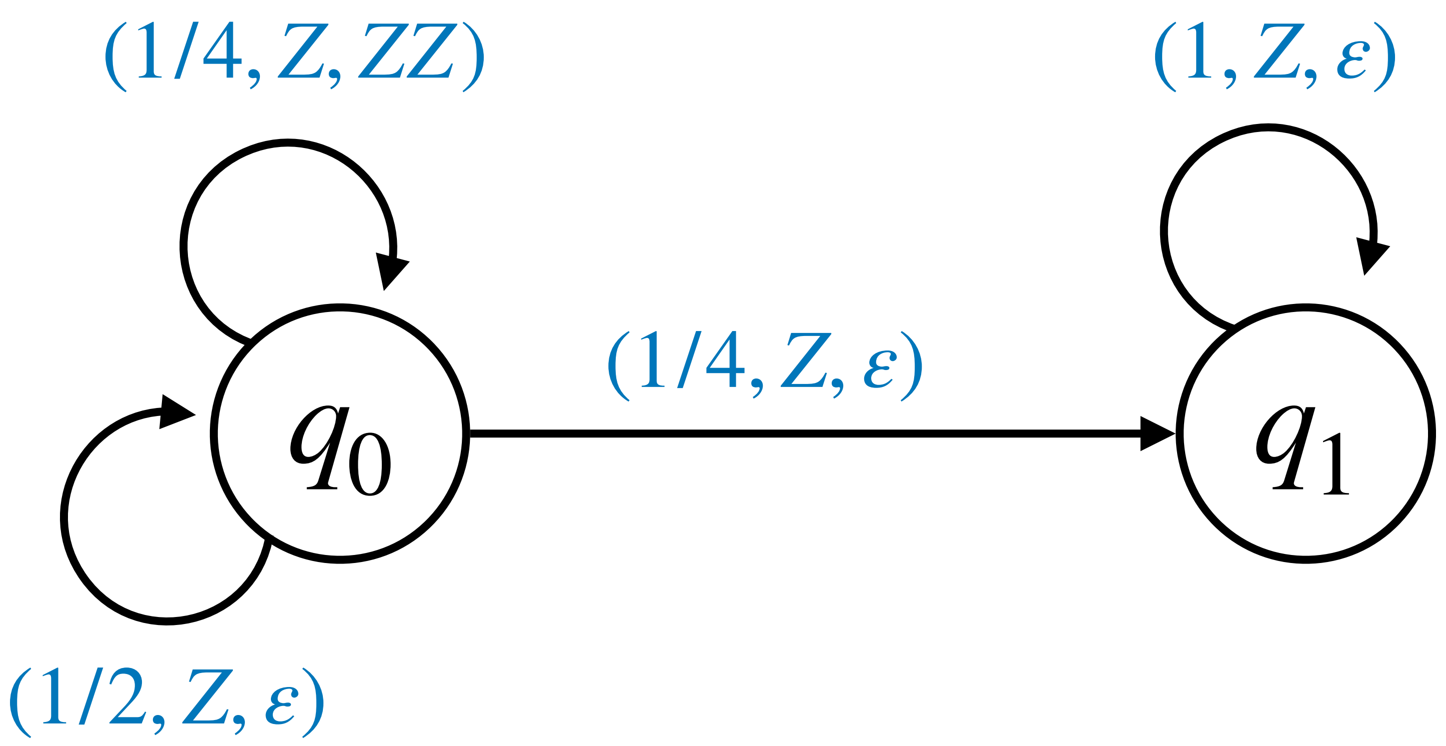

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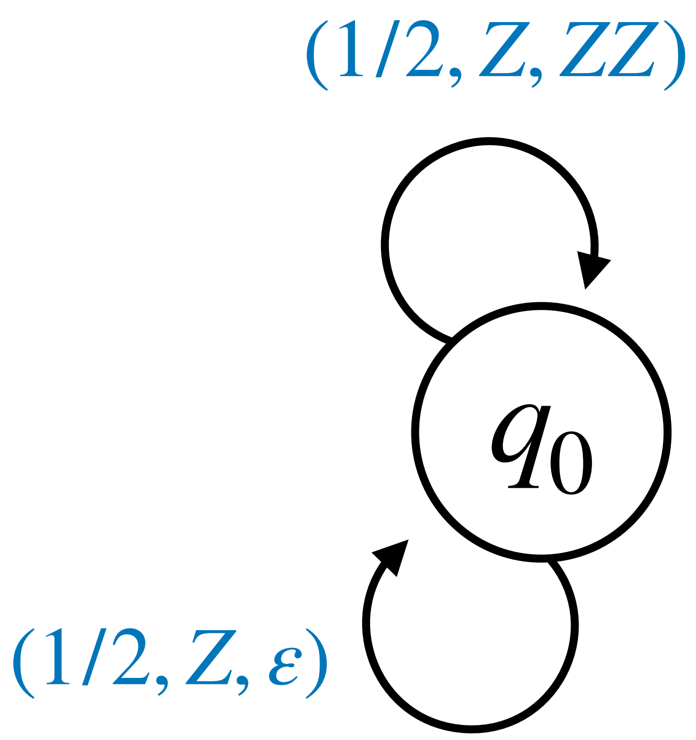

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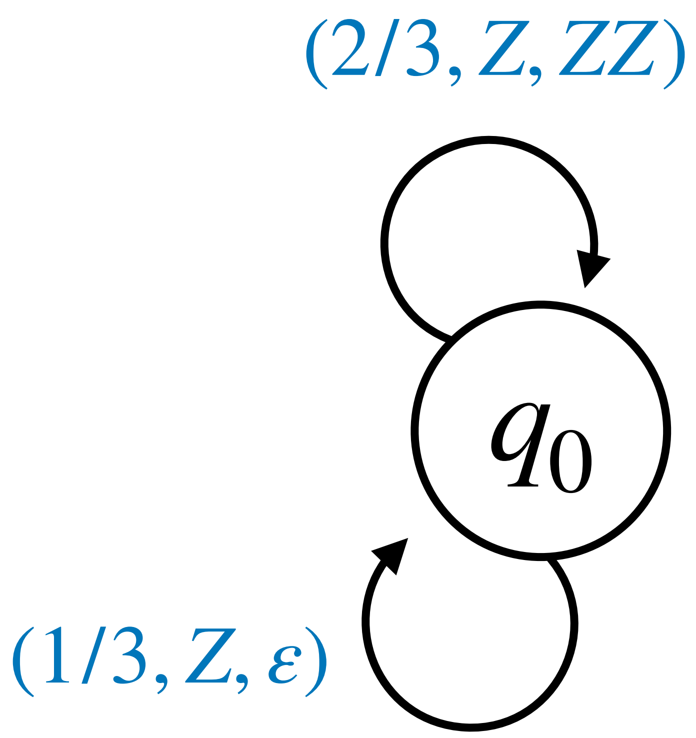

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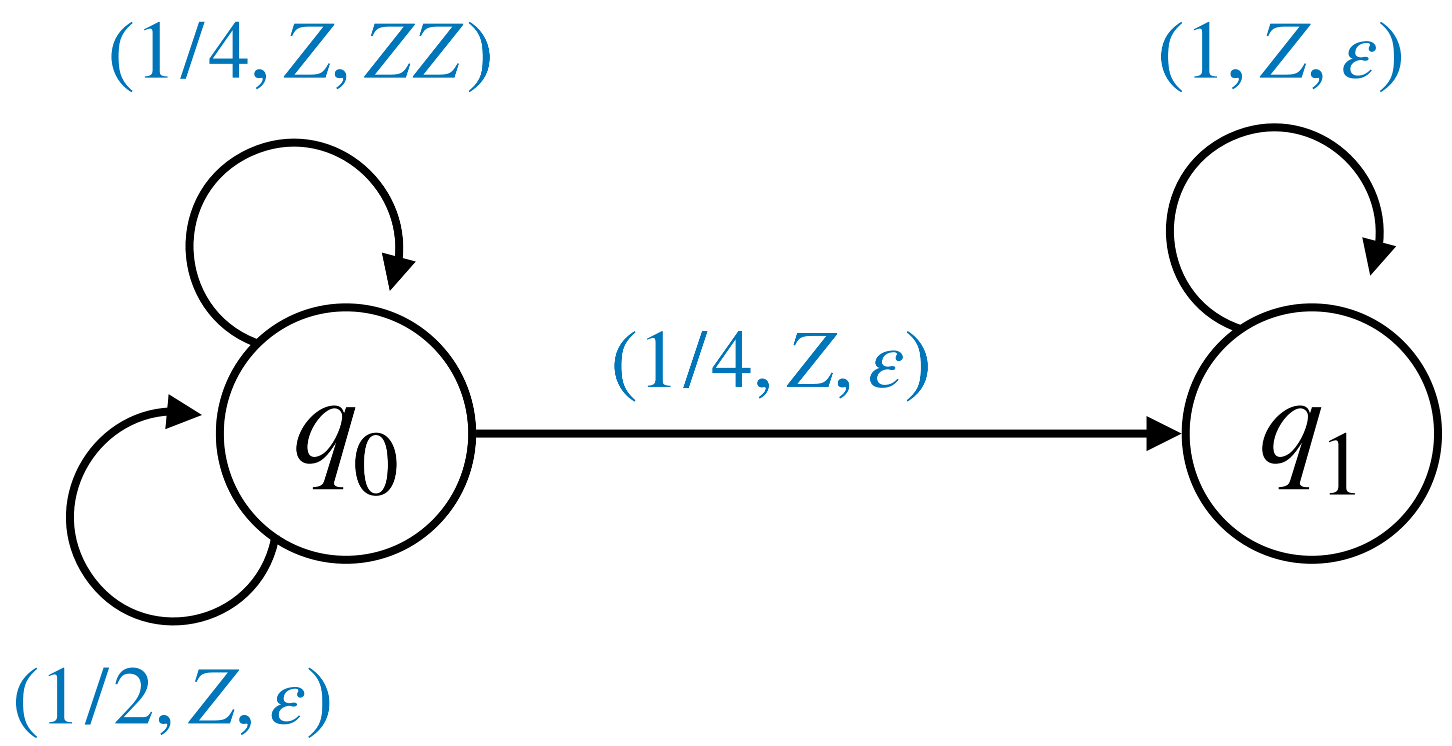

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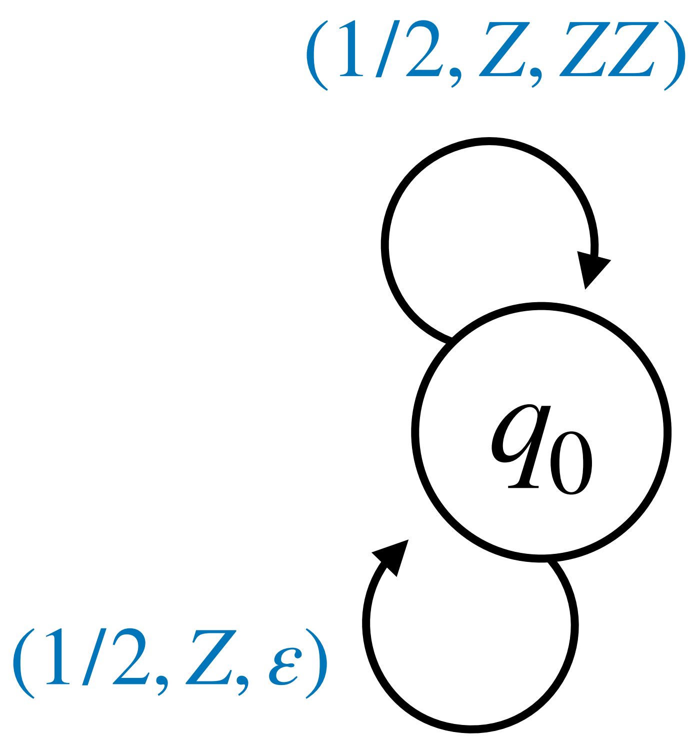

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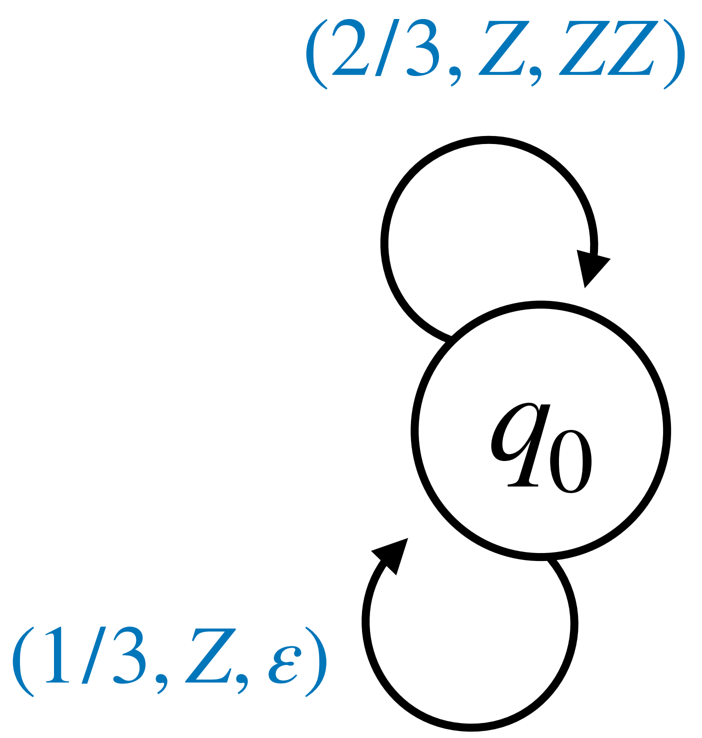
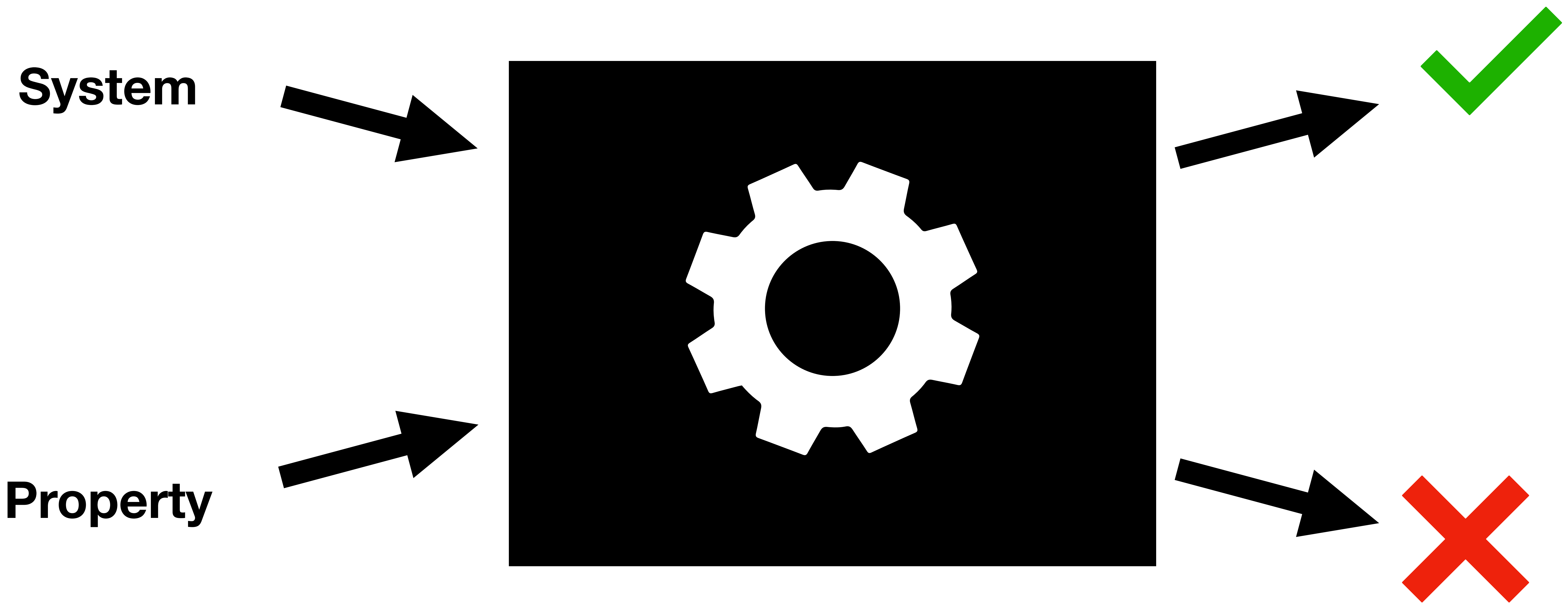


not AST

All decidable in PSPACE by reduction to $\exists\mathbb{R}$ [Esparza et al. LICS ’04 + ’05]

Do you trust your verification tool?



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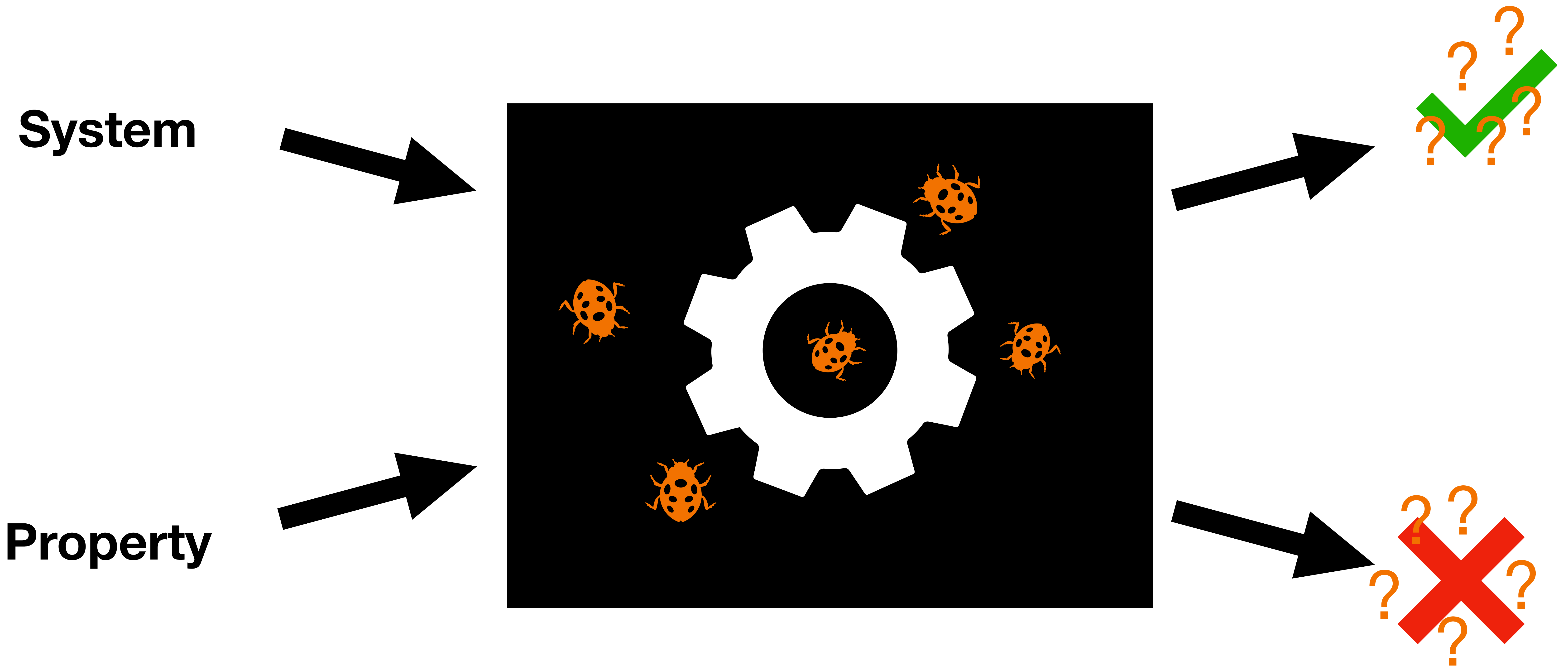

Main Result: Certificates for PAST

A pPDA Δ terminates with probability 1 in finite expected runtime (PAST)

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$$\Longleftrightarrow$$

there exist **rational** vectors $\overrightarrow{u} \in \mathbb{Q}^n_{\geq 0}$, $\vec{r} \in \mathbb{Q}^m_{\geq 0}$ such that

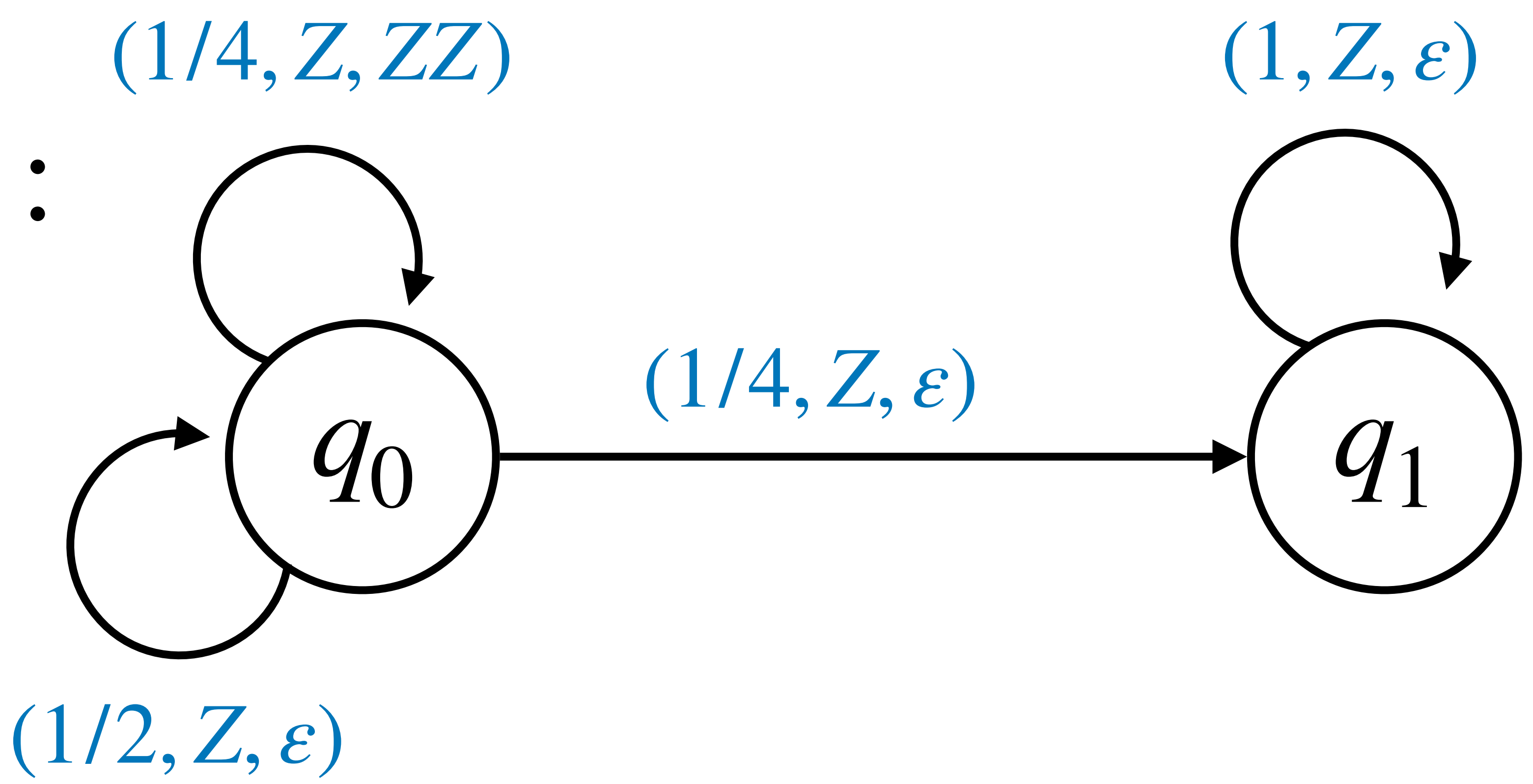
(1) $\vec{f}_\Delta(\overrightarrow{u}) \leq \overrightarrow{u}$

(2) $M_\Delta(\overrightarrow{u})\vec{r} + \overrightarrow{1} \leq \vec{r}$

where $\vec{f}_\Delta \in \mathbb{Q}_{\geq 0}[x_1, \ldots, x_n]^n$ and $M_\Delta \in \mathbb{Q}_{\geq 0}[x_1, \ldots, x_n]^{m \times m}$ can be constructed in polynomial time in the size of Δ.

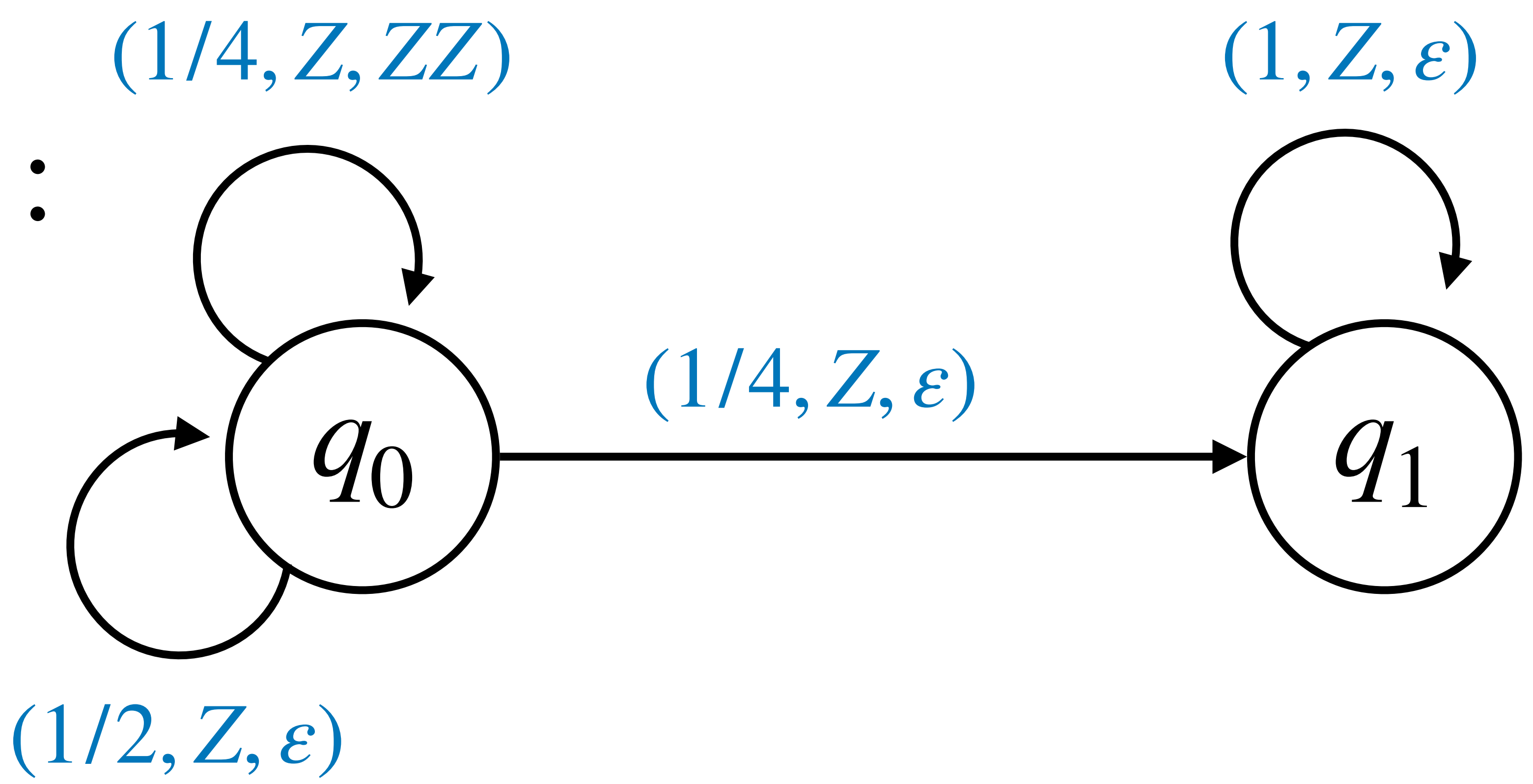
Certifying PAST — Concrete Example

$\Delta :$



Certifying PAST — Concrete Example

Δ :



Read off from Δ:

$$\vec{f}_\Delta = \begin{pmatrix} \frac{1}{4}x_0^2 + \frac{1}{2} \\ \frac{1}{4}x_0x_1 + \frac{1}{4}x_1 + \frac{1}{4} \end{pmatrix}$$

$$M_\Delta = \begin{pmatrix} \frac{1}{4} + x_0 & \frac{1}{4} + x_1 \\ 0 & 0 \end{pmatrix}$$

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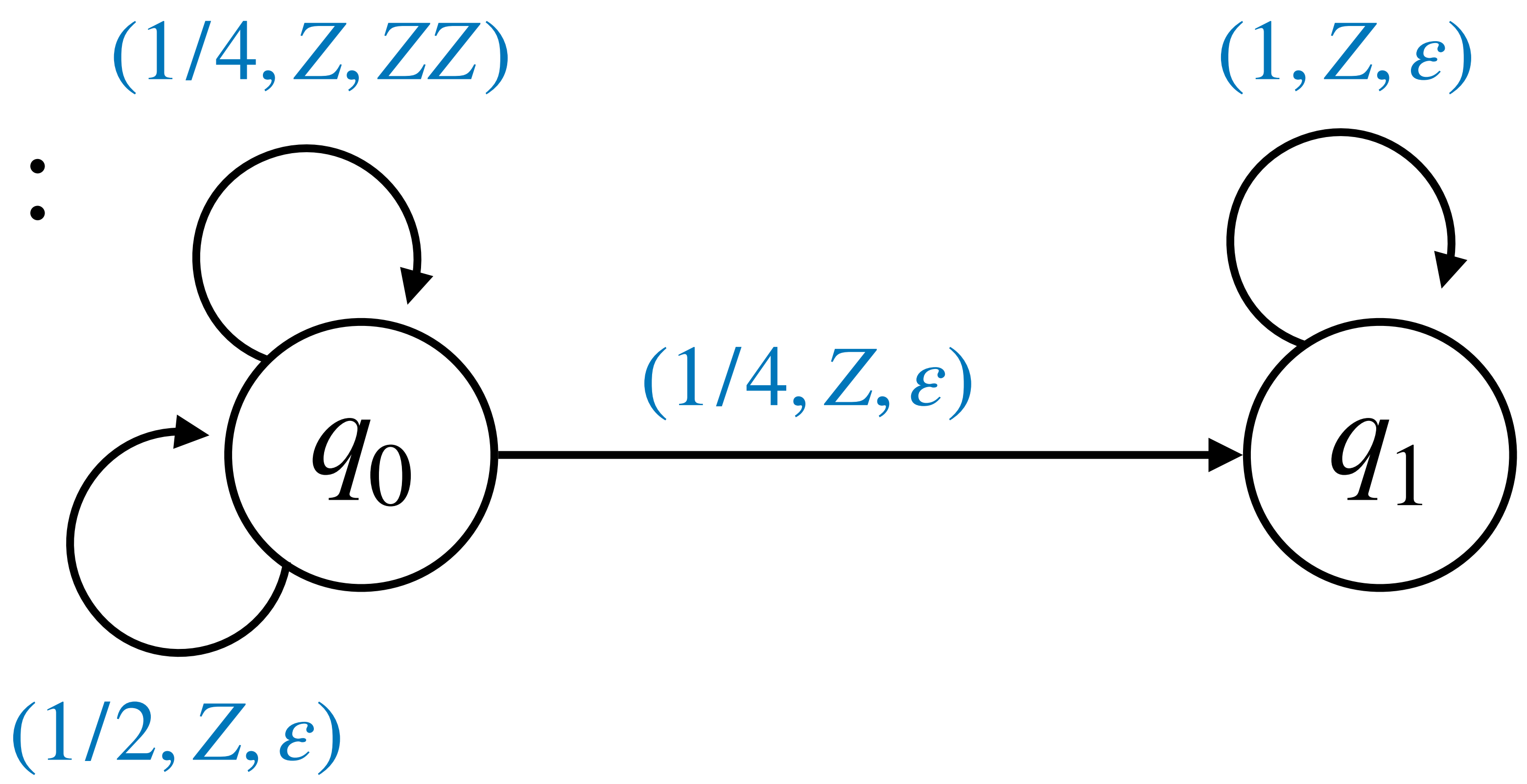

Certificate: $\vec{u} = \begin{pmatrix} \frac{3}{5} \\ \frac{1}{2} \end{pmatrix}, \vec{r} = \begin{pmatrix} \frac{45}{14} \\ 1 \end{pmatrix}$

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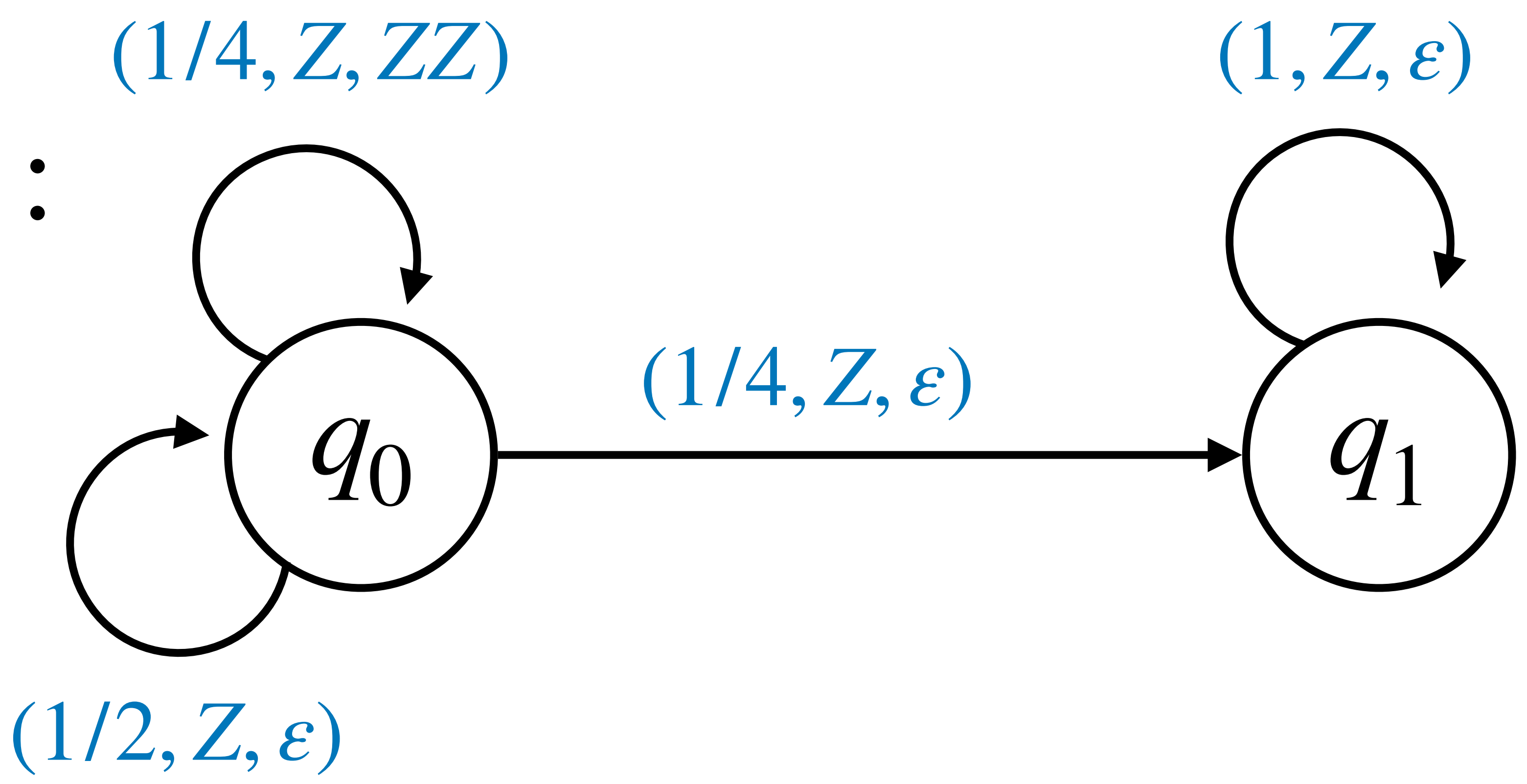

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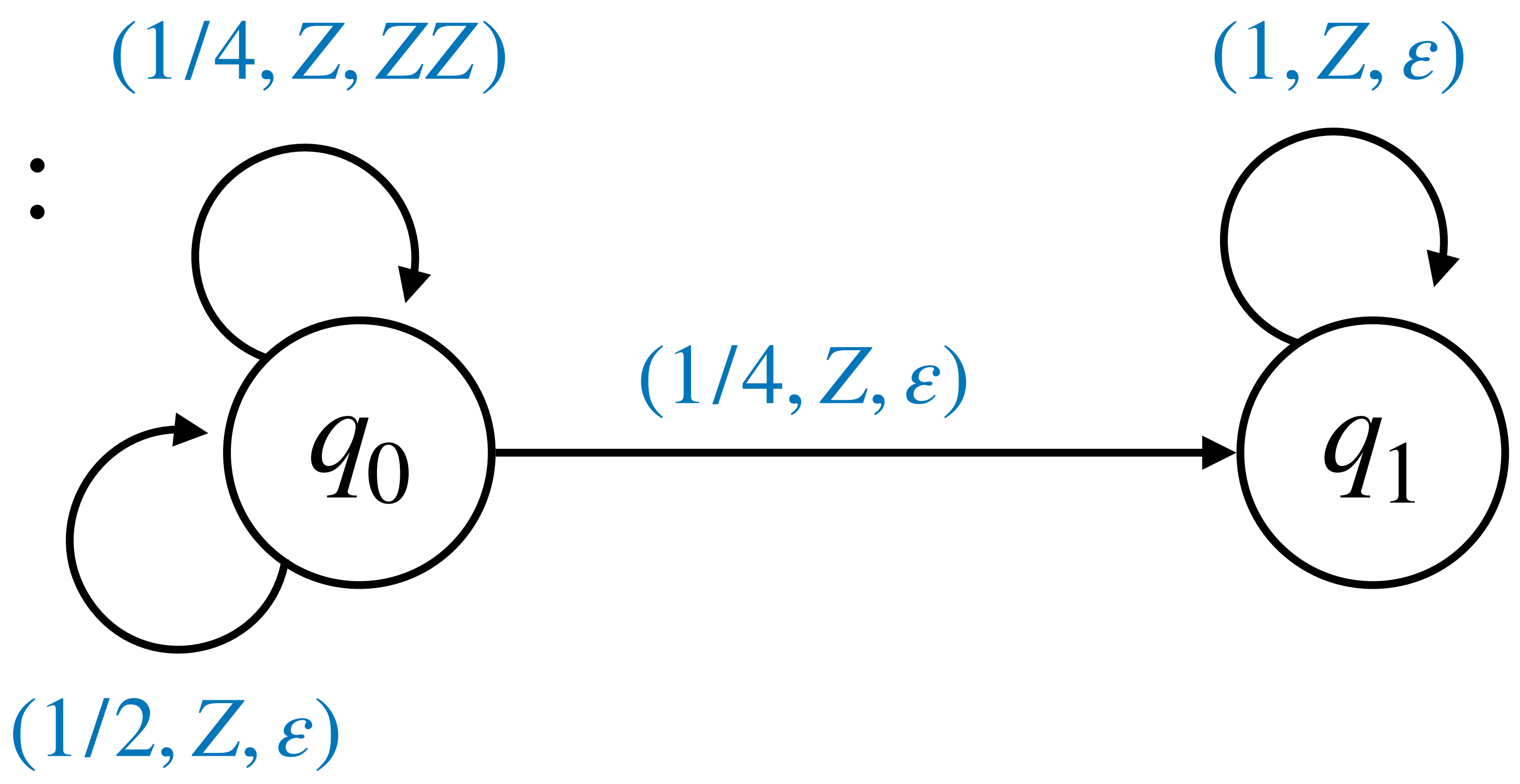

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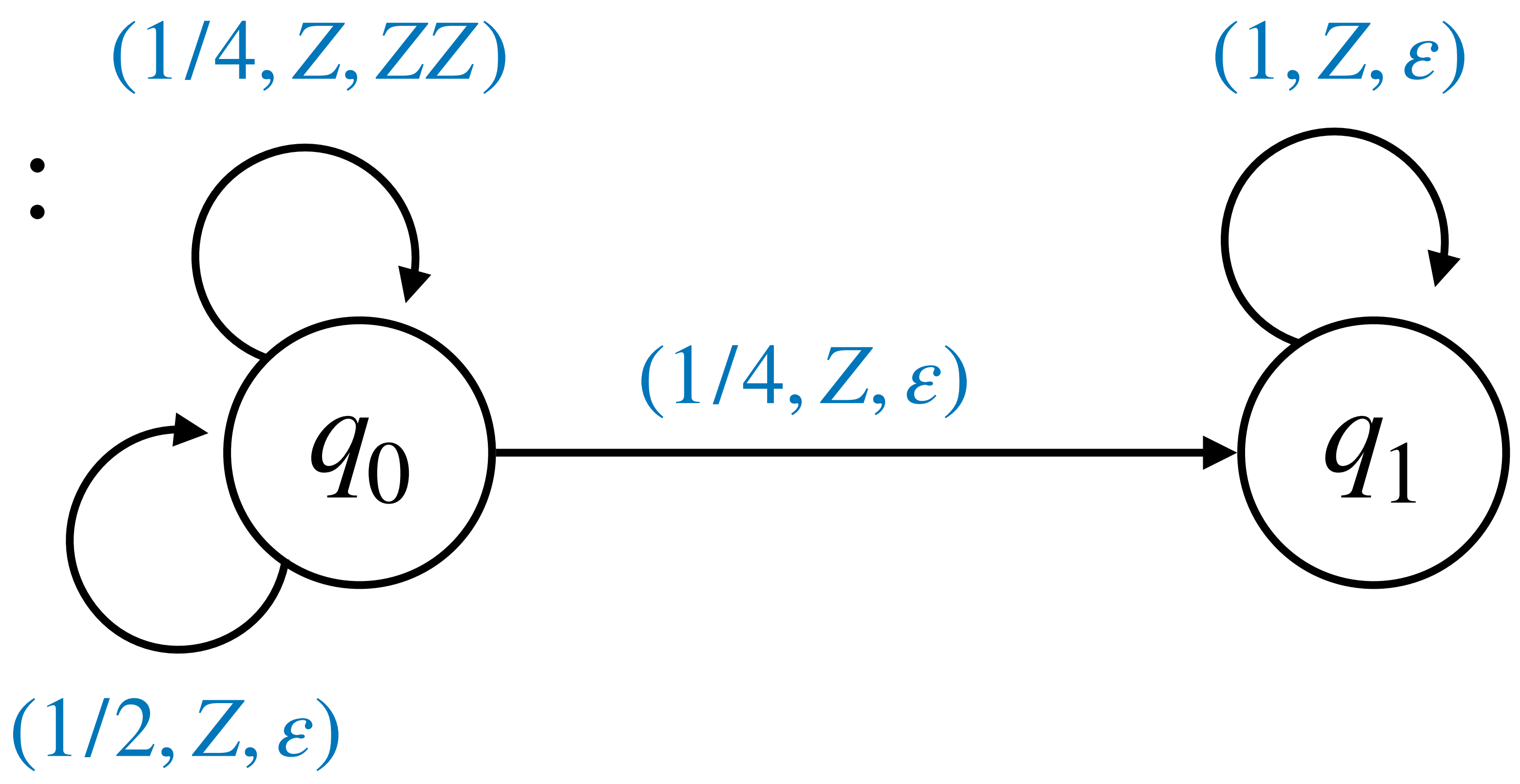

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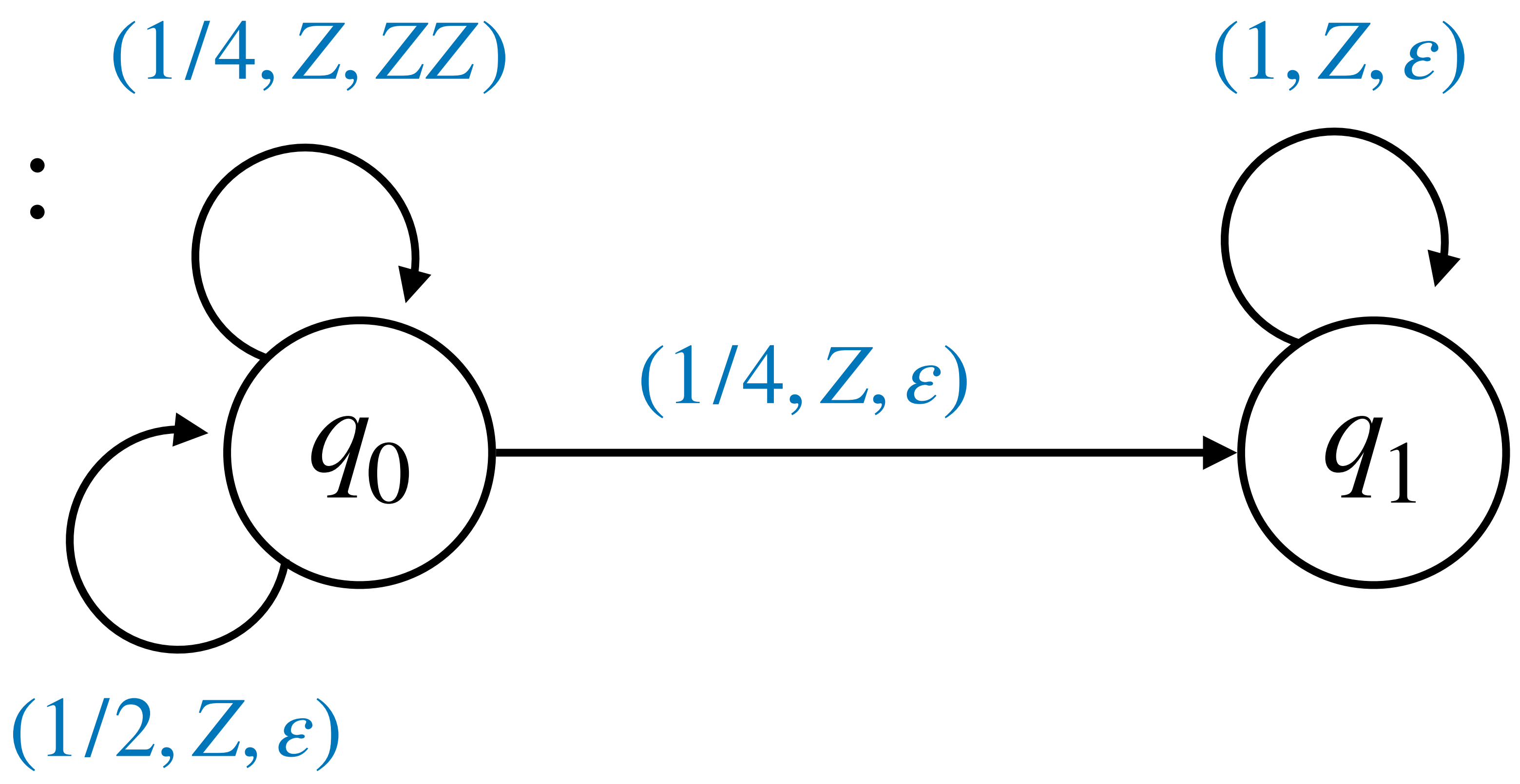

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This proves PAST (soundness).

For every PAST pPDA we can find such a certificate (completeness).

Main Result: Certificates for PAST

A pPDA Δ terminates with probability 1 in finite expected runtime (PAST)

$\Longleftarrow$ (soundness)

there exist **rational** vectors $\vec{u} \in \mathbb{Q}^n_{\geq 0}$, $\vec{r} \in \mathbb{Q}^m_{\geq 0}$ such that

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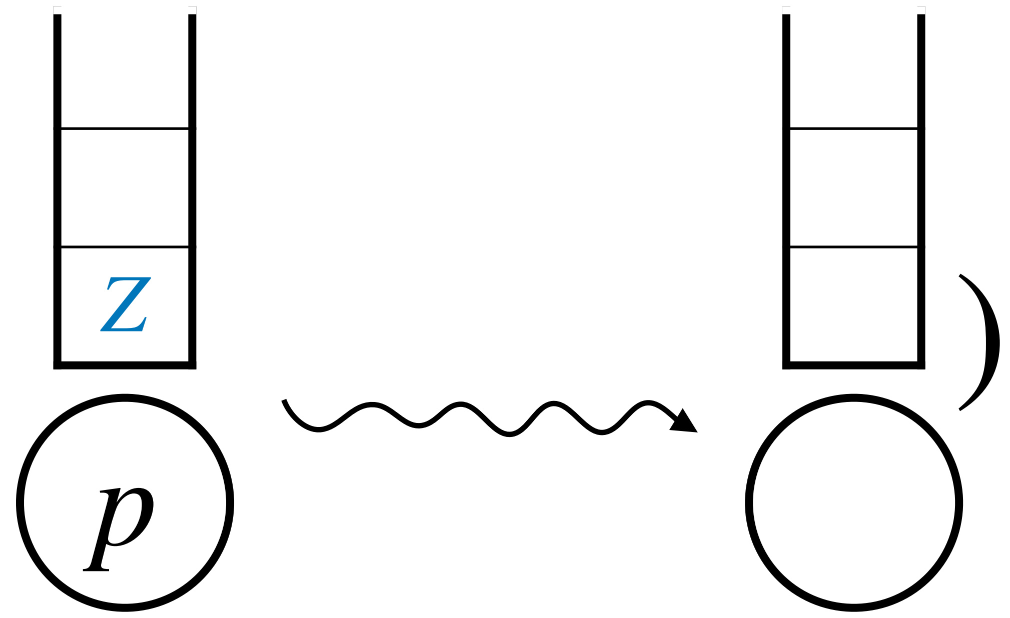
Characterizing the Expected Runtime (*ert*)

$ert[pZ] = E[len($  $)]$

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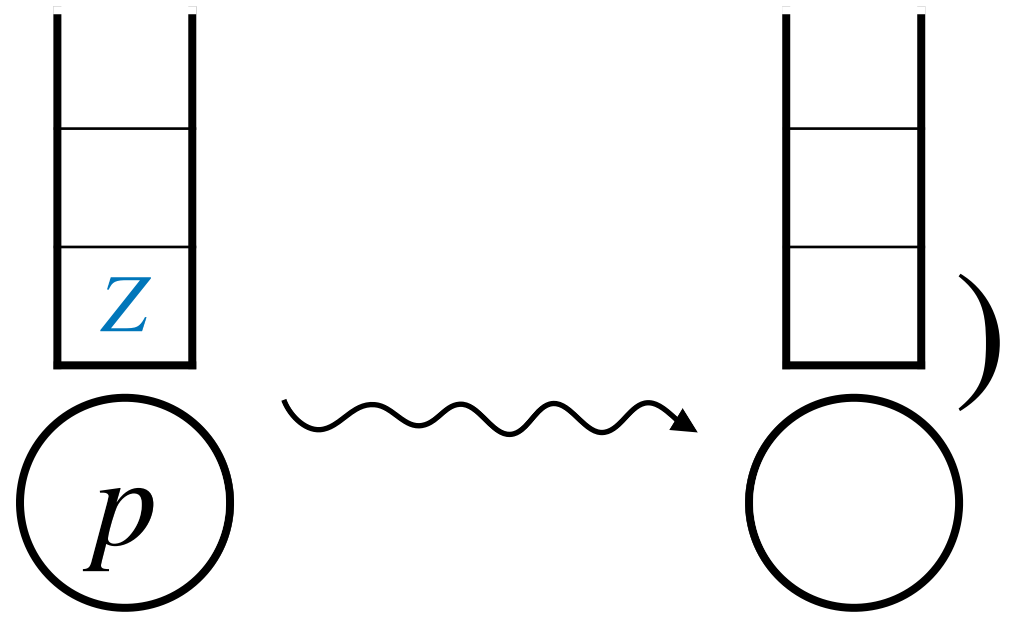
$ert[pZ] = E[len($  $)]$

$$ert[pZ] = 1 + \sum_{pZ \xrightarrow{a} rY} a \cdot ert[rY] + \sum_{pZ \xrightarrow{a} rYX} a \cdot \left(ert[rY] + \sum_{t} [rYt] \cdot ert[tX]\right)$$

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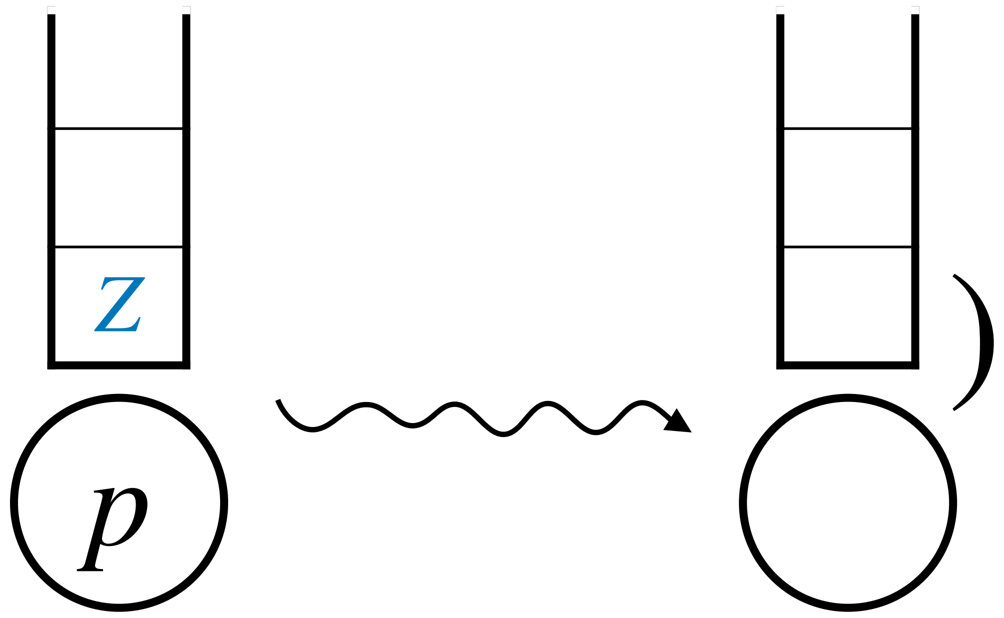
$ert[pZ] = E[len($  $)]$

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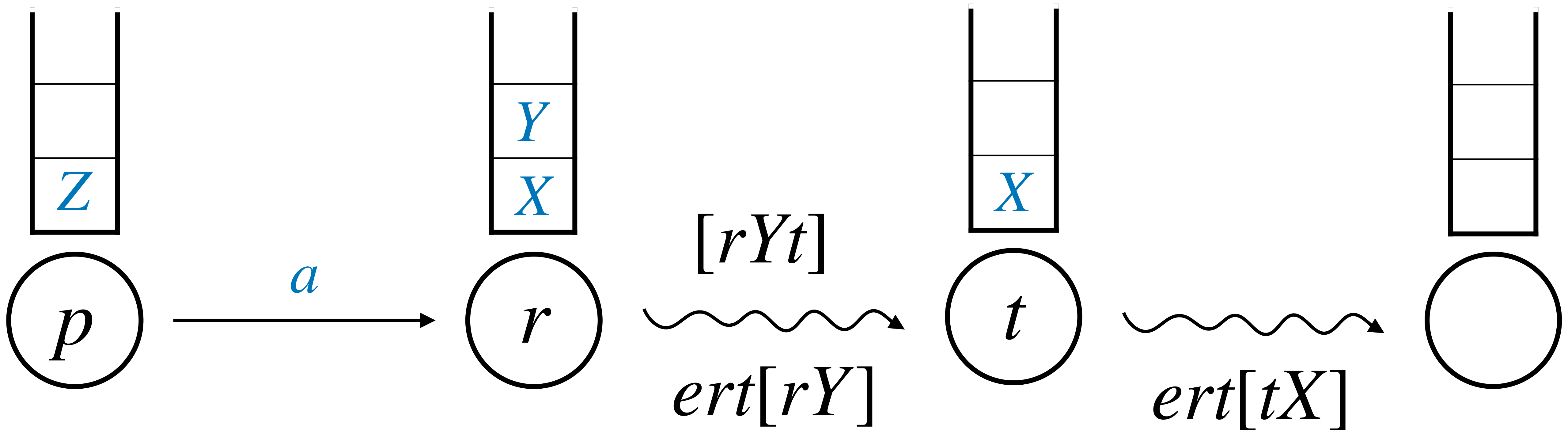

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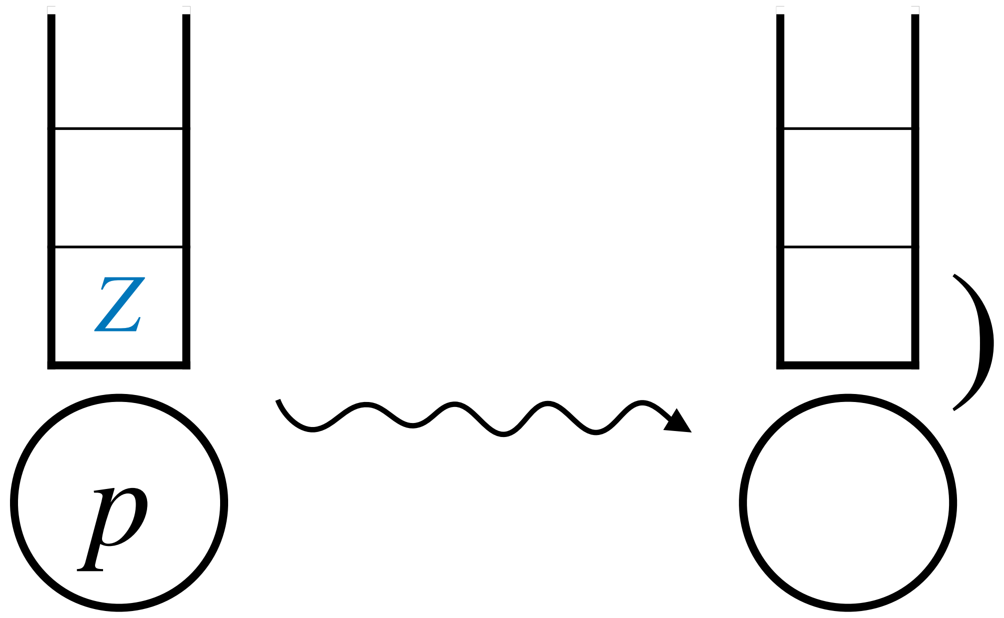
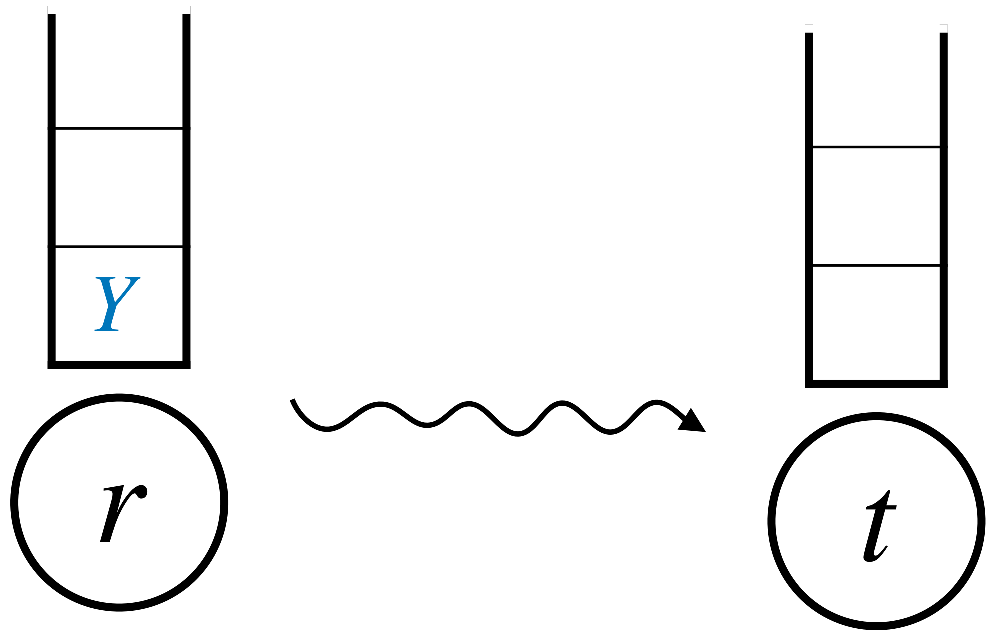
$ert[pZ] = E[len($  $)]$

$[rYt] = Pr[$  $]$

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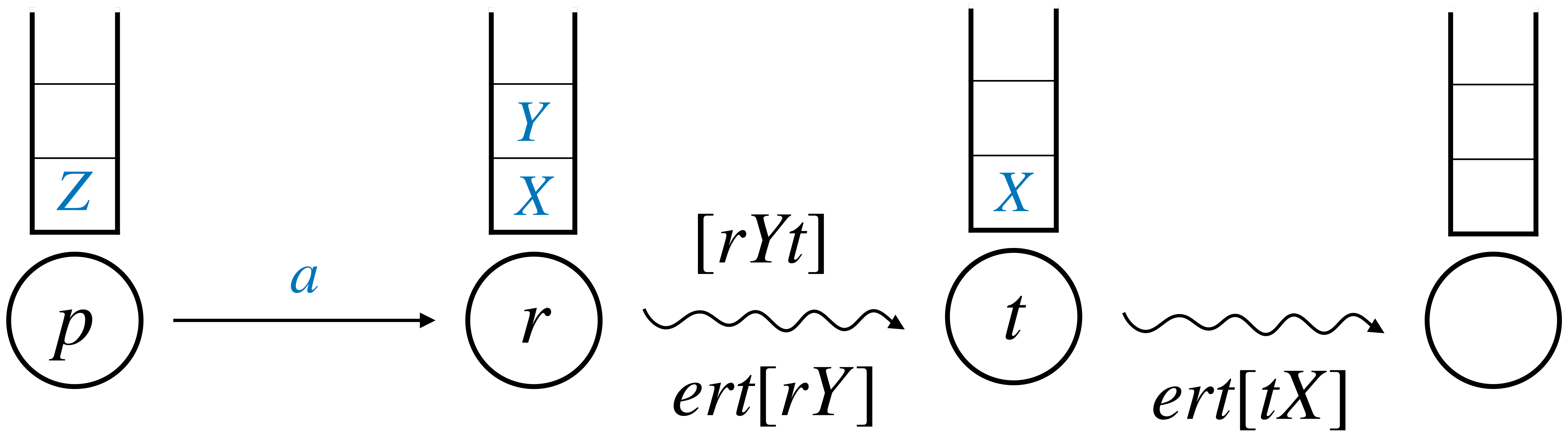

Characterizing the Expected Runtime

Theorem
The linear equation system

$$\forall p, Z \qquad ert[pZ] = 1 + \sum_{pZ \xrightarrow{a} rY} a \cdot ert[rY] + \sum_{pZ \xrightarrow{a} rYX} a \cdot \left(ert[rY] + \sum_{t} [rYt] \cdot ert[tX]\right)$$

has a solution in $\mathbb{R}_{\geq 0}$ iff the pPDA is PAST.

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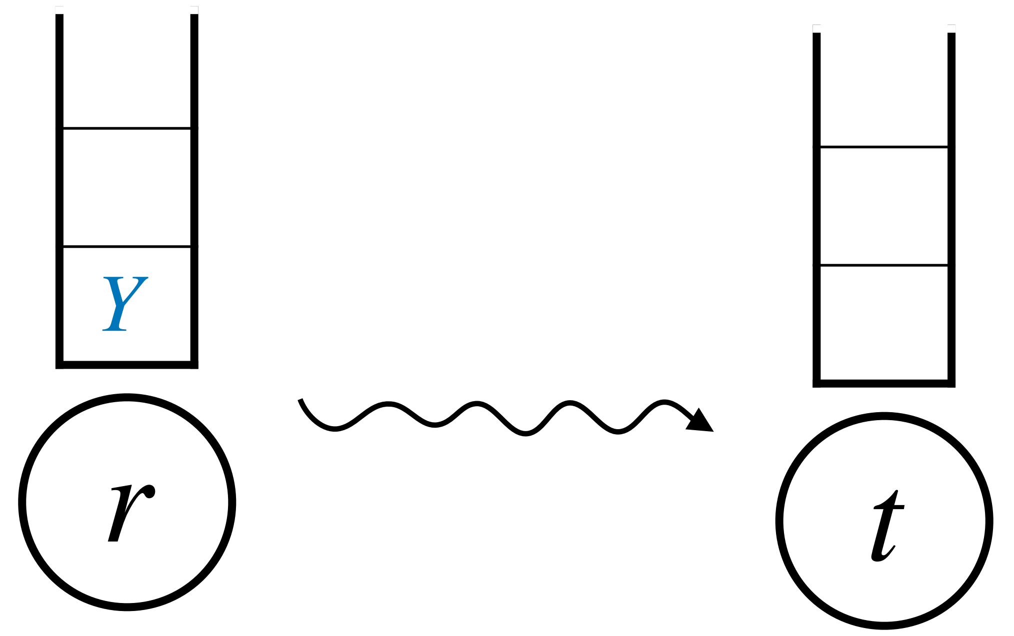
$[rYt] = Pr[$  $]$

Characterizing the Expected Runtime

Matrix $M_\Delta(\overrightarrow{x})$ from certificate condition

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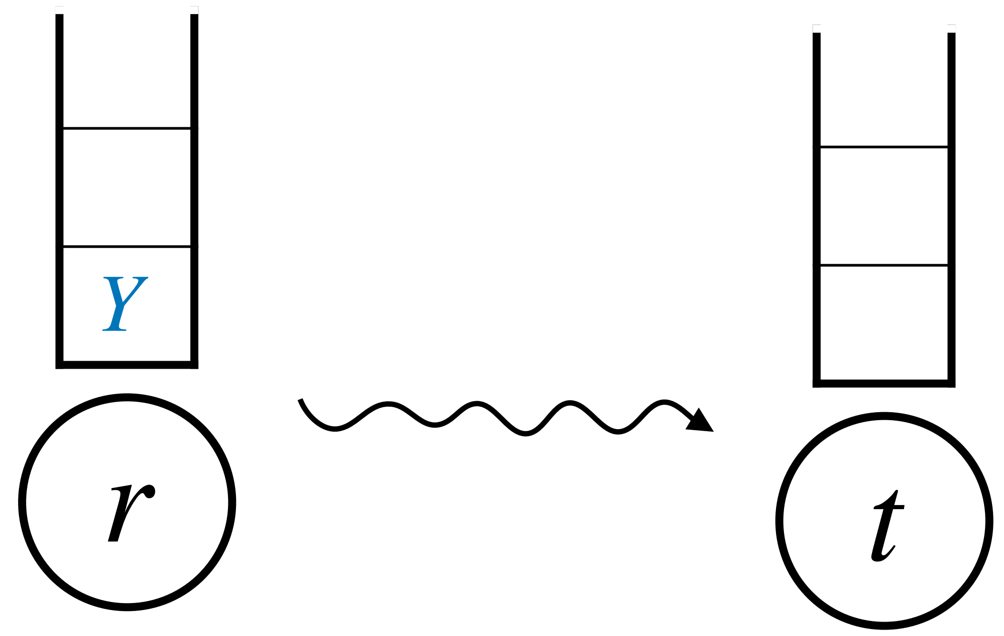
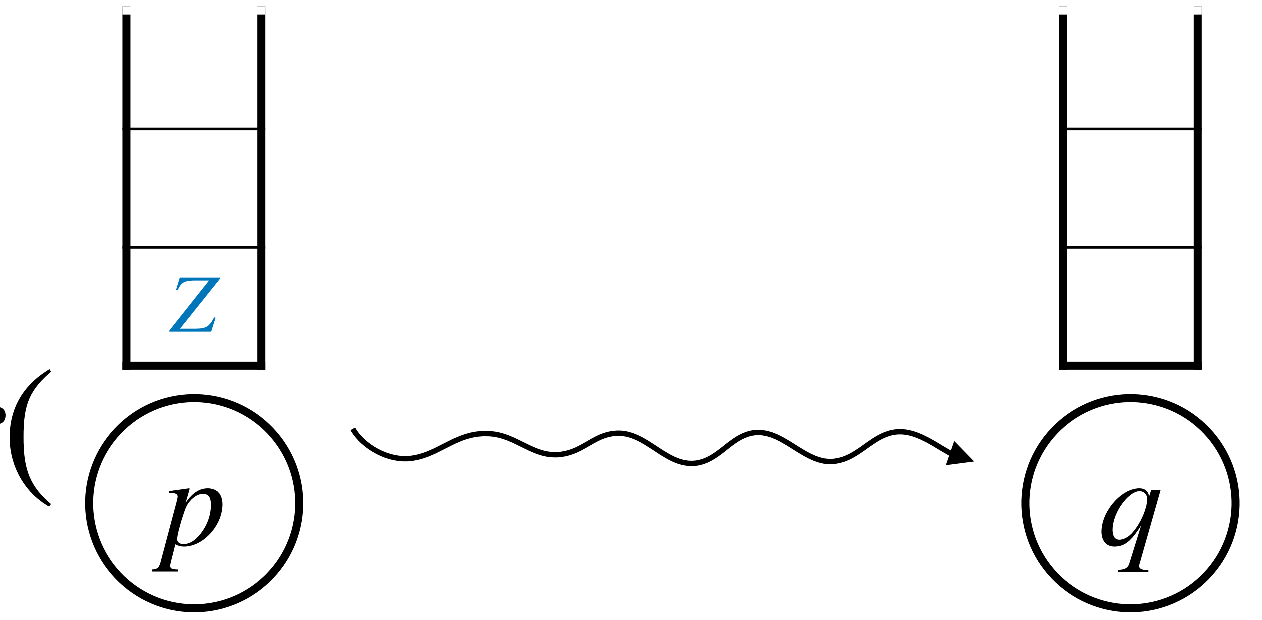
$[rYt] = Pr[$  $]$

pPDA ➡ Polynomial Equations

[Esparza et al. ’04]

$[pZq] = Pr($  $)$

Probabilities $[pZq]$ are least solution ≥ 0 of

$$[pZq] \;=\; \sum_{pZ \xrightarrow{a} qY} a \cdot [rYq] + \sum_{pZ \xrightarrow{a} rXY} \sum_{t \in Q} a \cdot [rYt] \cdot [tXq] + \sum_{pZ \xrightarrow{a} q\epsilon} a$$

pPDA ➞ Polynomial Equations

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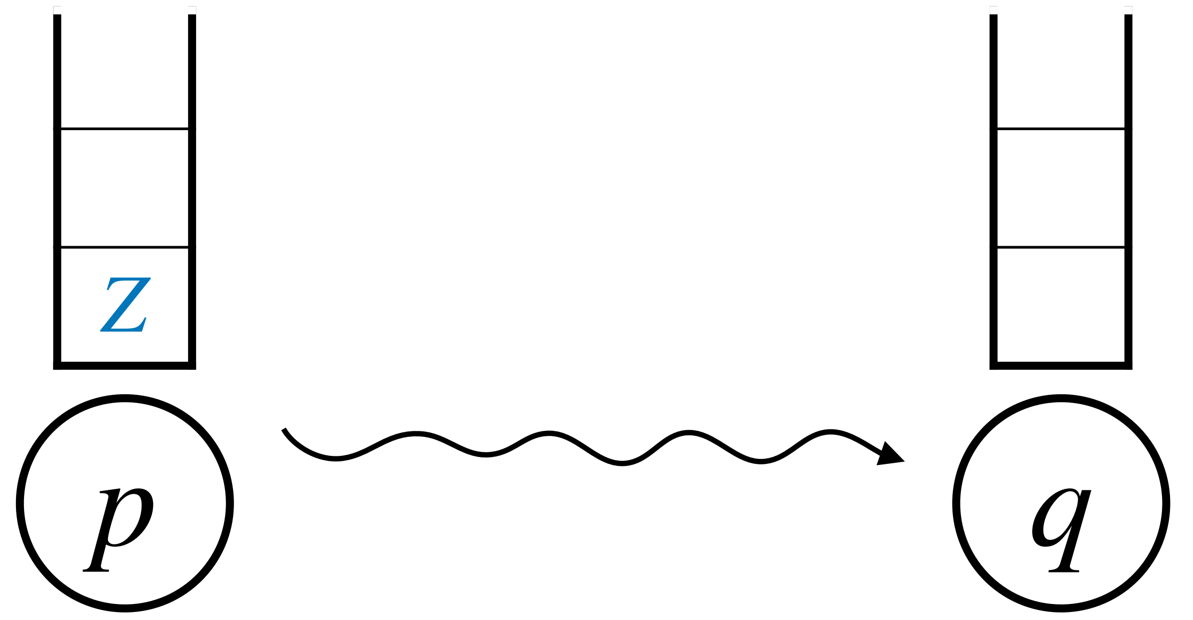
$[pZq] = Pr($  $)$

Probabilities $[pZq]$ are least solution ≥ 0 of

$$[pZq] \;=\; \sum_{pZ\xrightarrow{a}qY} a\cdot[rYq] + \sum_{pZ\xrightarrow{a}rXY}\;\sum_{t\in Q} a\cdot[rYt]\cdot[tXq] + \sum_{pZ\xrightarrow{a}q\epsilon} a$$

Polynomials $\vec{f}_\Delta$ from certificate condition

Certificates for Upper Bounds on [pZq]

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For all $\overrightarrow{u} \in \mathbb{R}^n_{\geq 0}$: $\vec{f}(\overrightarrow{u}) \leq \overrightarrow{u} \implies \text{lfp}\, \vec{f} \leq \overrightarrow{u}$

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Main Result: Certificates for PAST

A pPDA Δ terminates with probability 1 in finite expected runtime (PAST)

$\Longleftarrow$ ✓

there exist **rational** vectors $\overrightarrow{u} \in \mathbb{Q}^n_{\geq 0}$, $\vec{r} \in \mathbb{Q}^m_{\geq 0}$ such that

(1) $\vec{f}_\Delta(\overrightarrow{u}) \leq \overrightarrow{u}$

(2) $M_\Delta(\overrightarrow{u})\vec{r} + \overrightarrow{1} \leq \vec{r}$

where $\vec{f}_\Delta \in \mathbb{Q}_{\geq 0}[x_1, \ldots, x_n]^n$ and $M_\Delta \in \mathbb{Q}_{\geq 0}[x_1, \ldots, x_n]^{m \times m}$ can be constructed in polynomial time in the size of Δ.

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Thank you! Questions?