Certificates for Probabilistic Pushdown Automata via Optimistic Value Iteration

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DFG Research Training Group 2236

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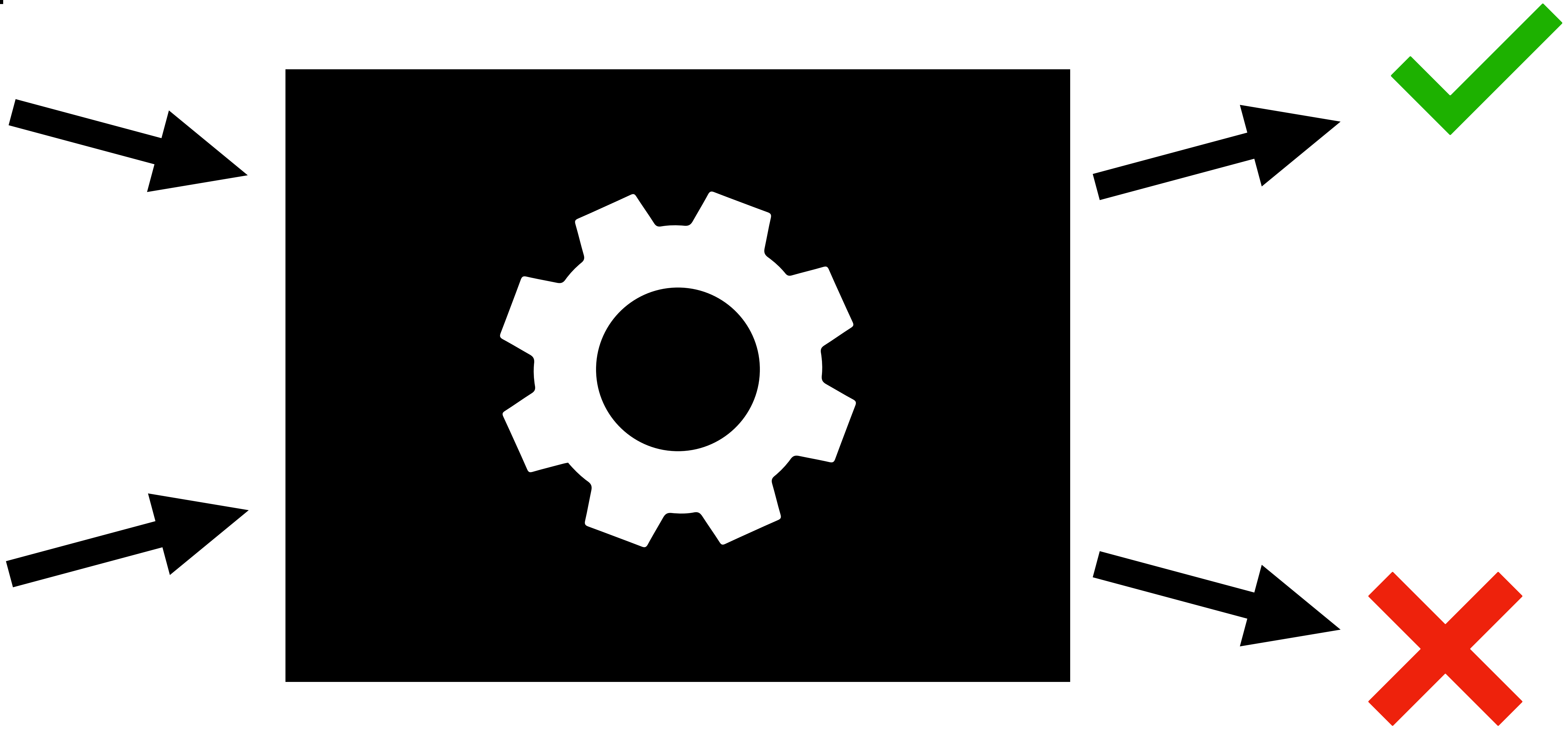
Probabilistic Model Checking

Probabilistic Model
- Markov chain
- MDP
- Probabilistic TA
- …

Property
- Reachability
- Safety
- LTL
- …



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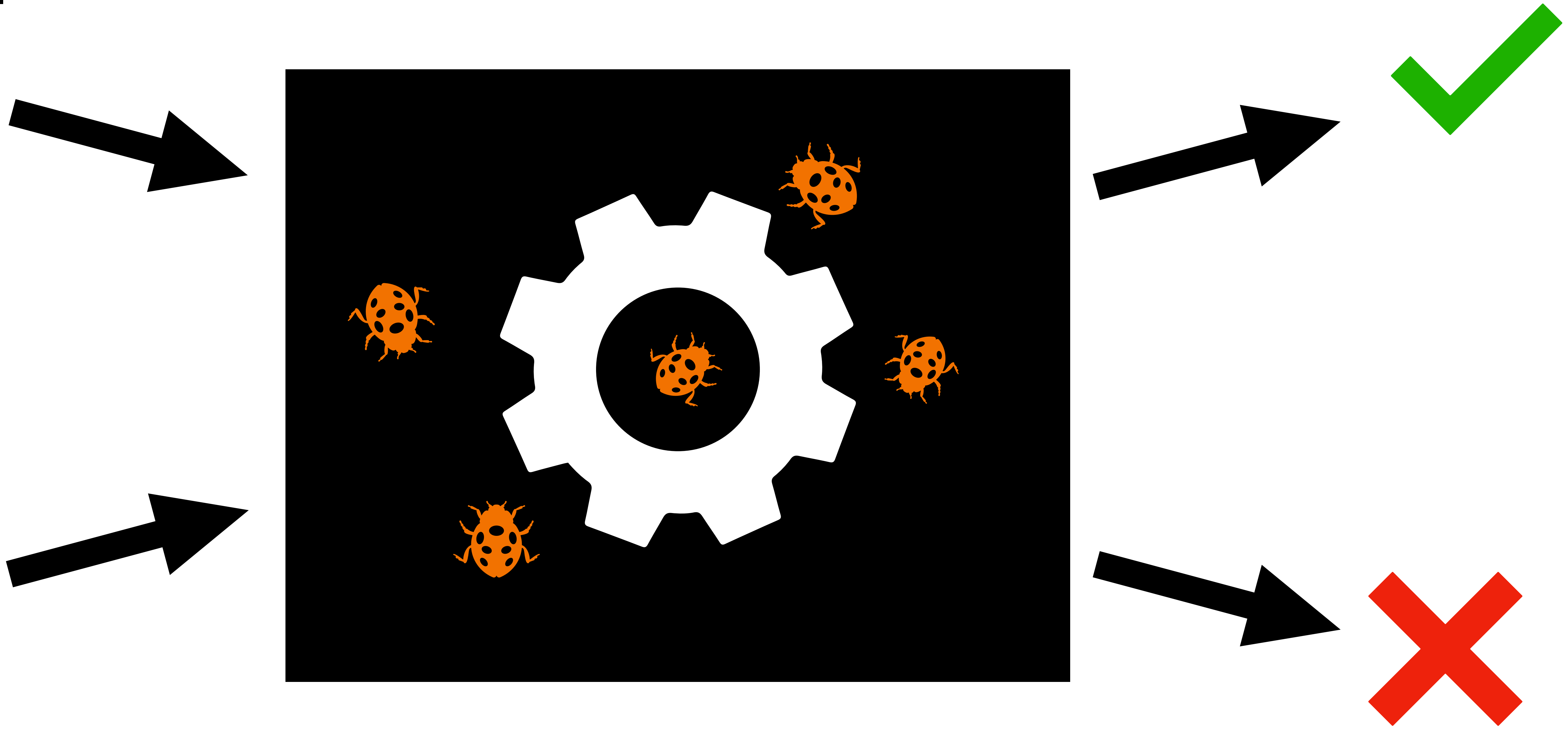

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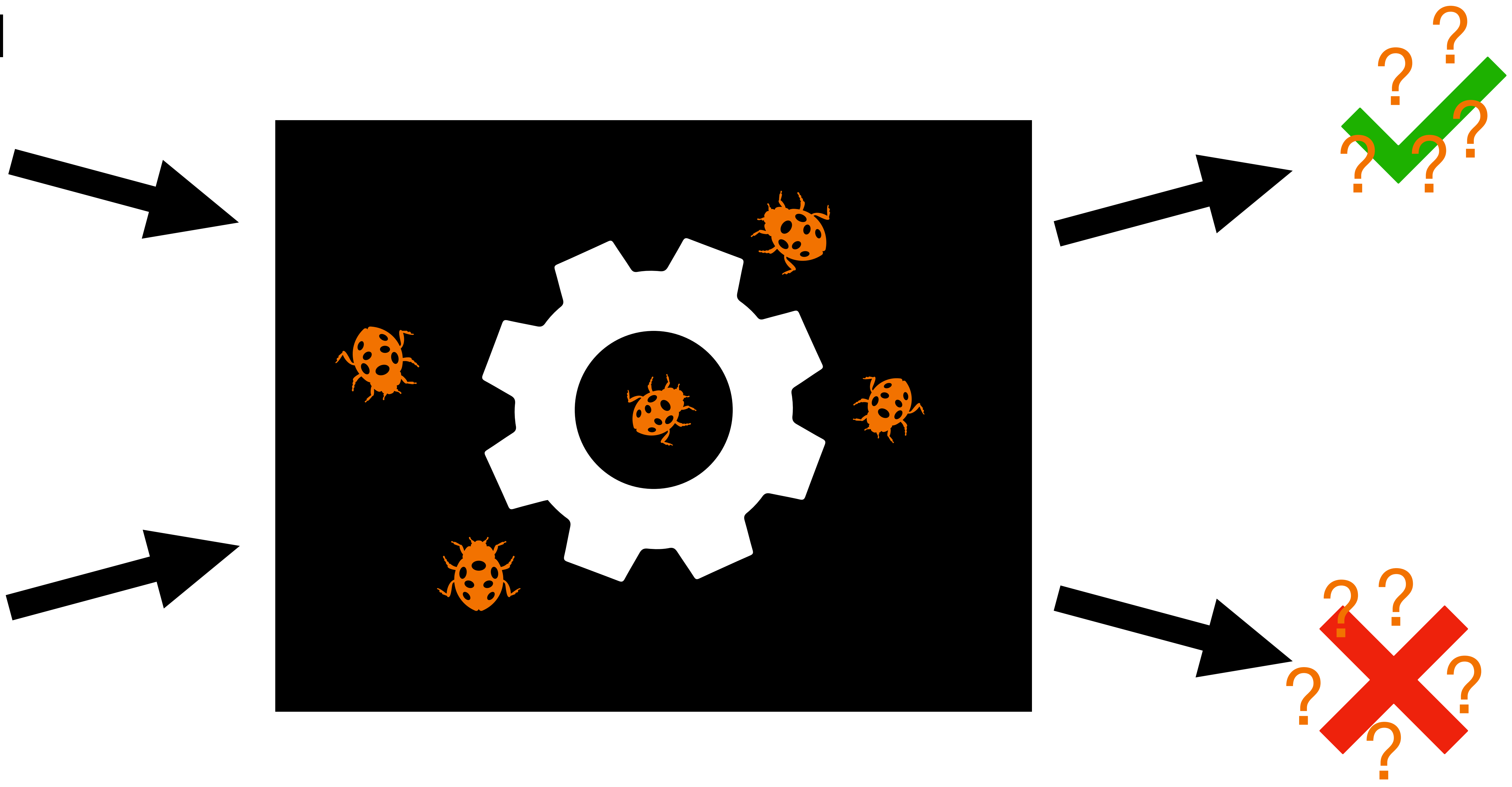

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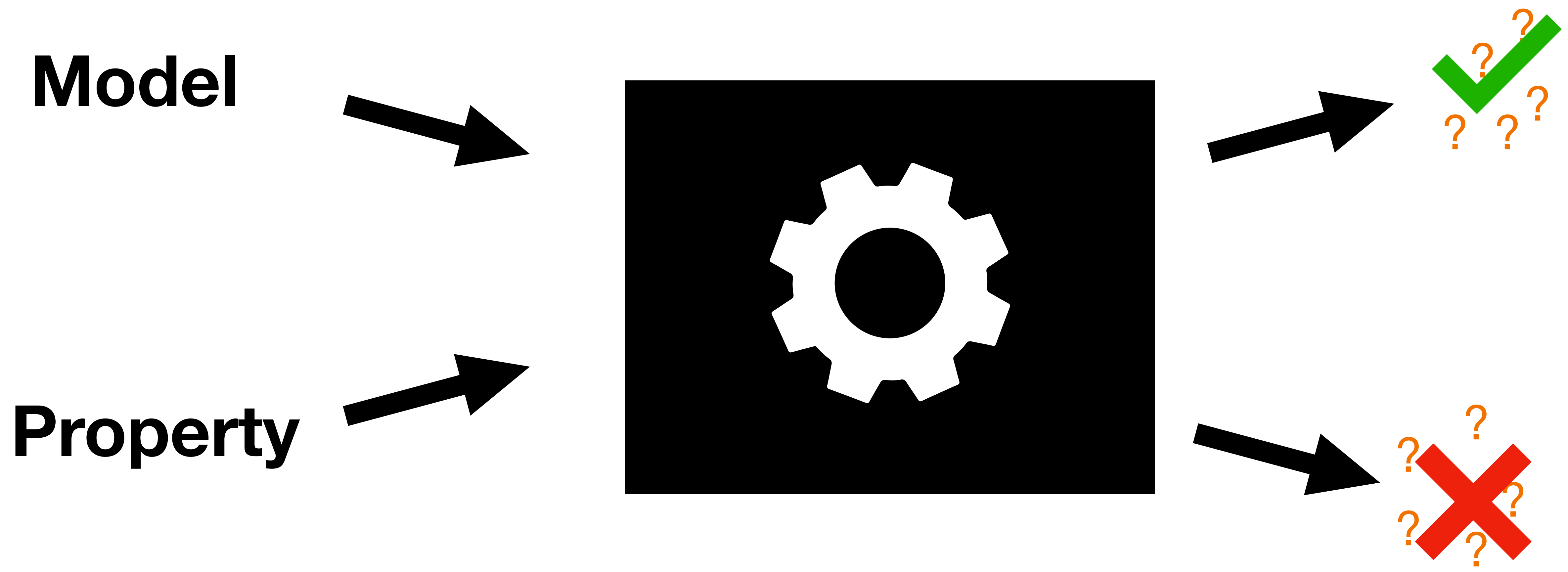

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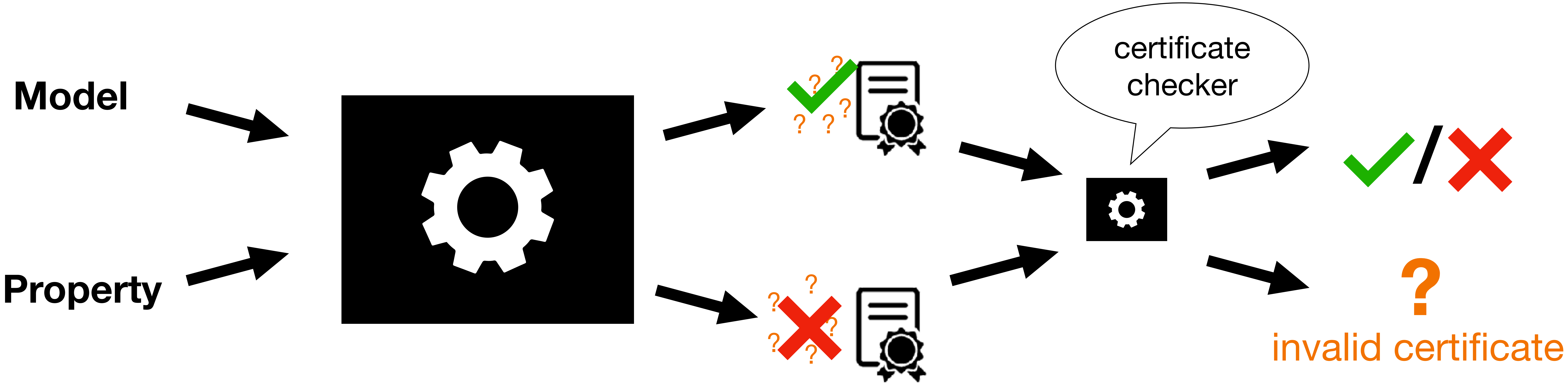

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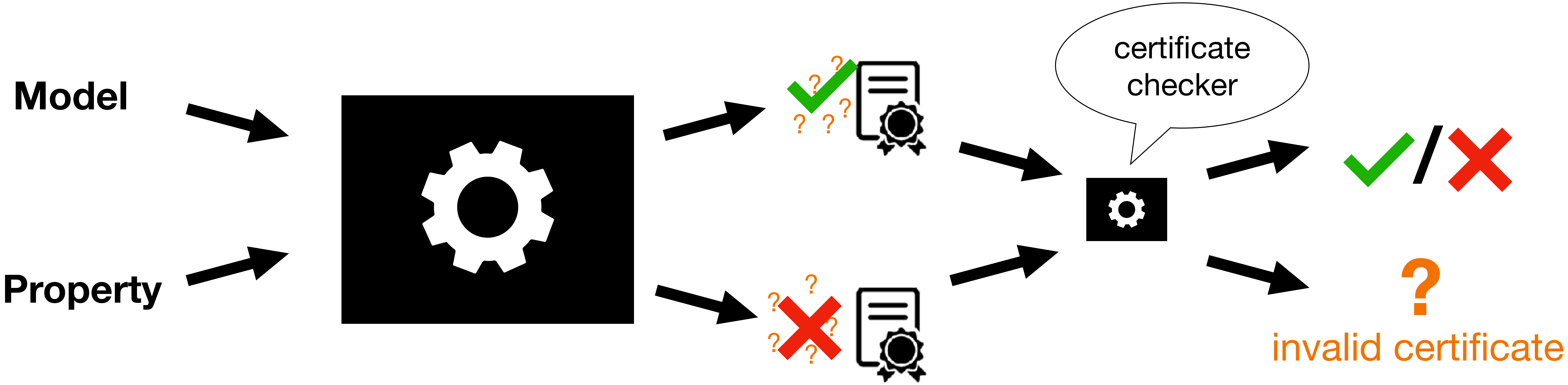


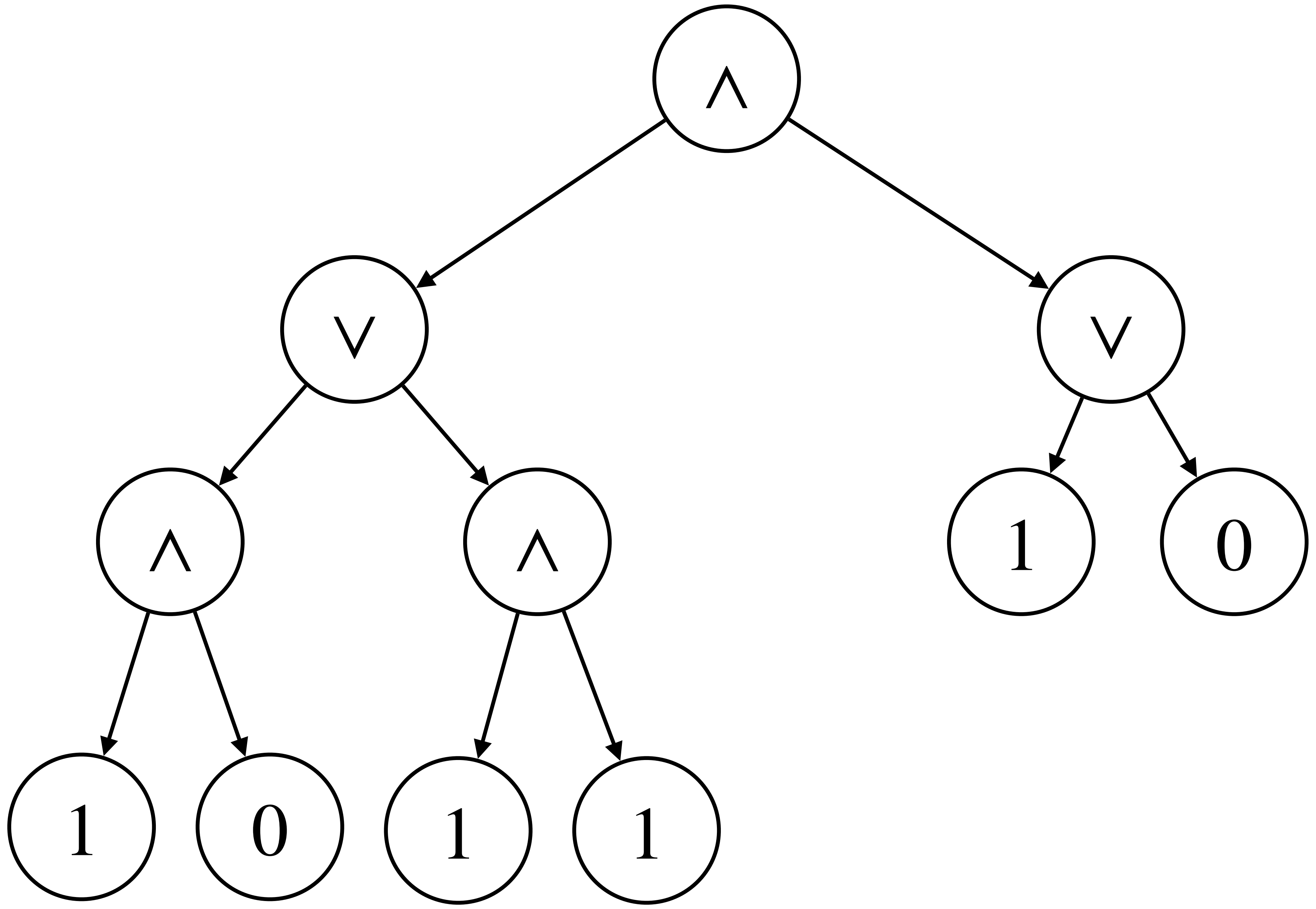
- Literature: certificates for MDP [Funke et al. '20], PTA [Jantsch et al. '20]

This talk:

Certifying Algorithms for
Probabilistic Pushdown Automata (pPDA)

Example: Random And-Or Trees

[Brázdil et al. '15]



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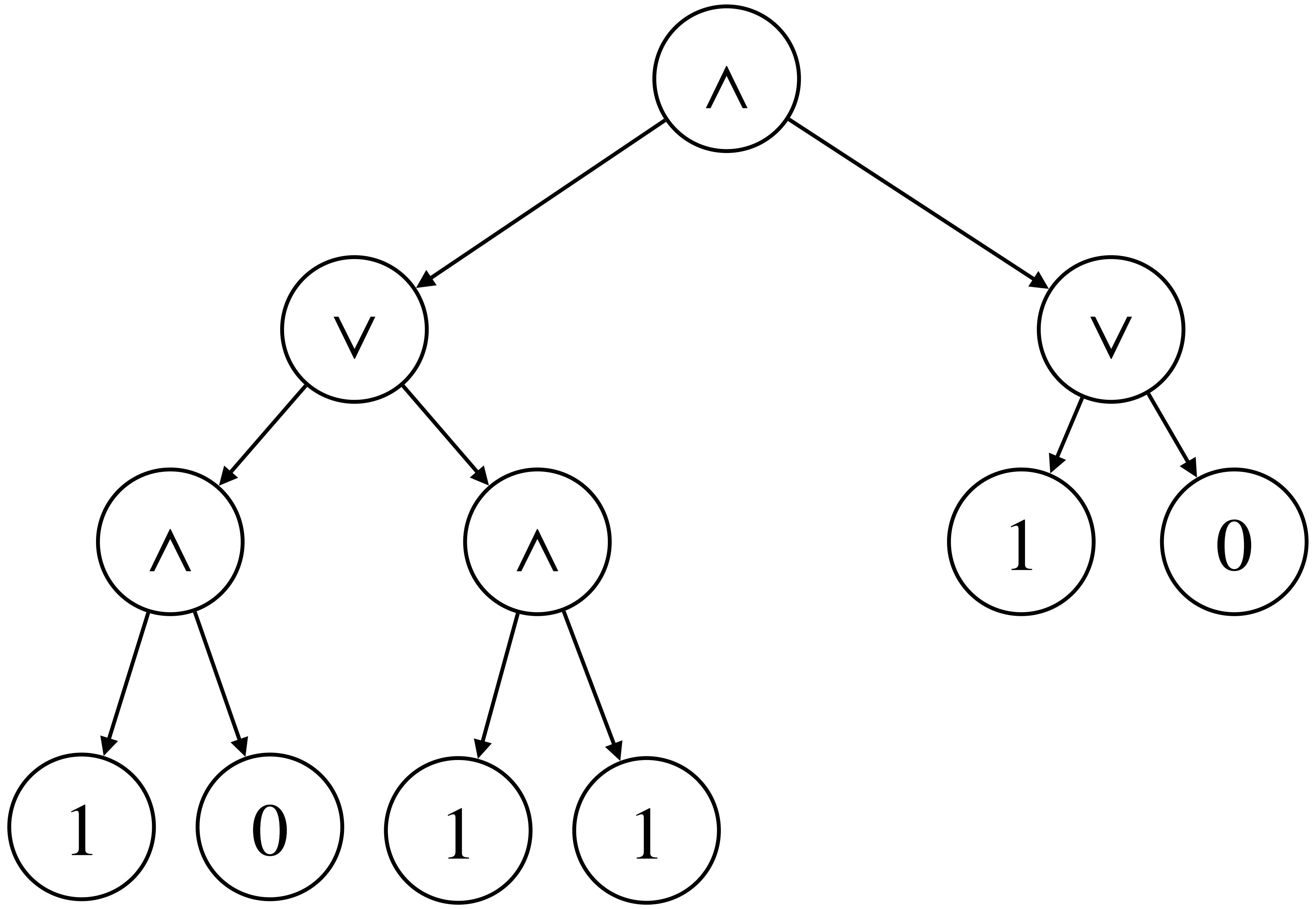

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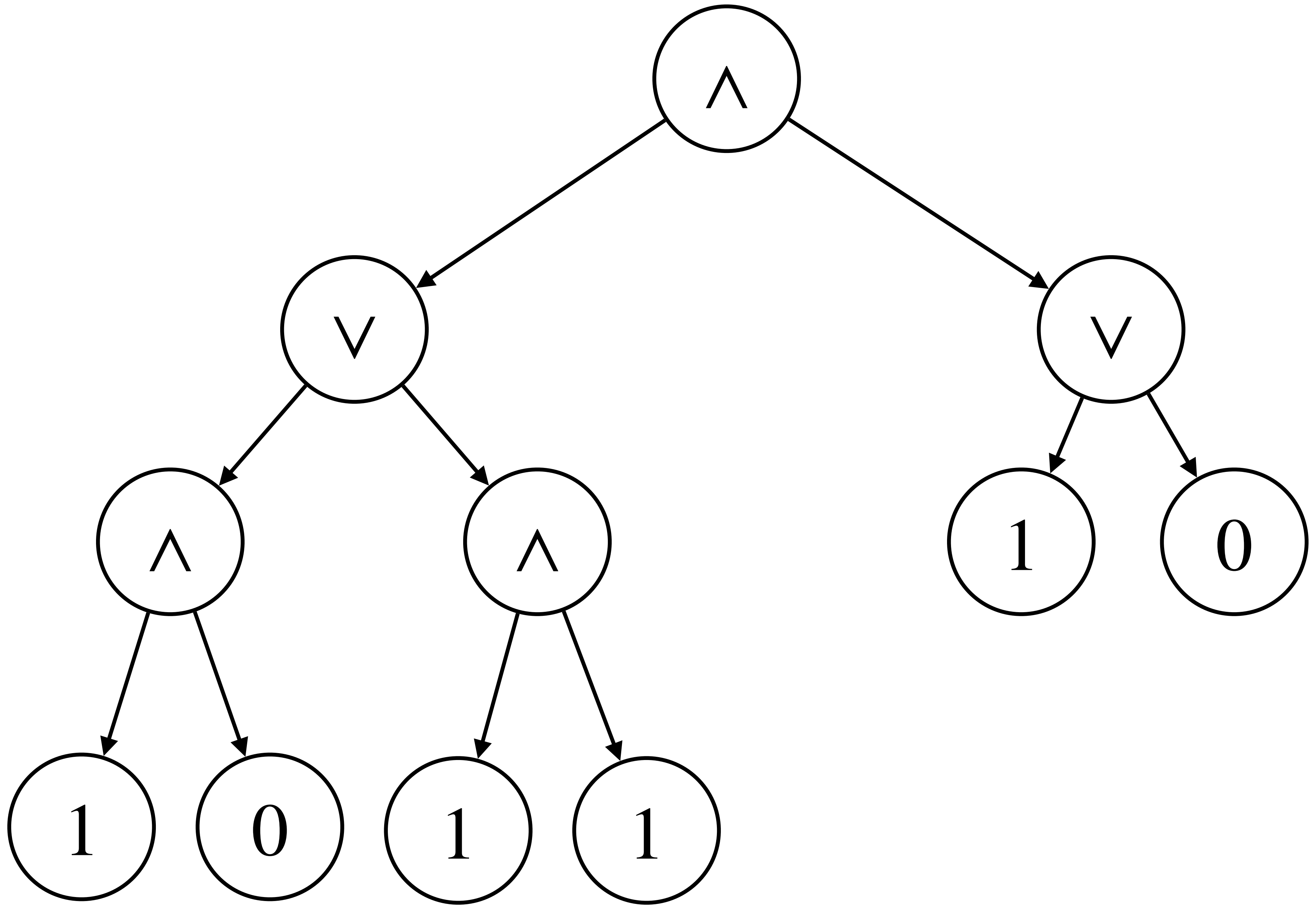

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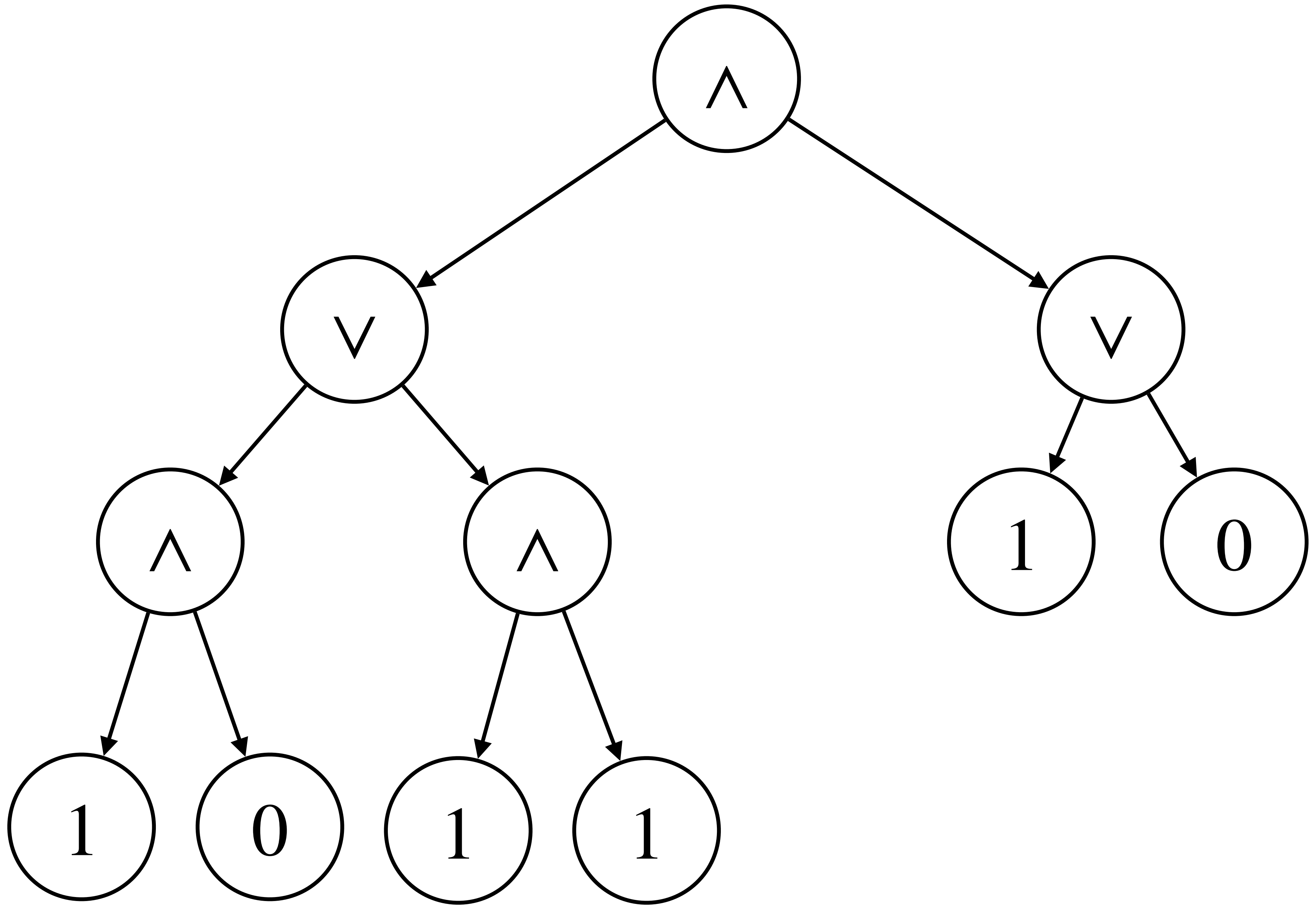

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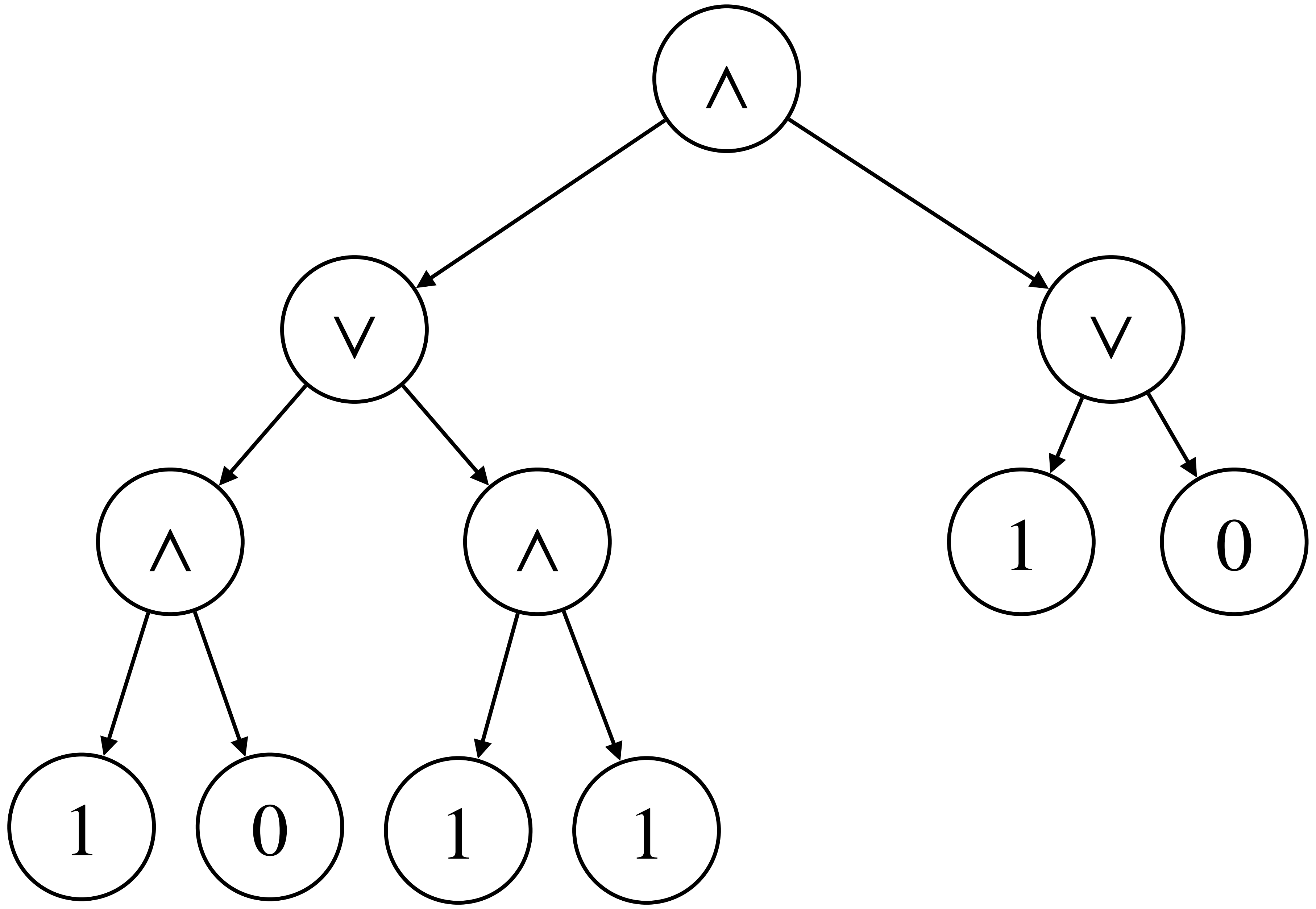

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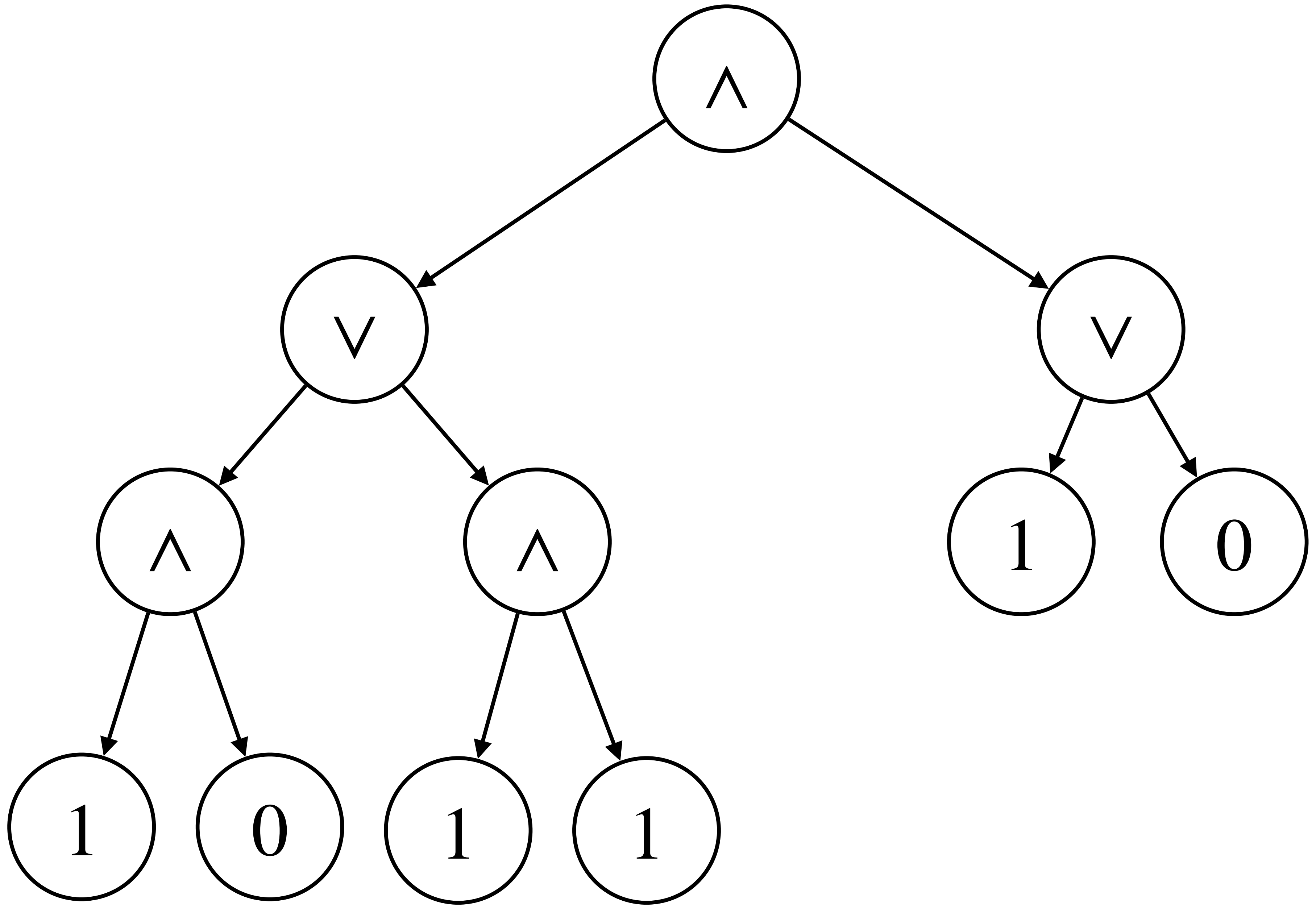
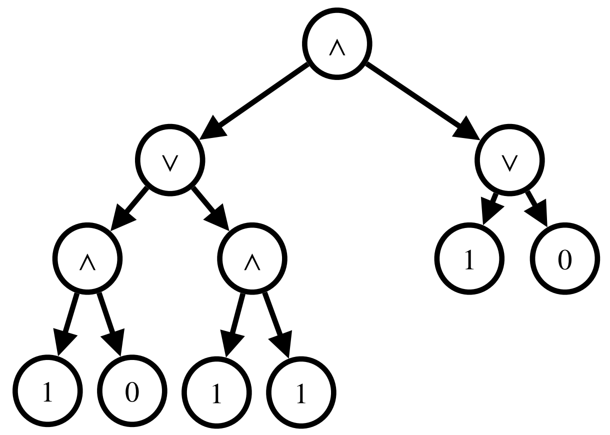


What is the probability that a random tree evaluates to true?

Example cont.



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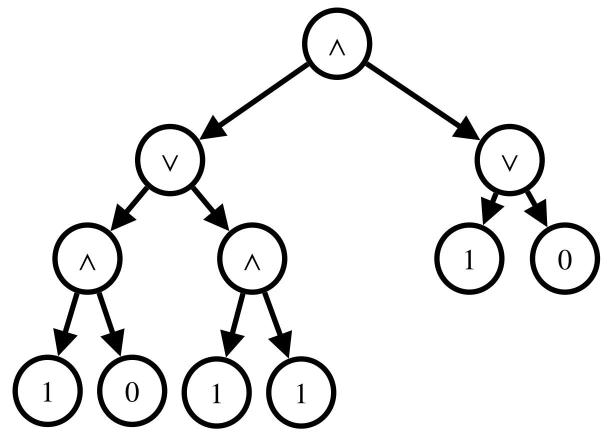

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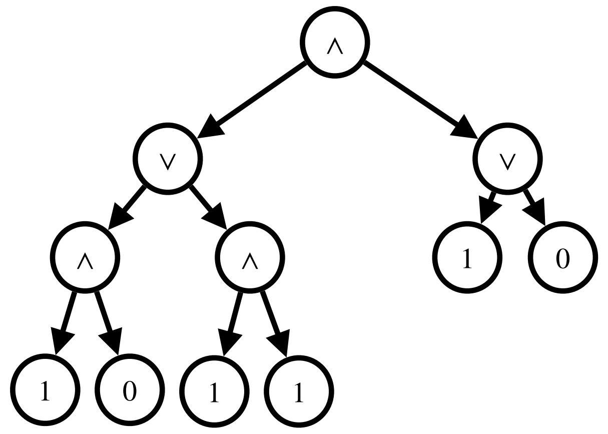

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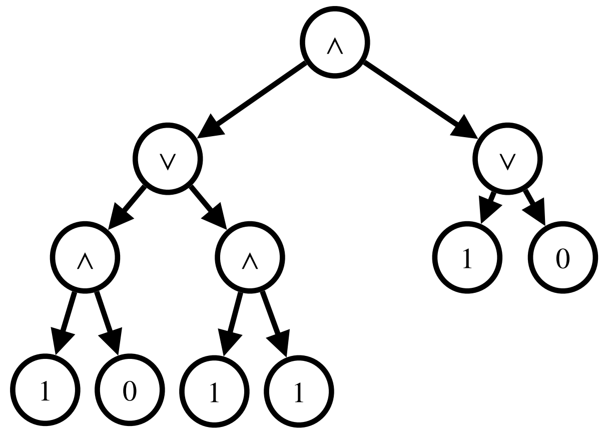

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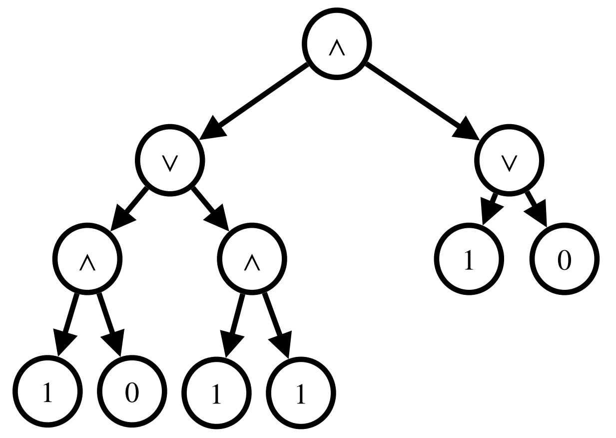

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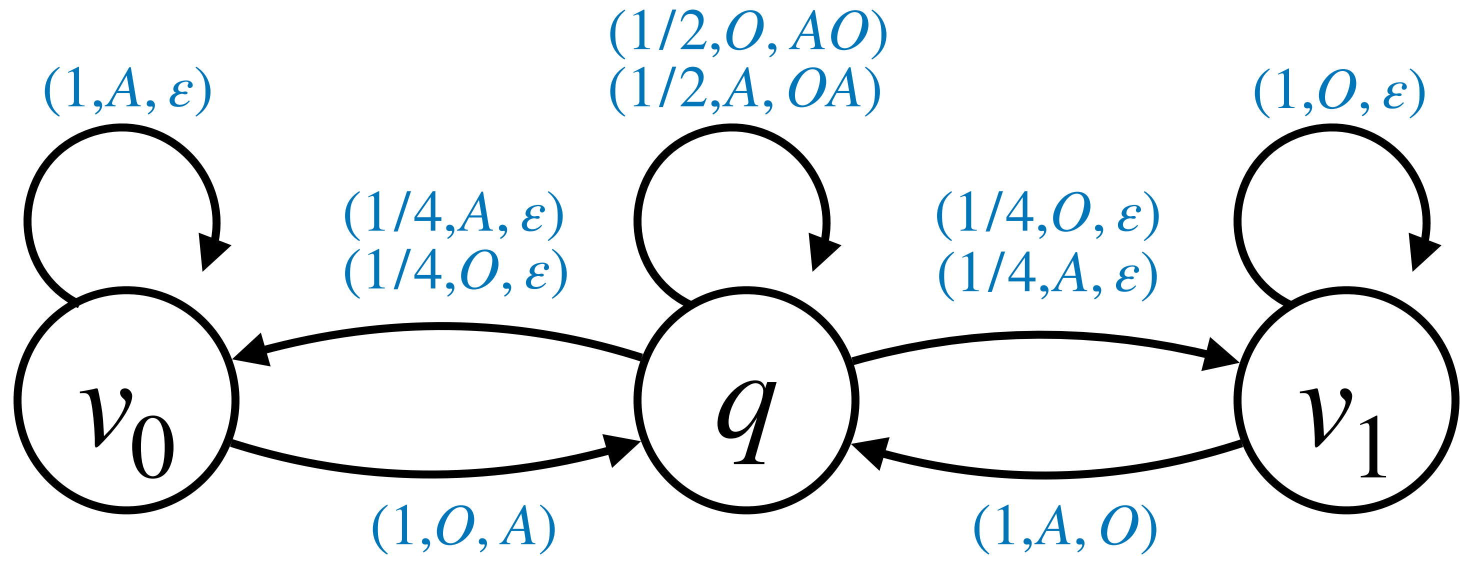

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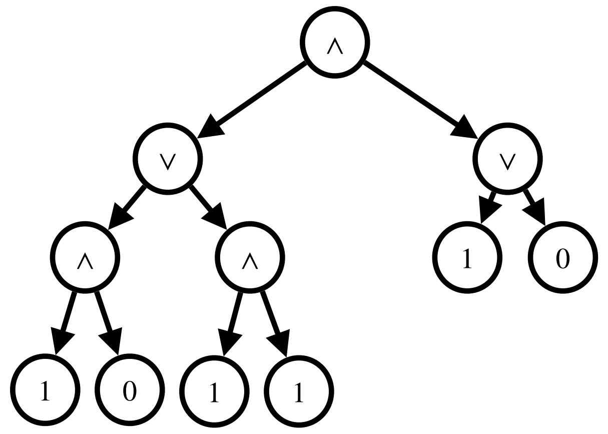

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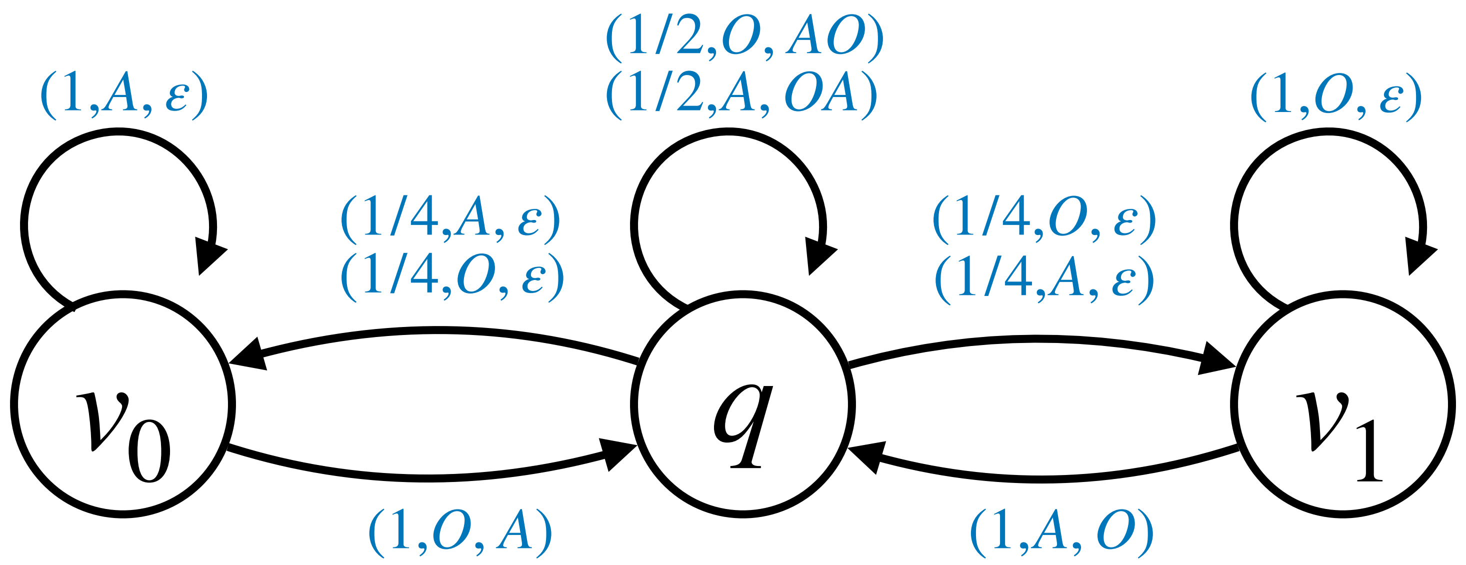

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$$Pr(V=0) \leq \frac{391}{933} \approx 0.42 \qquad Pr(V=1) \leq \frac{382}{657} \approx 0.58$$

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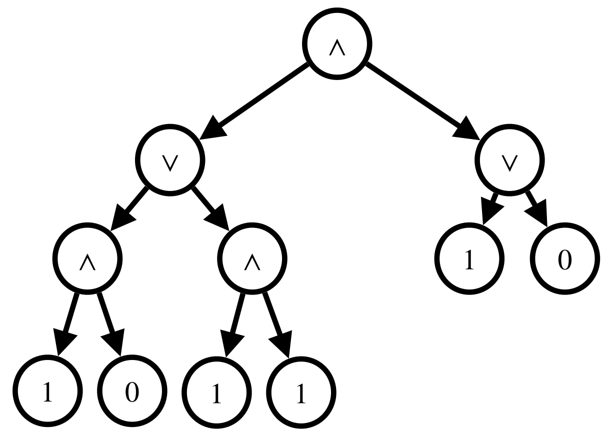

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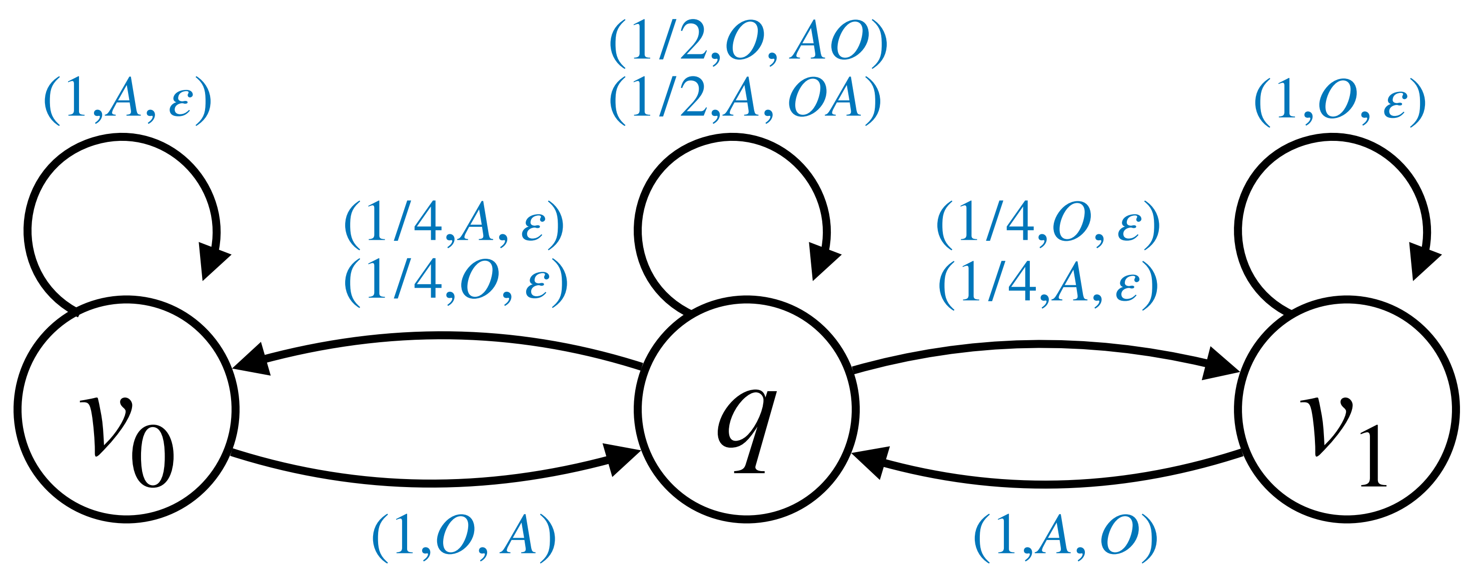

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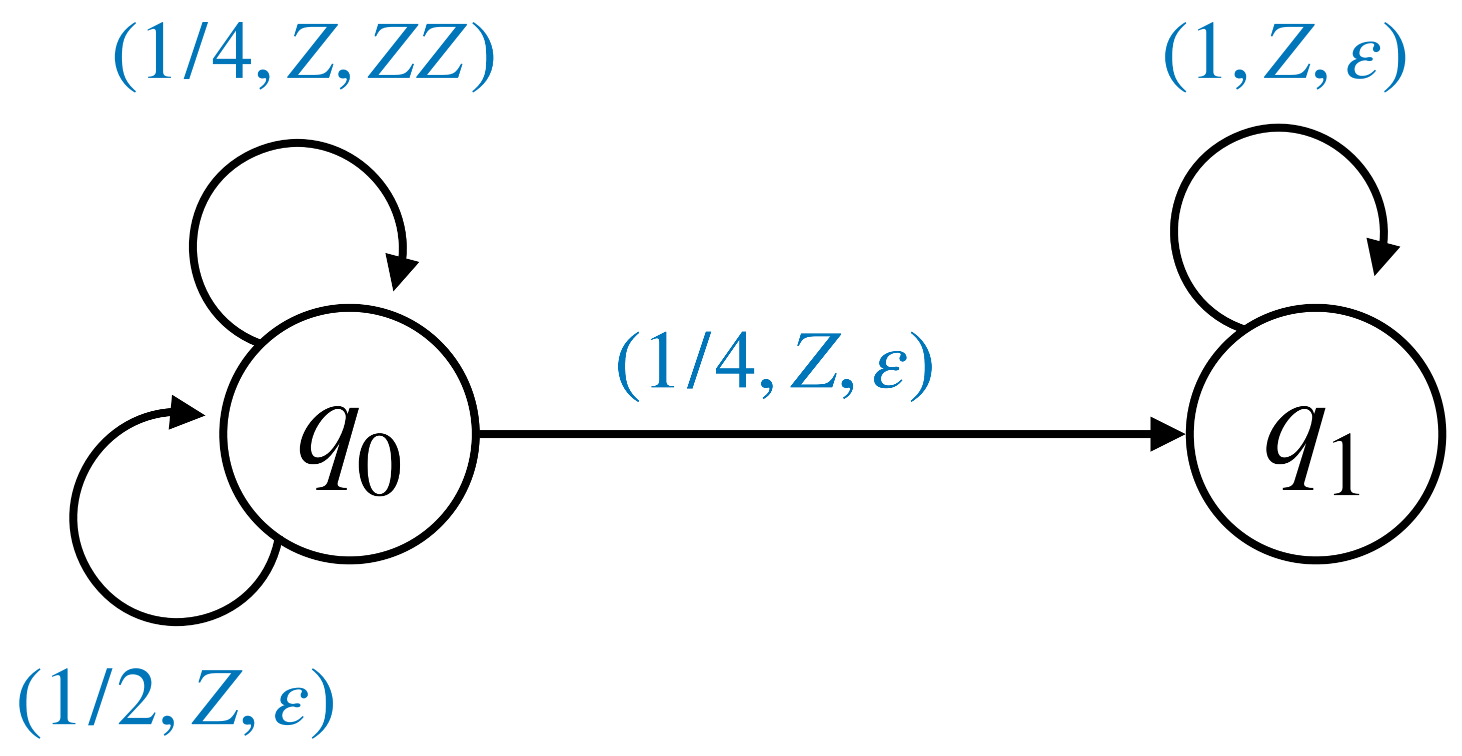
- Correctness of result can be easily checked independently → certificate!

Probabilistic Pushdown Automata [Esparza et al. '04]

pPDA



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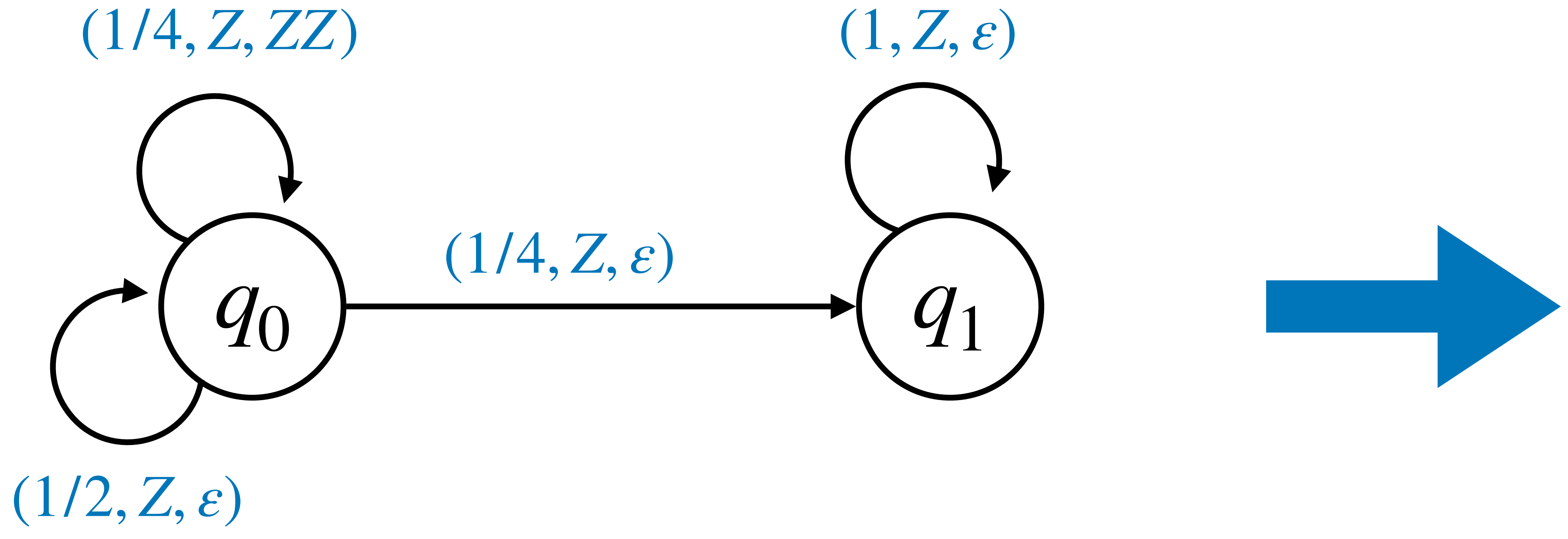

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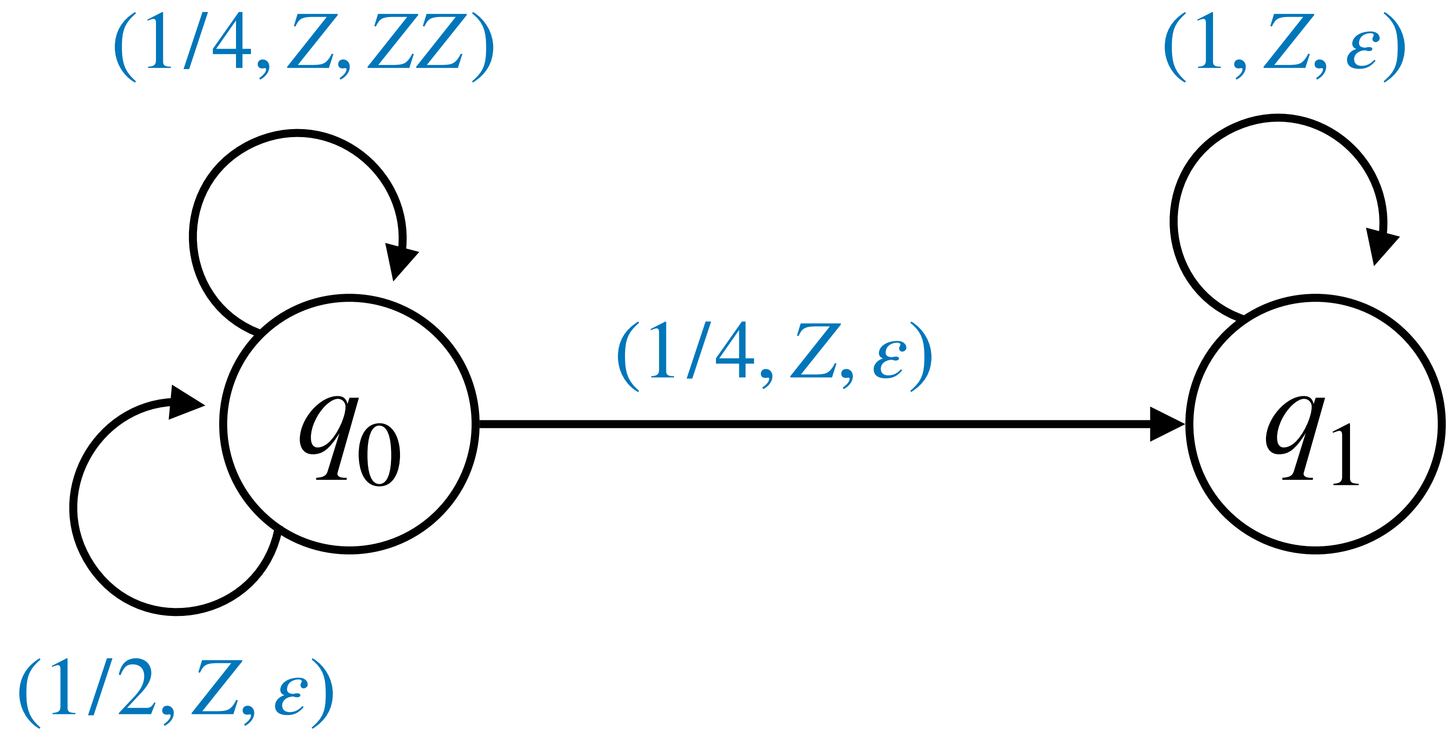

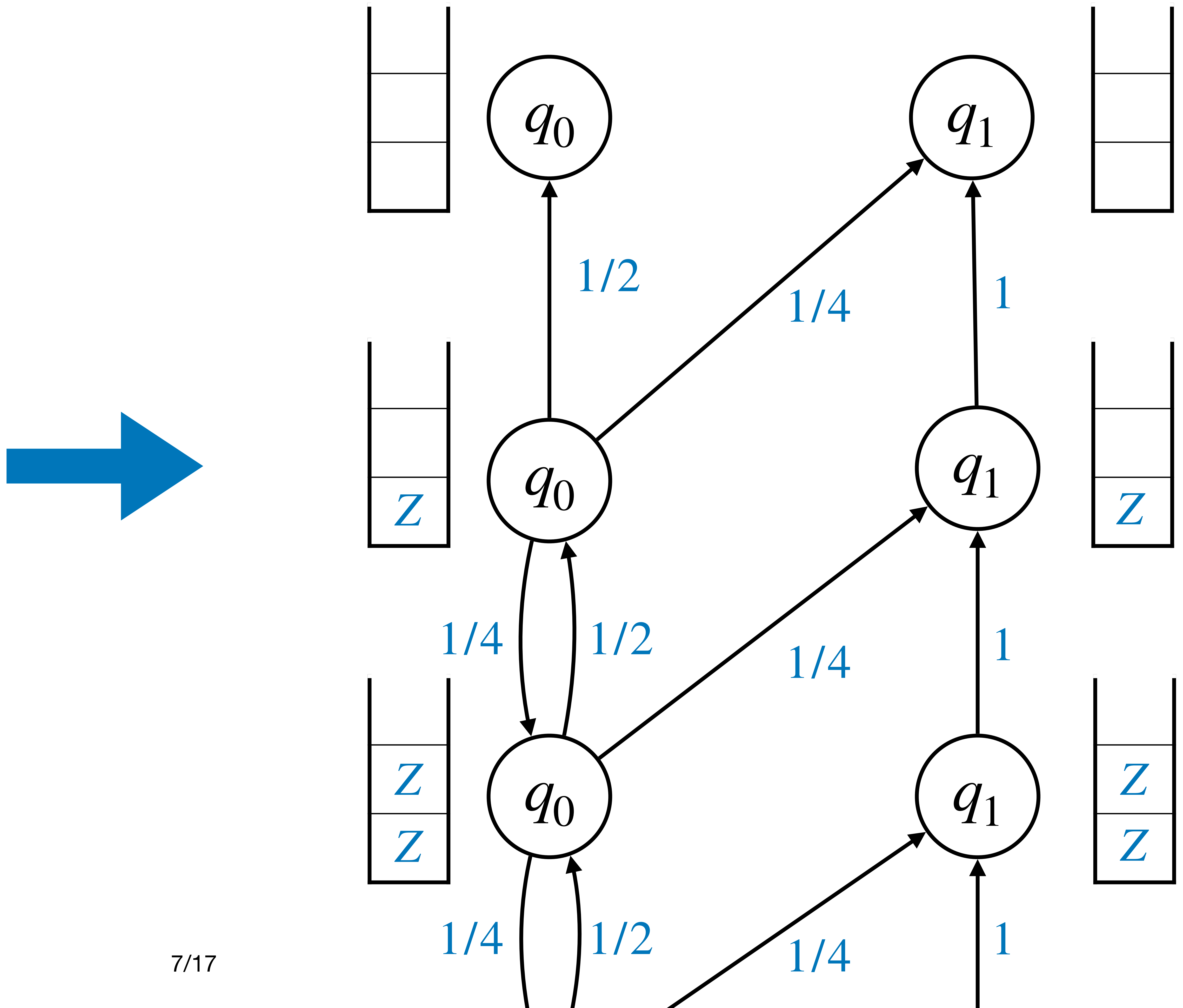

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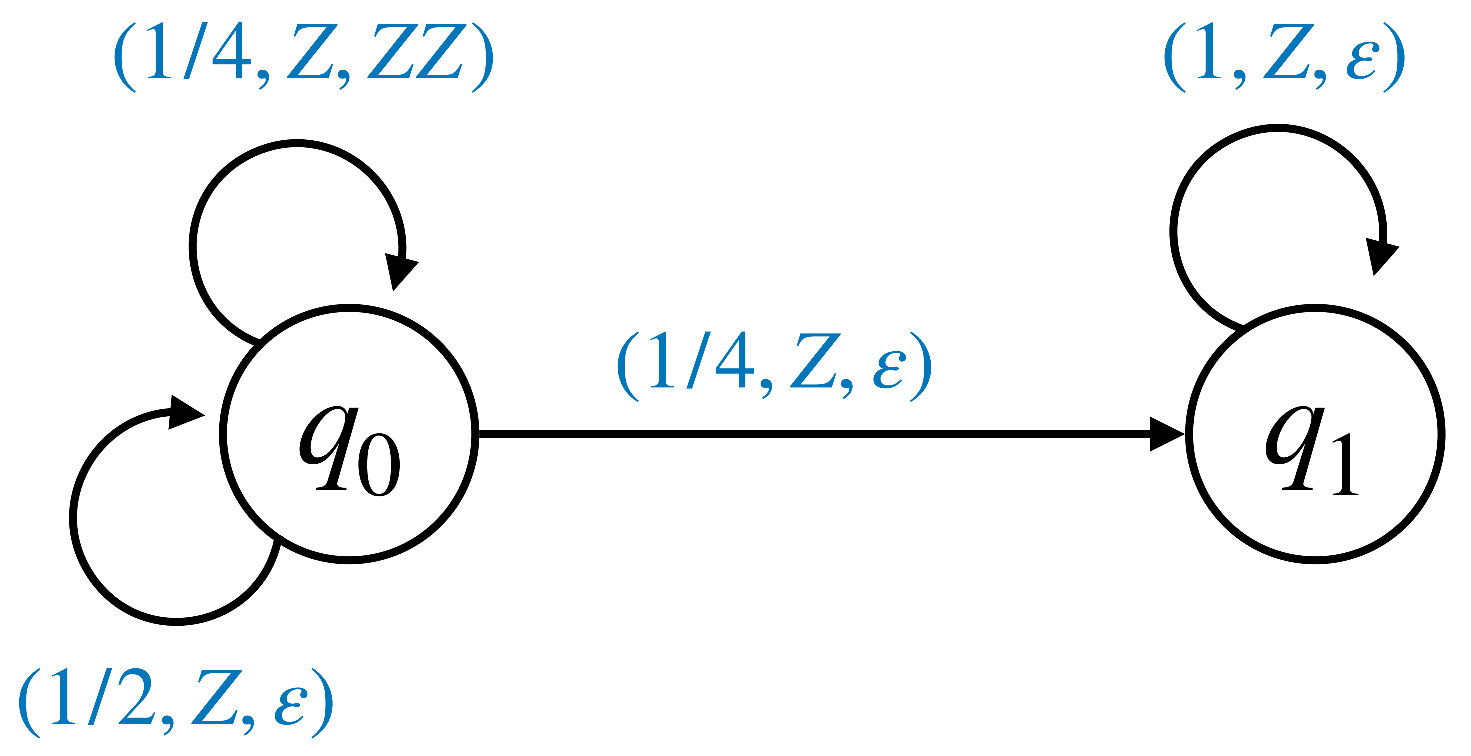

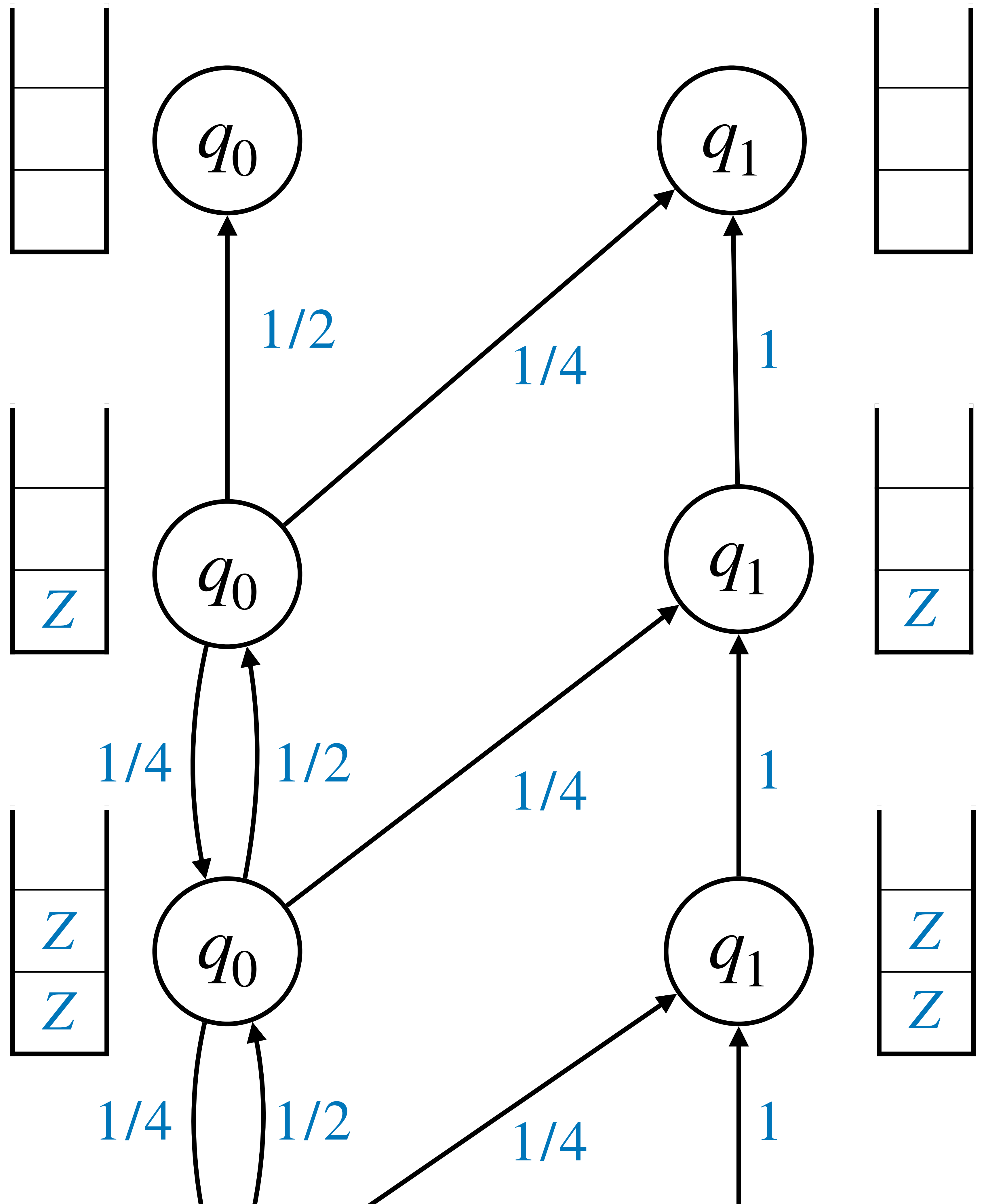

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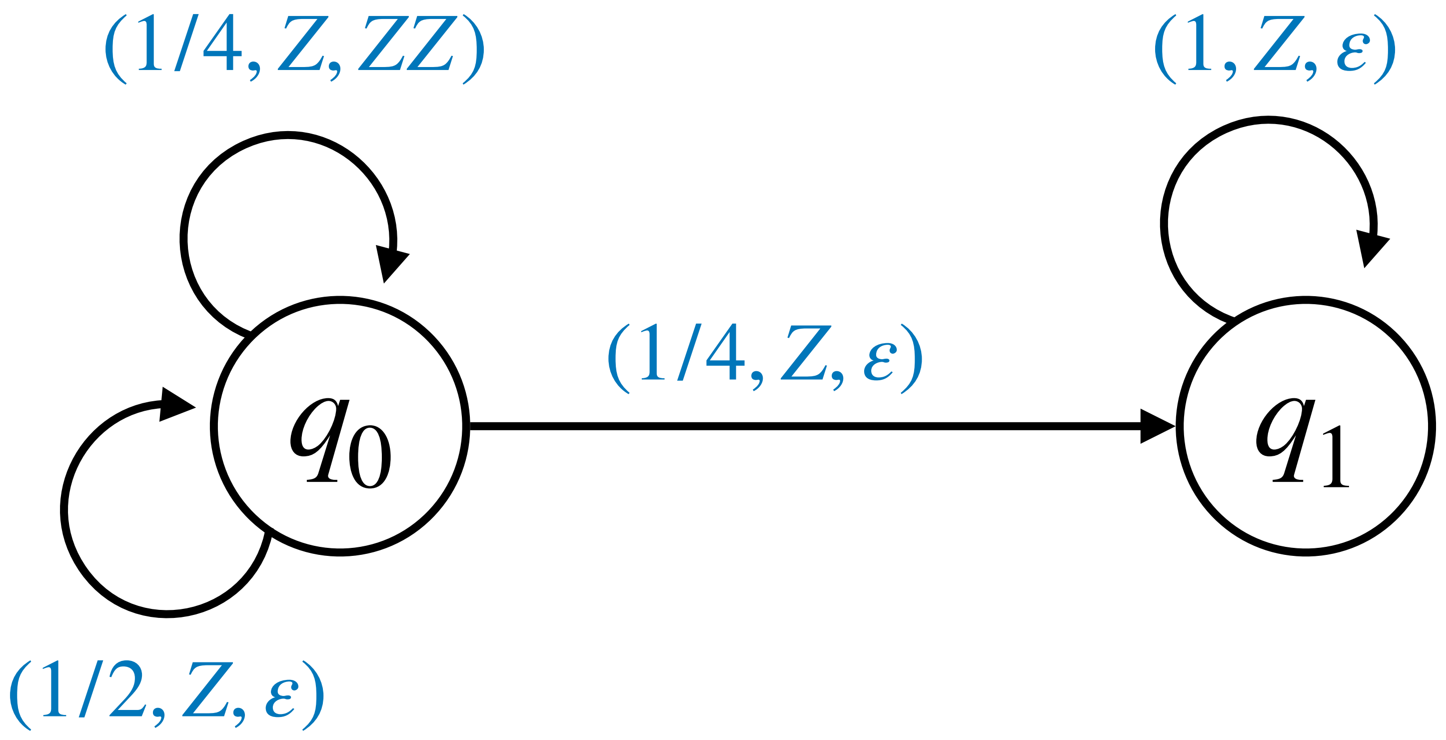

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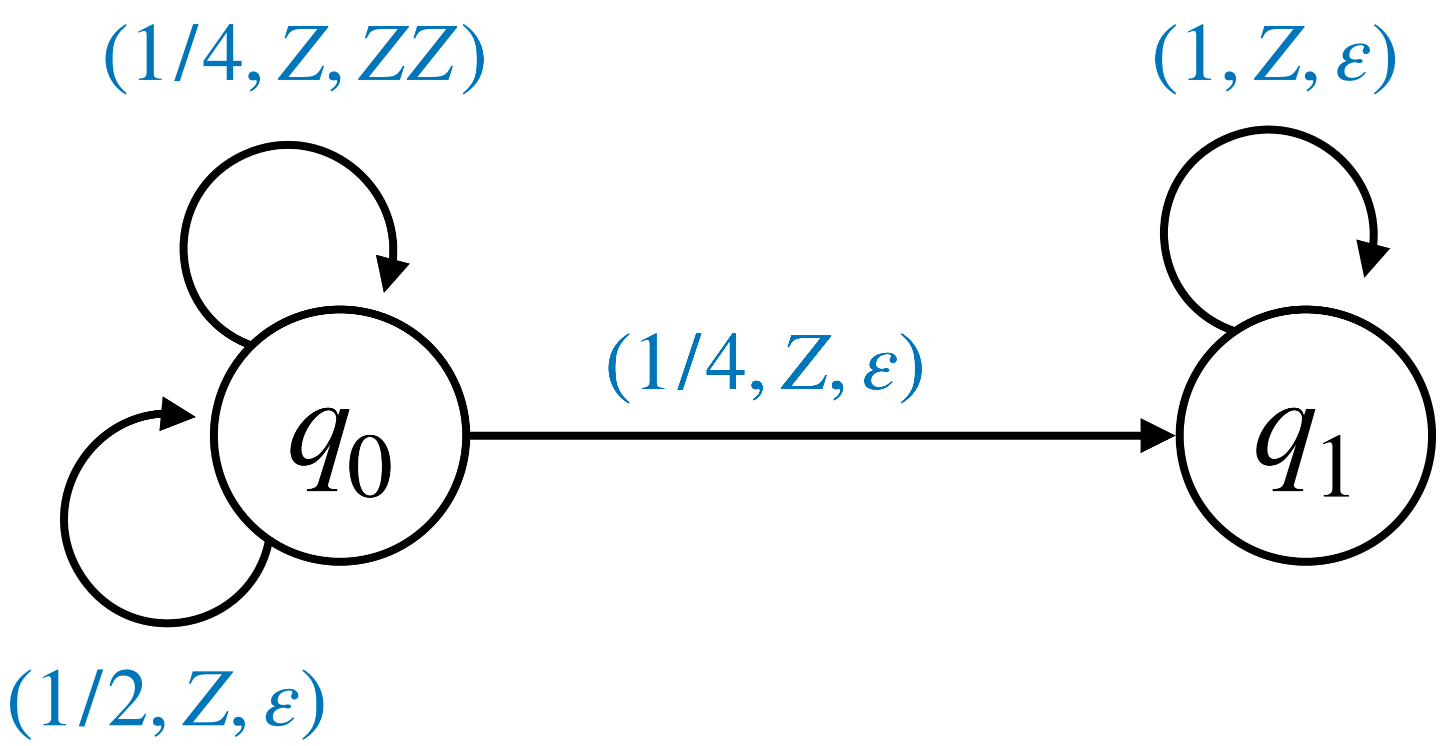
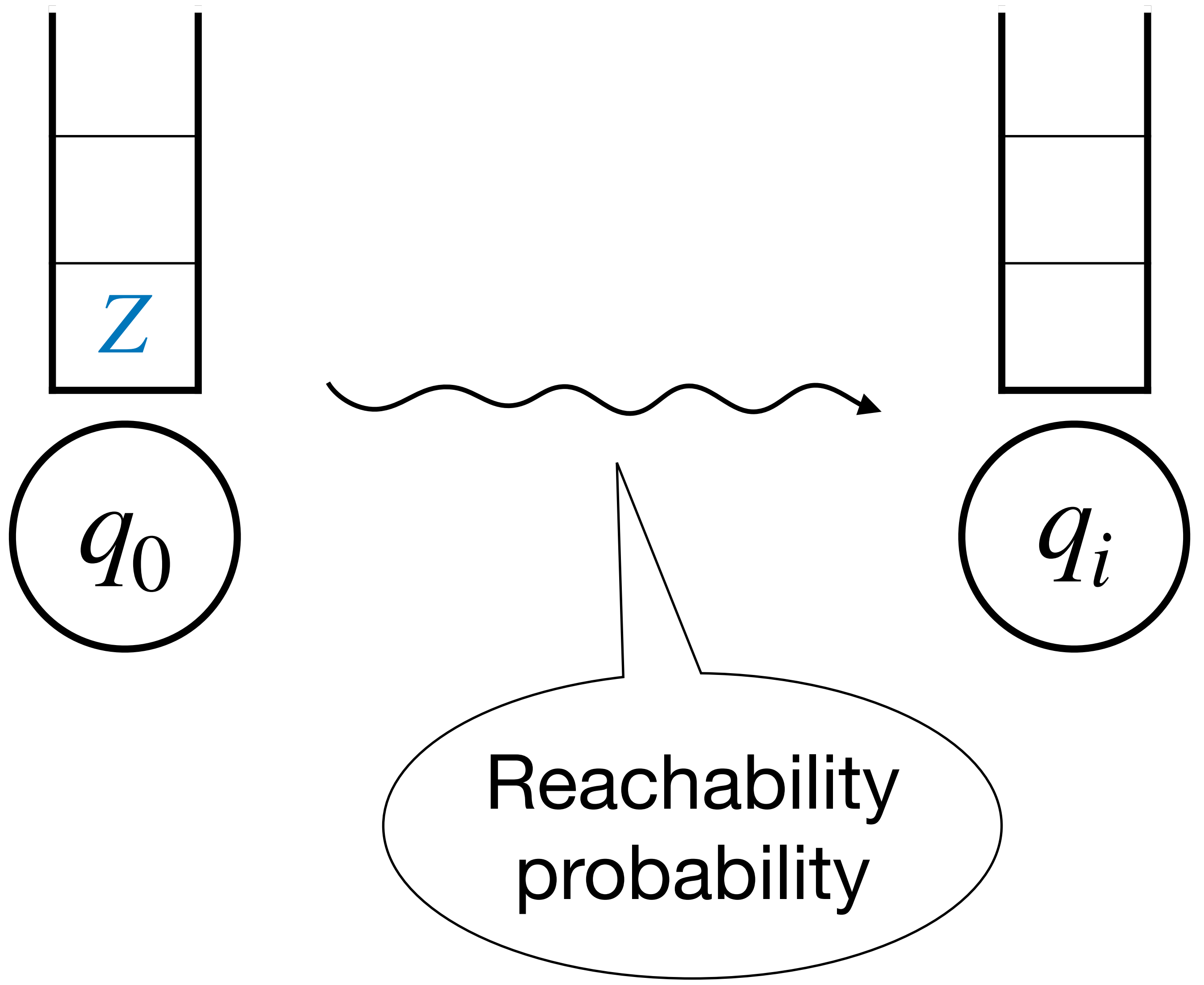


$[q_0 Z q_i] = Pr($  $)$

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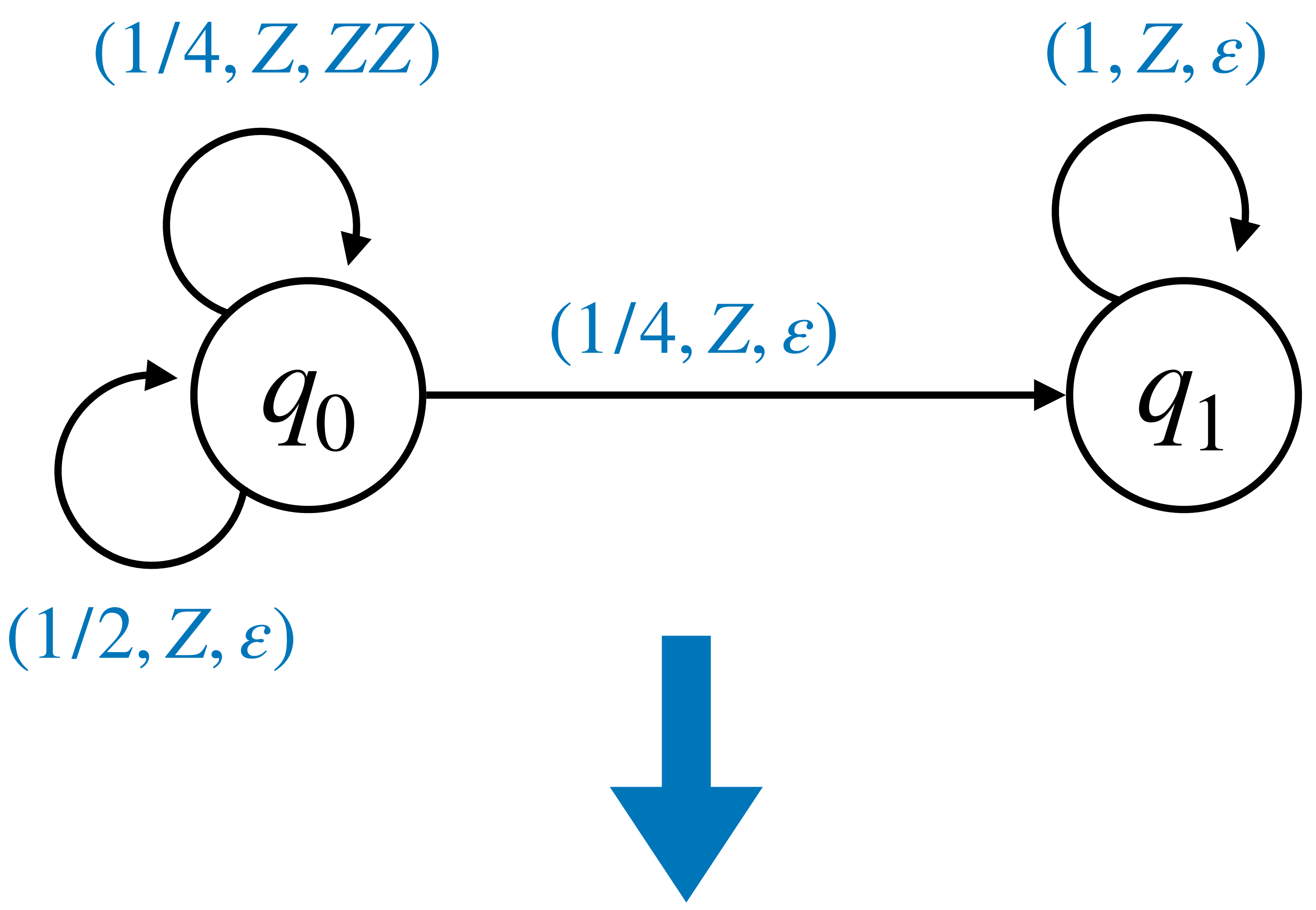

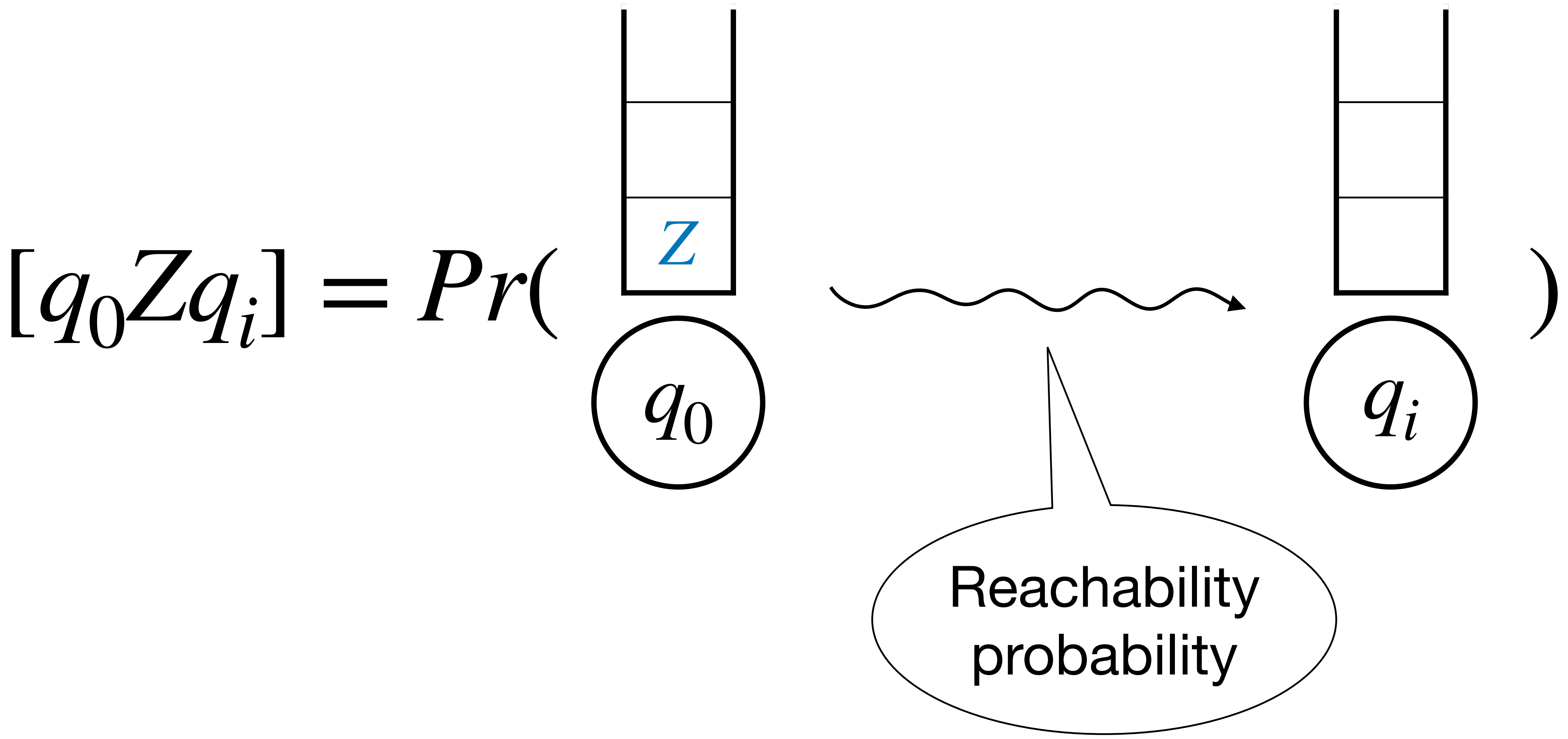

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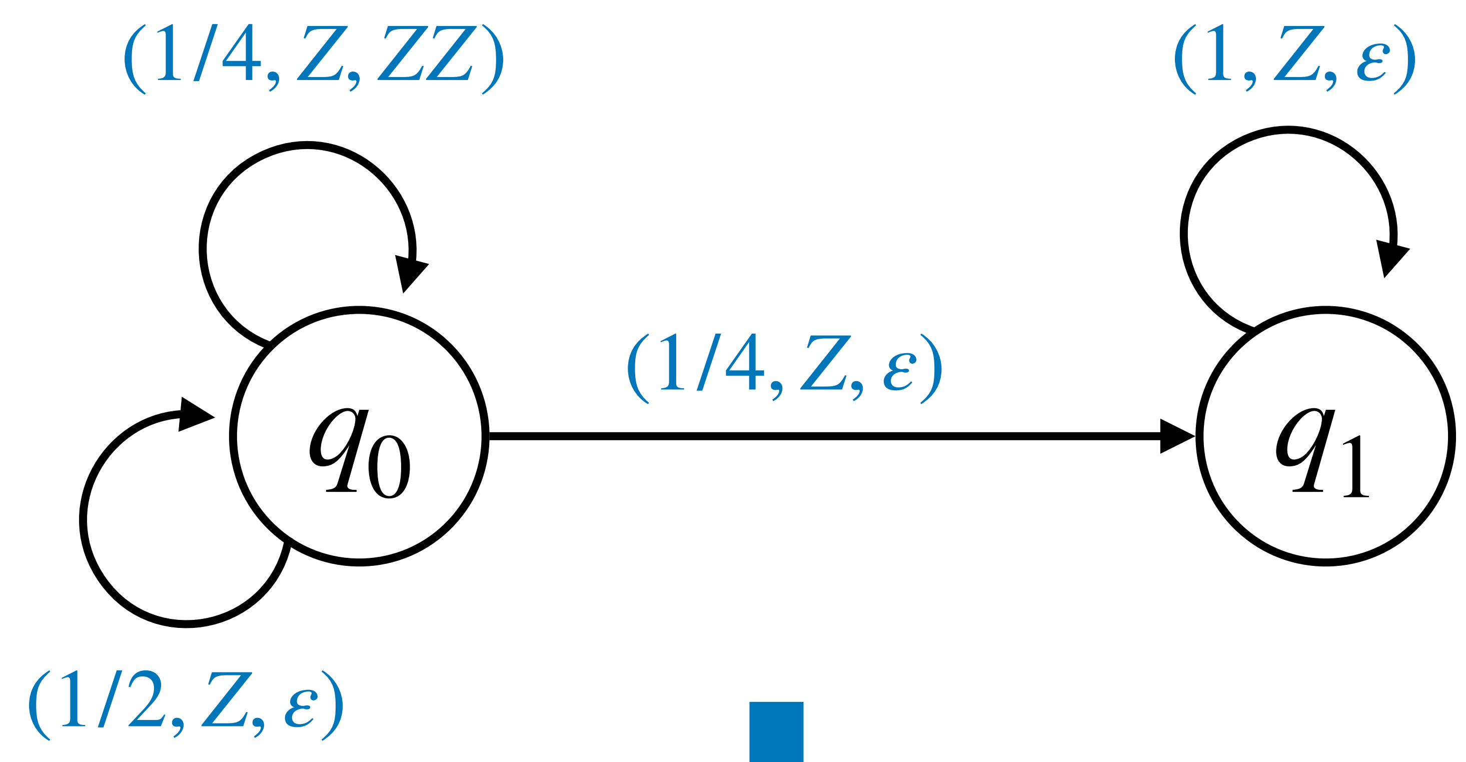
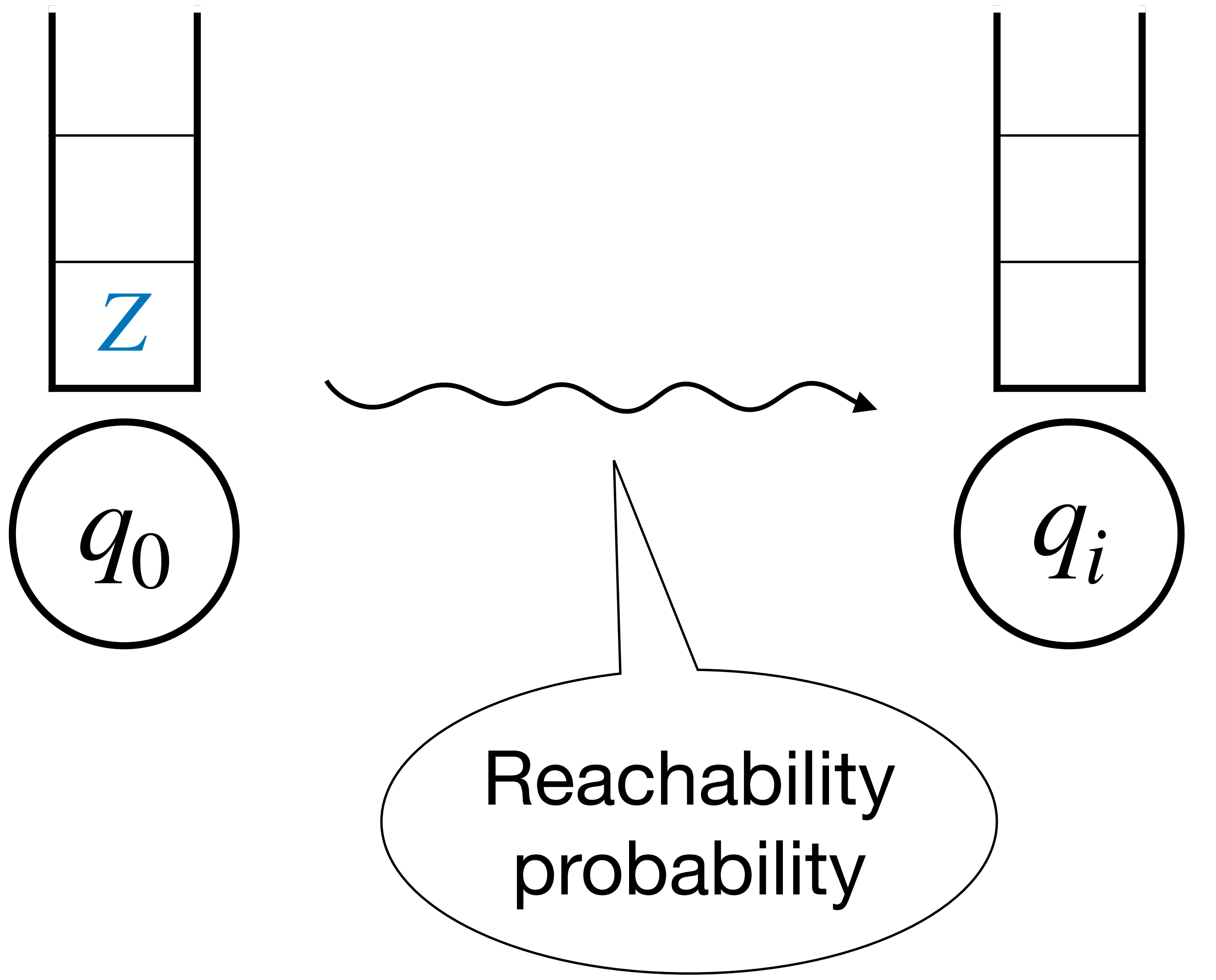


$[q_0Zq_i] = Pr($  $)$

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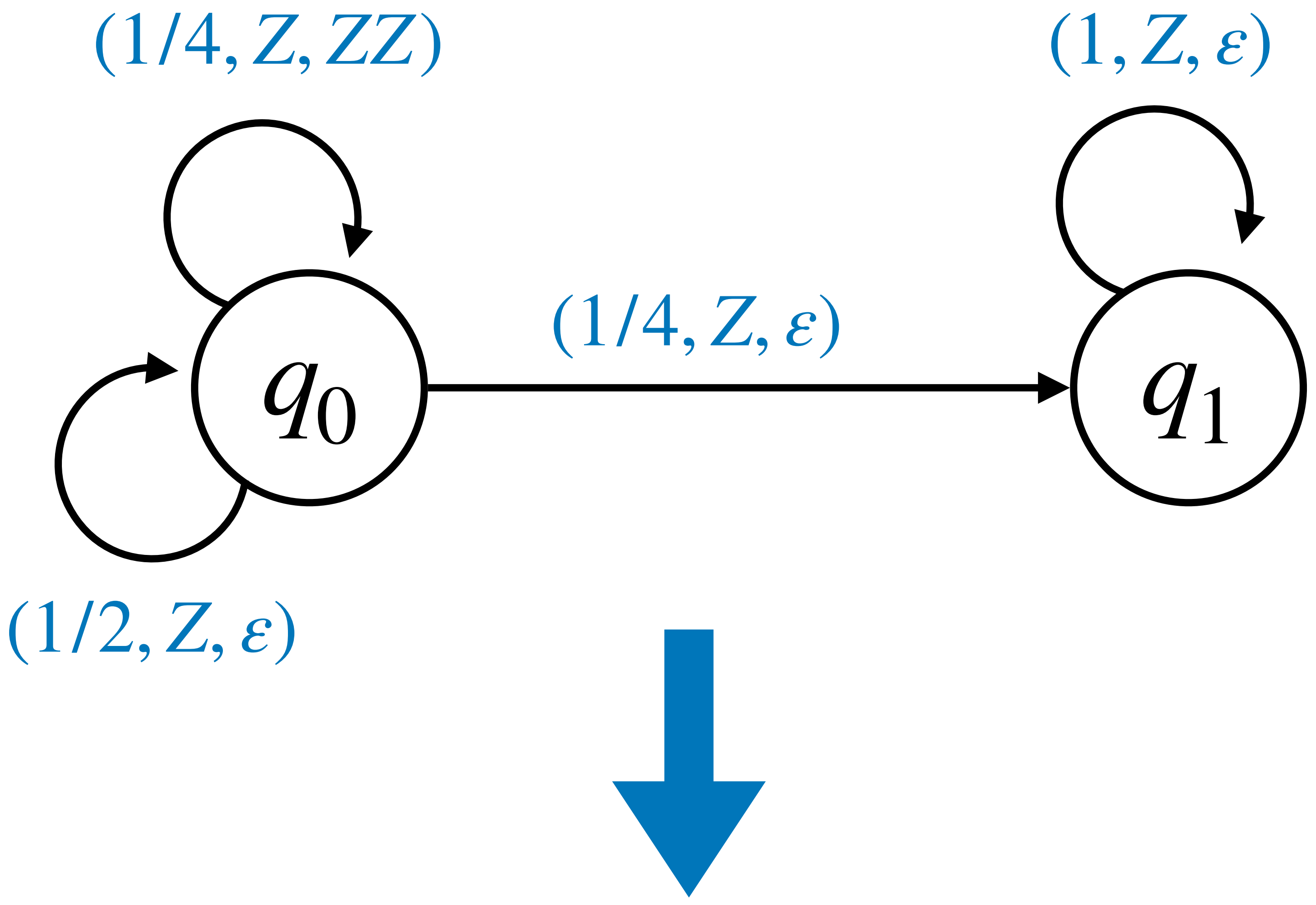
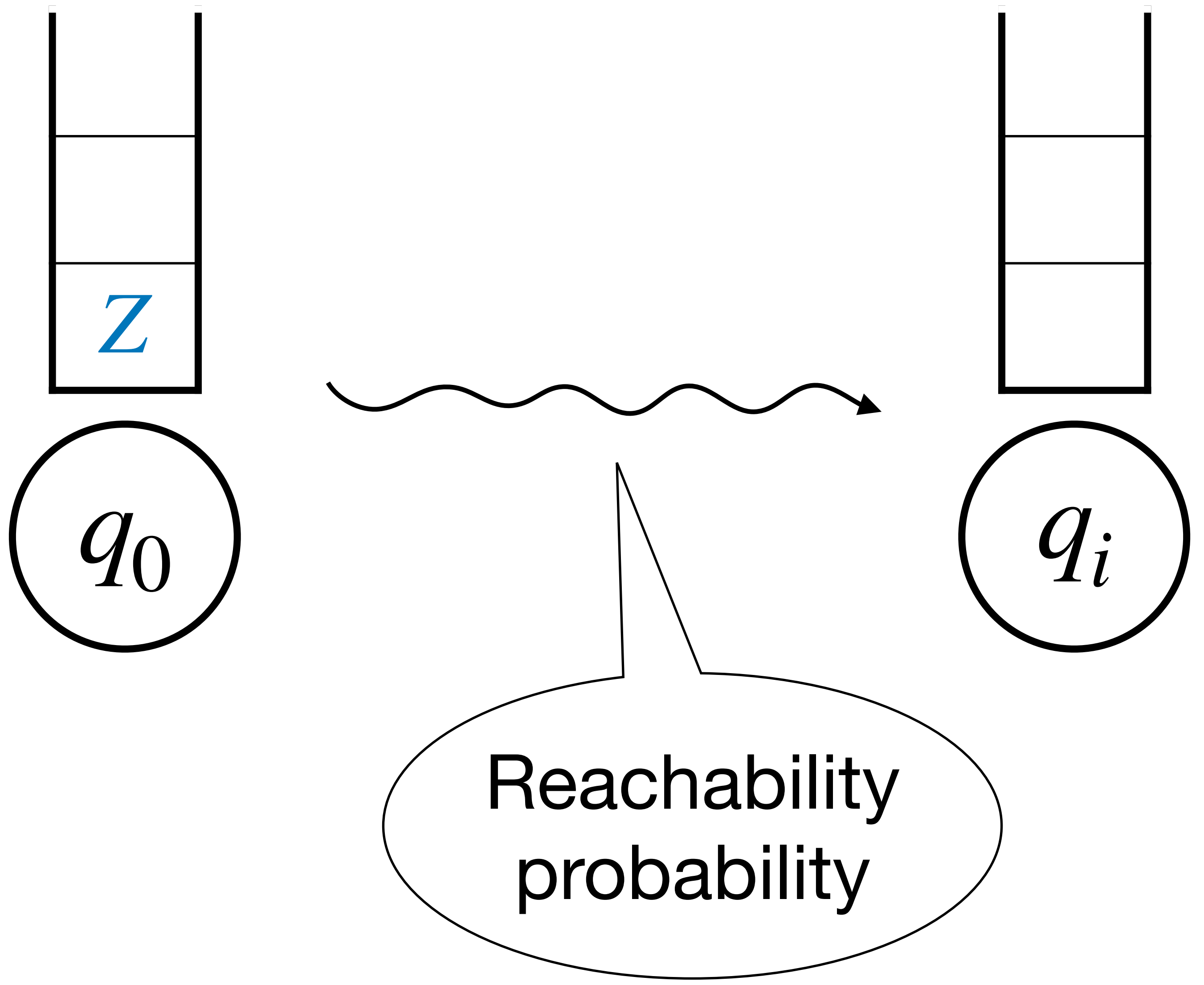


$[q_0 Z q_i] = Pr($  $)$

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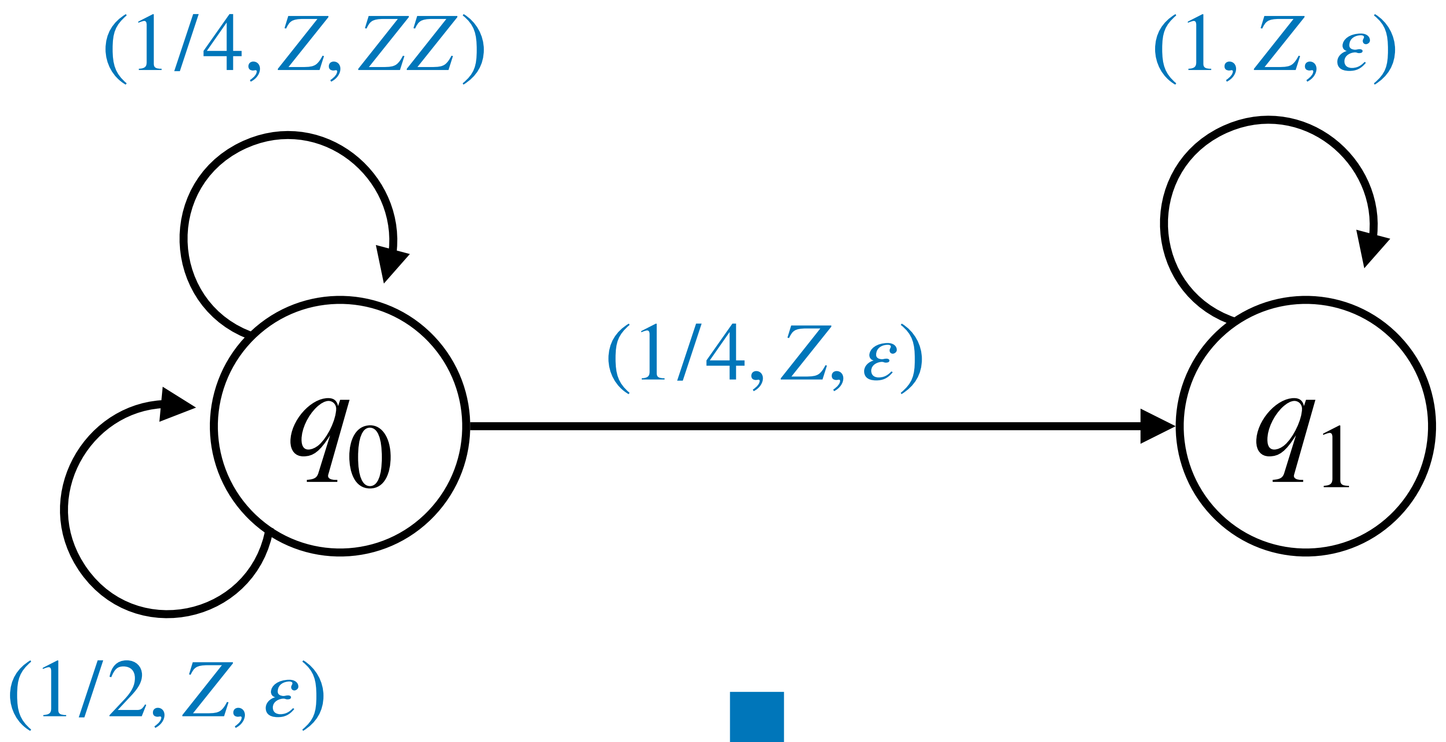

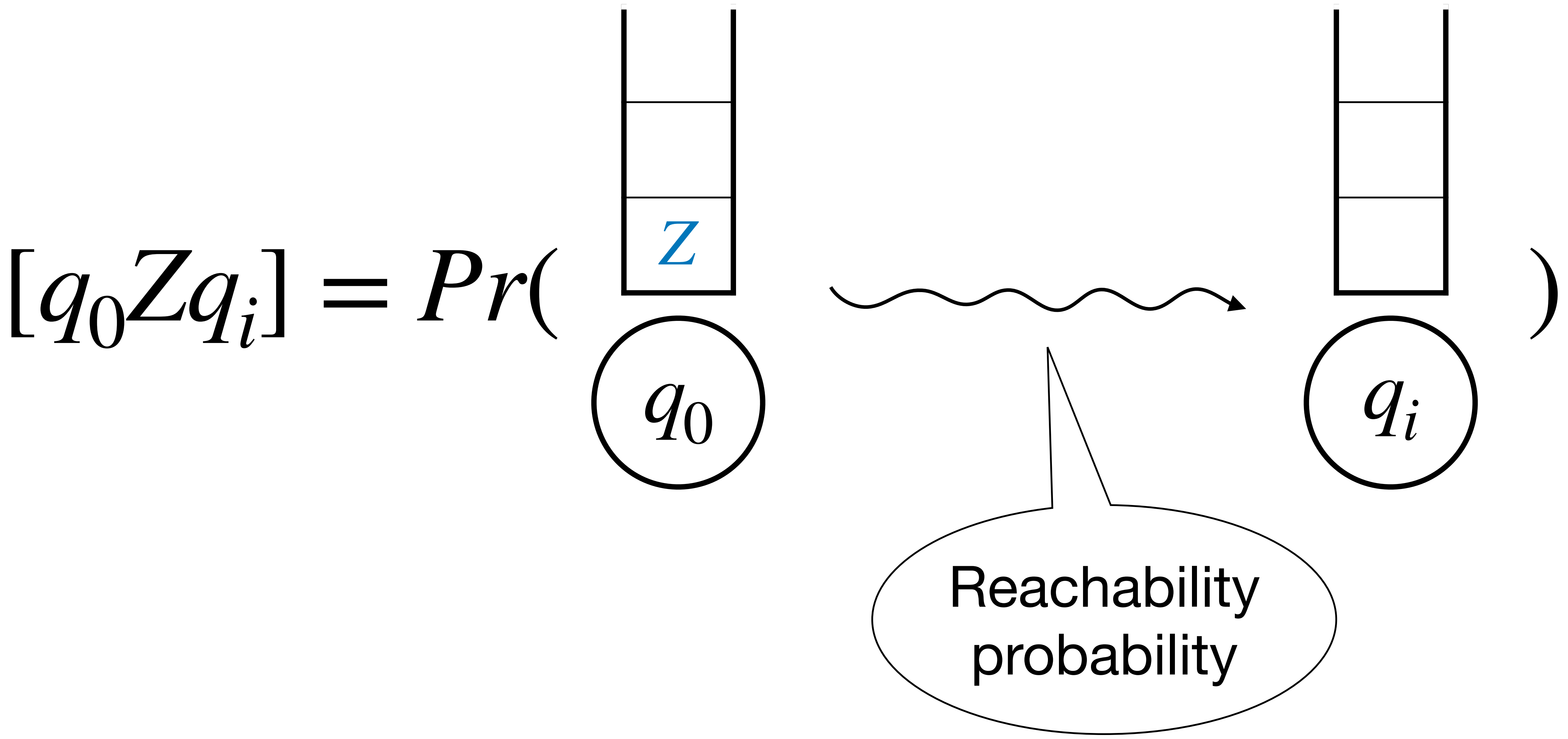

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- Problem: How to certify that approximation is "correct"?

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- This is unsound! Doesn't prove anything.

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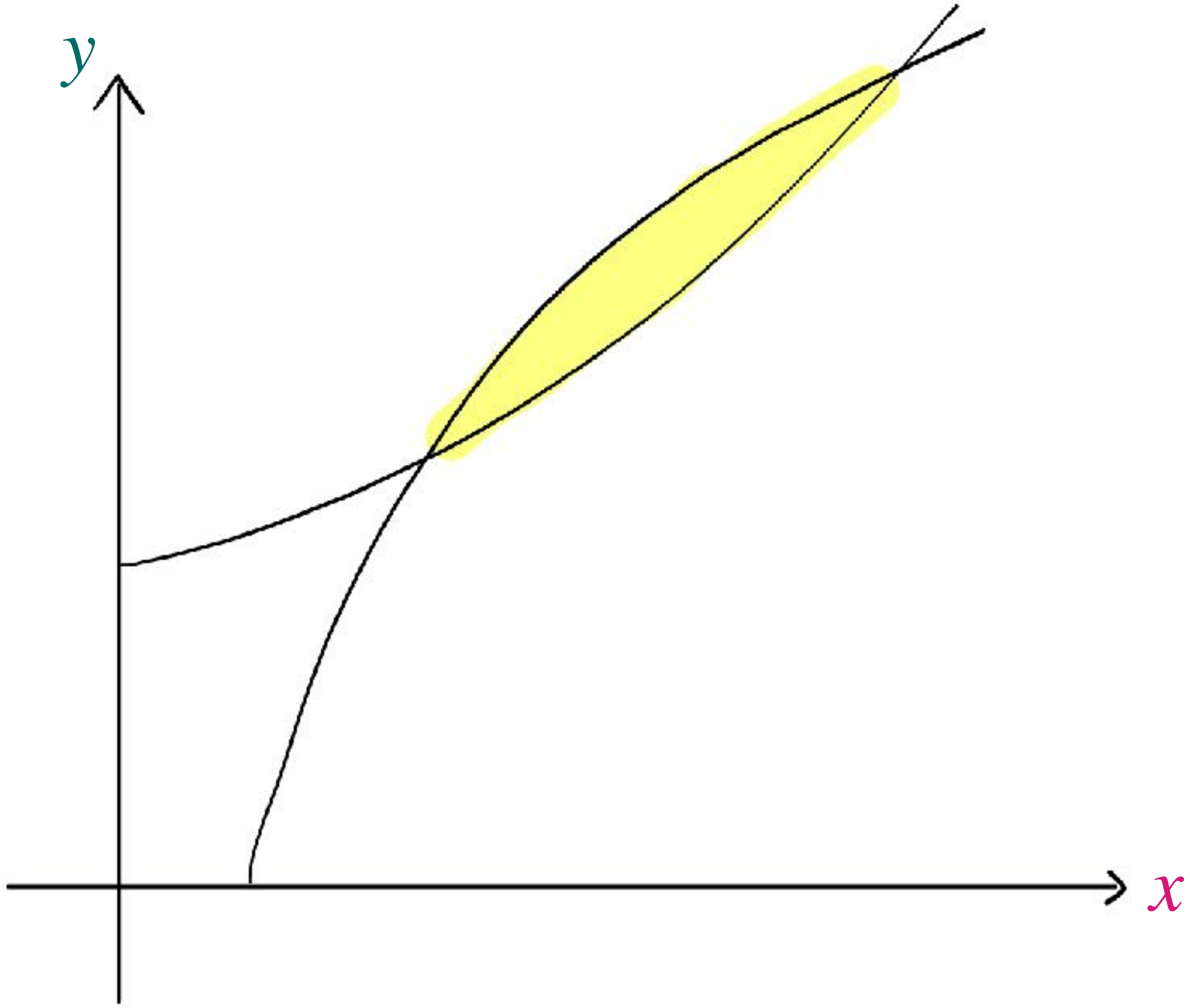

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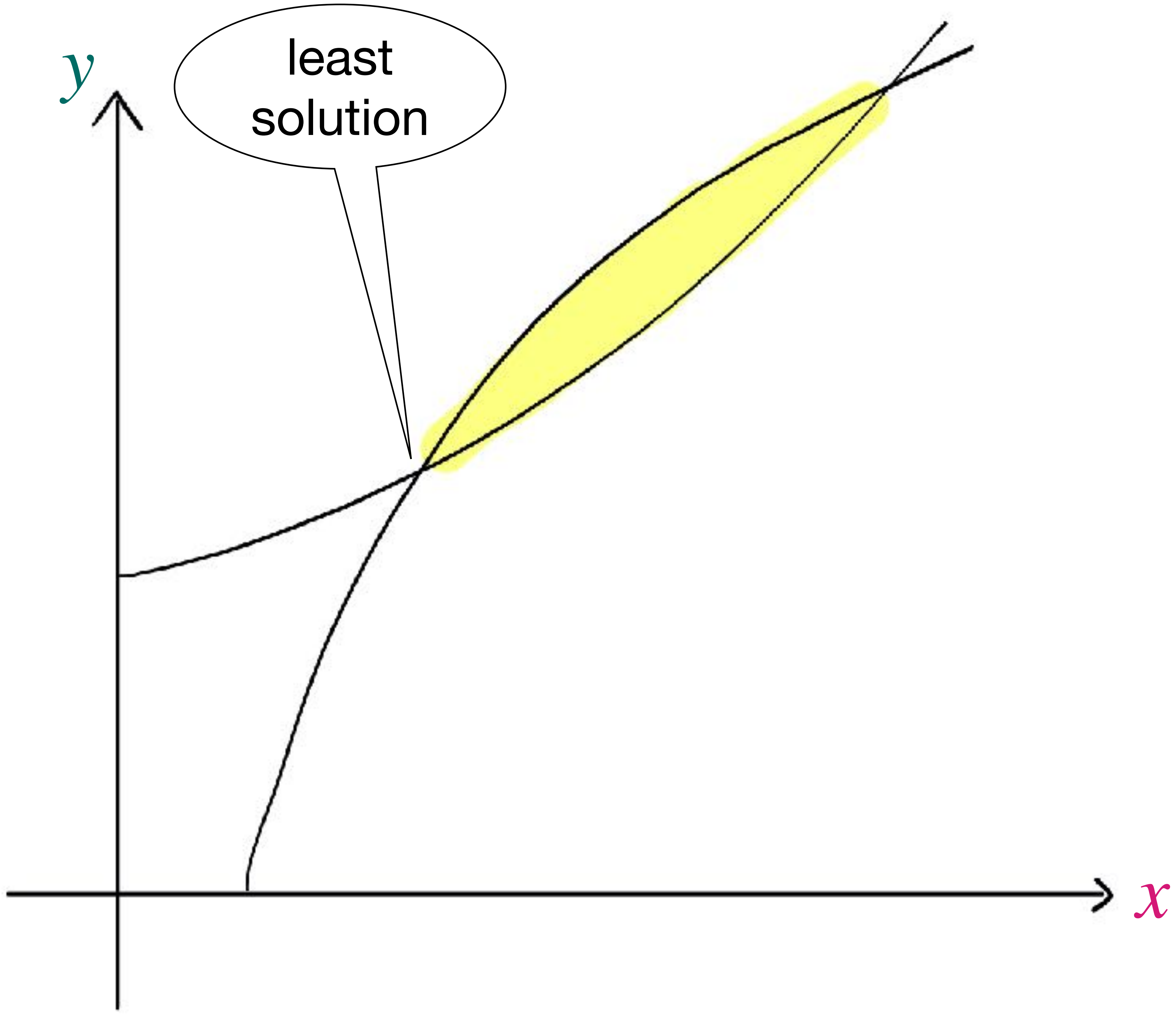

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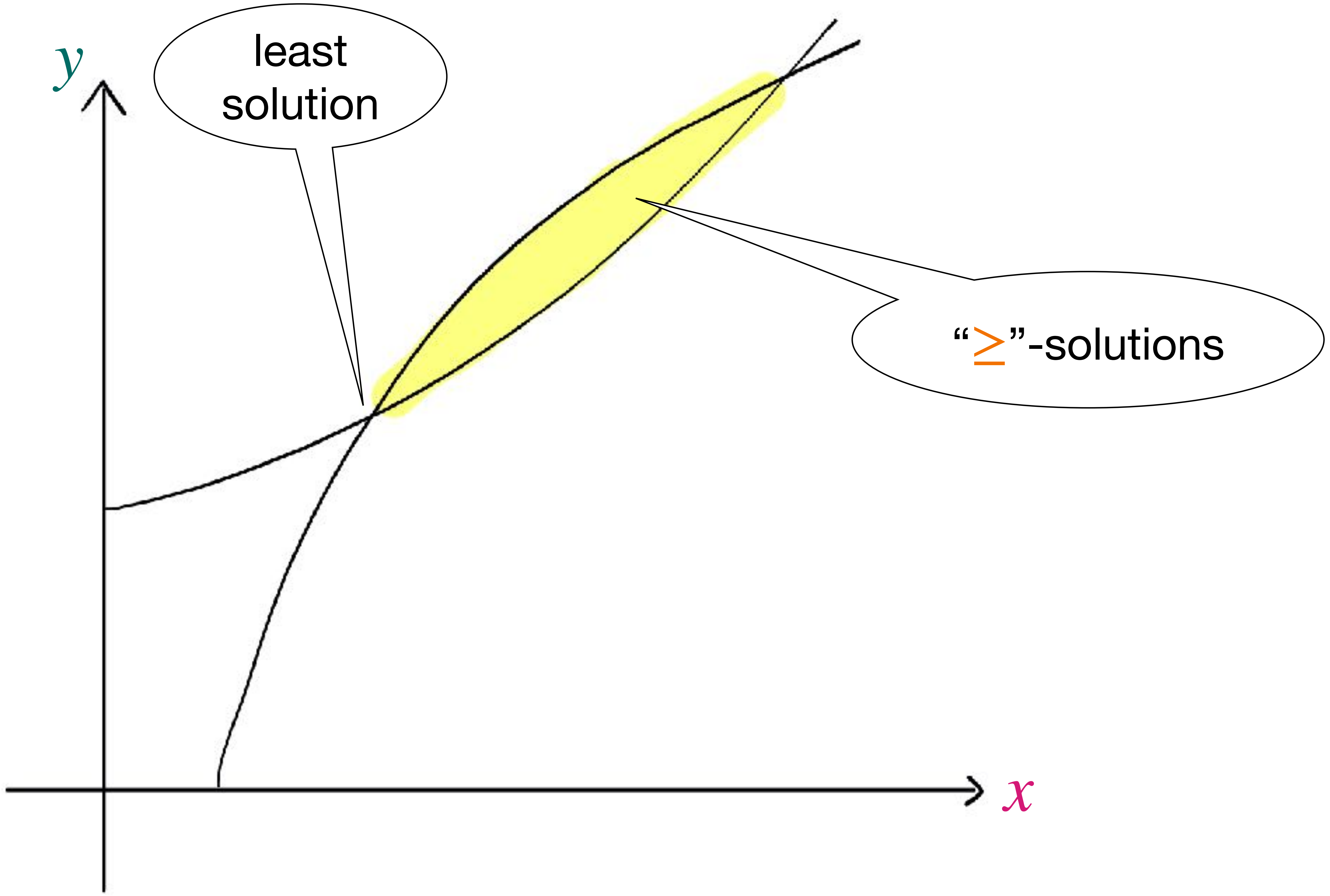

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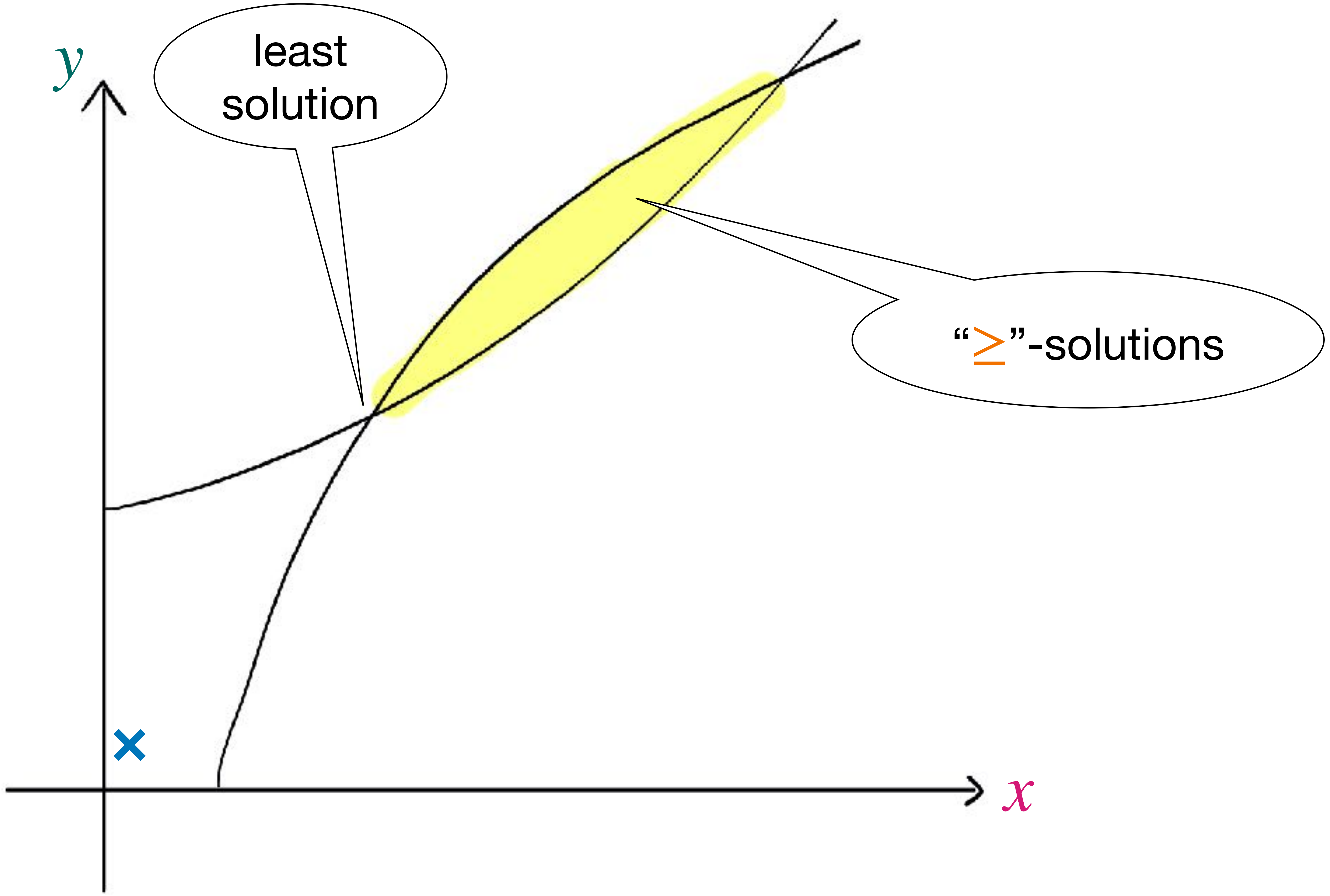

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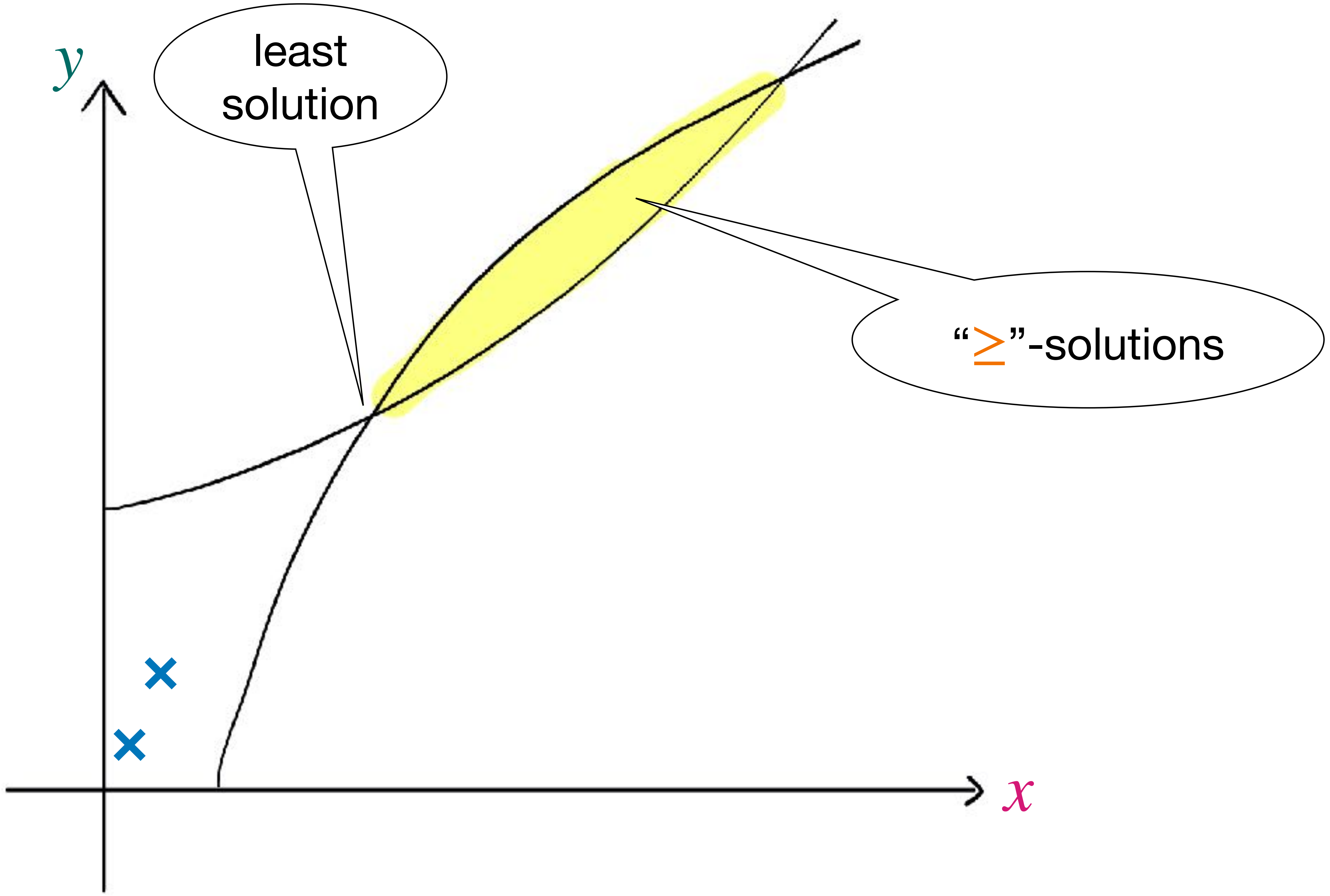

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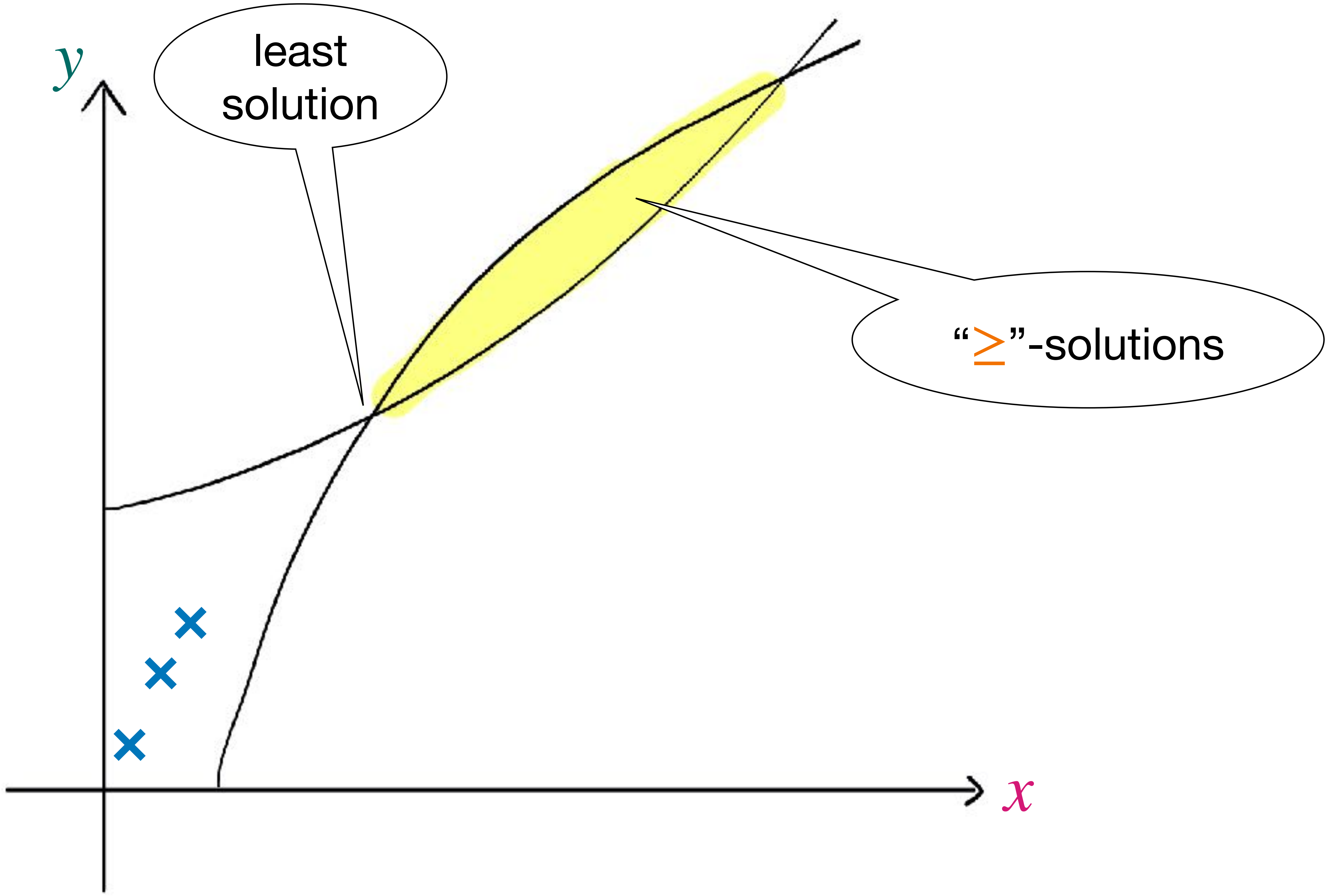

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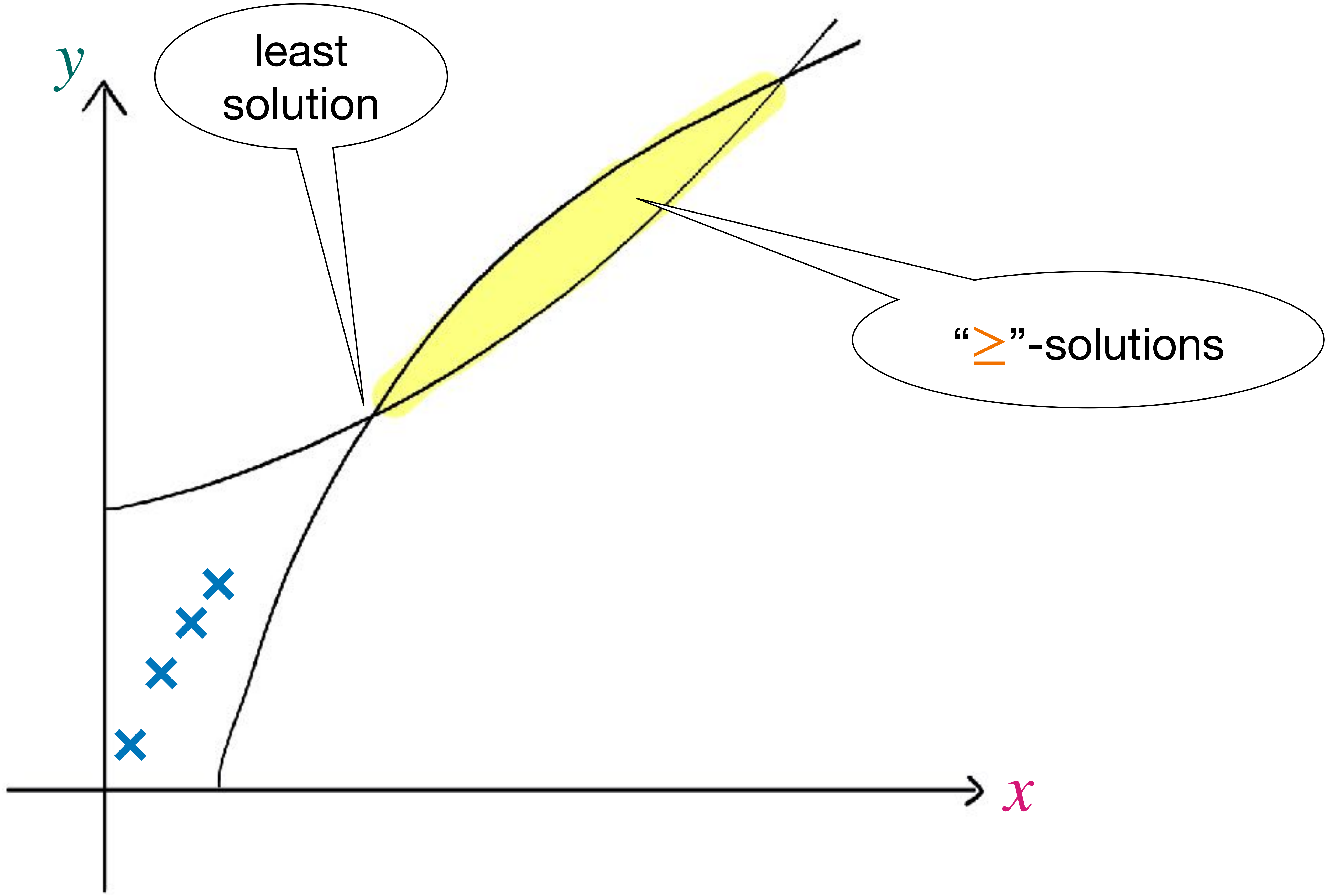

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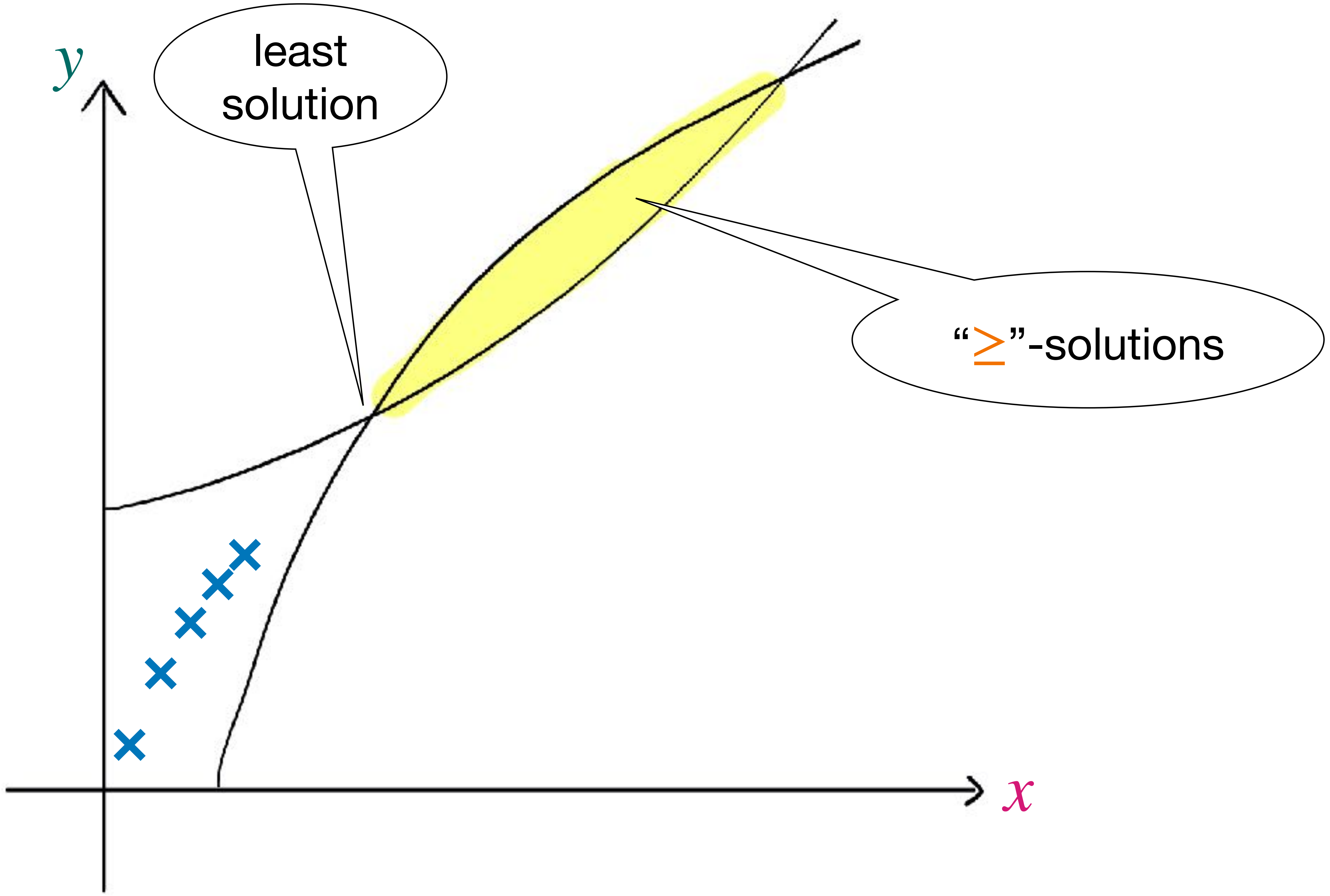

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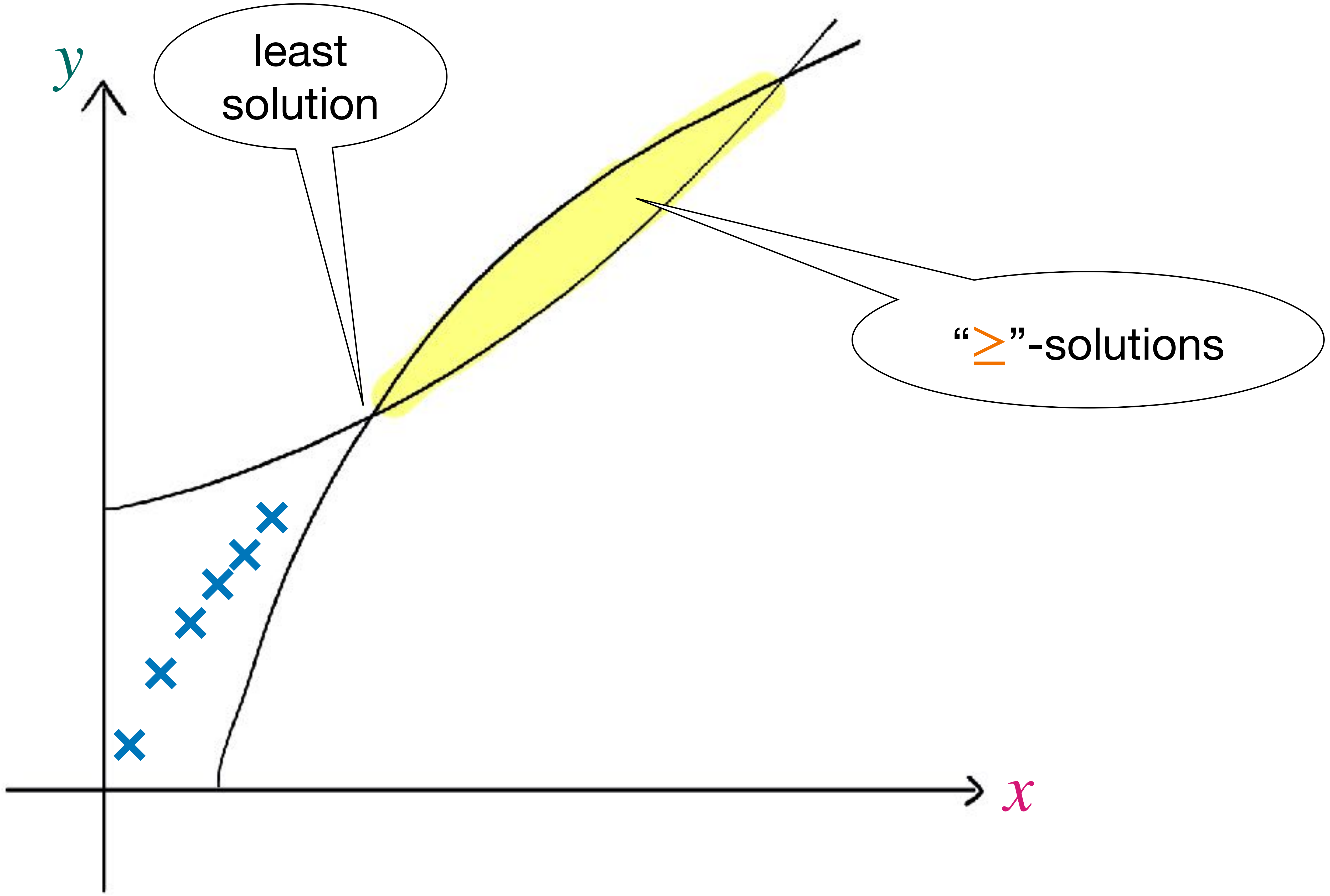

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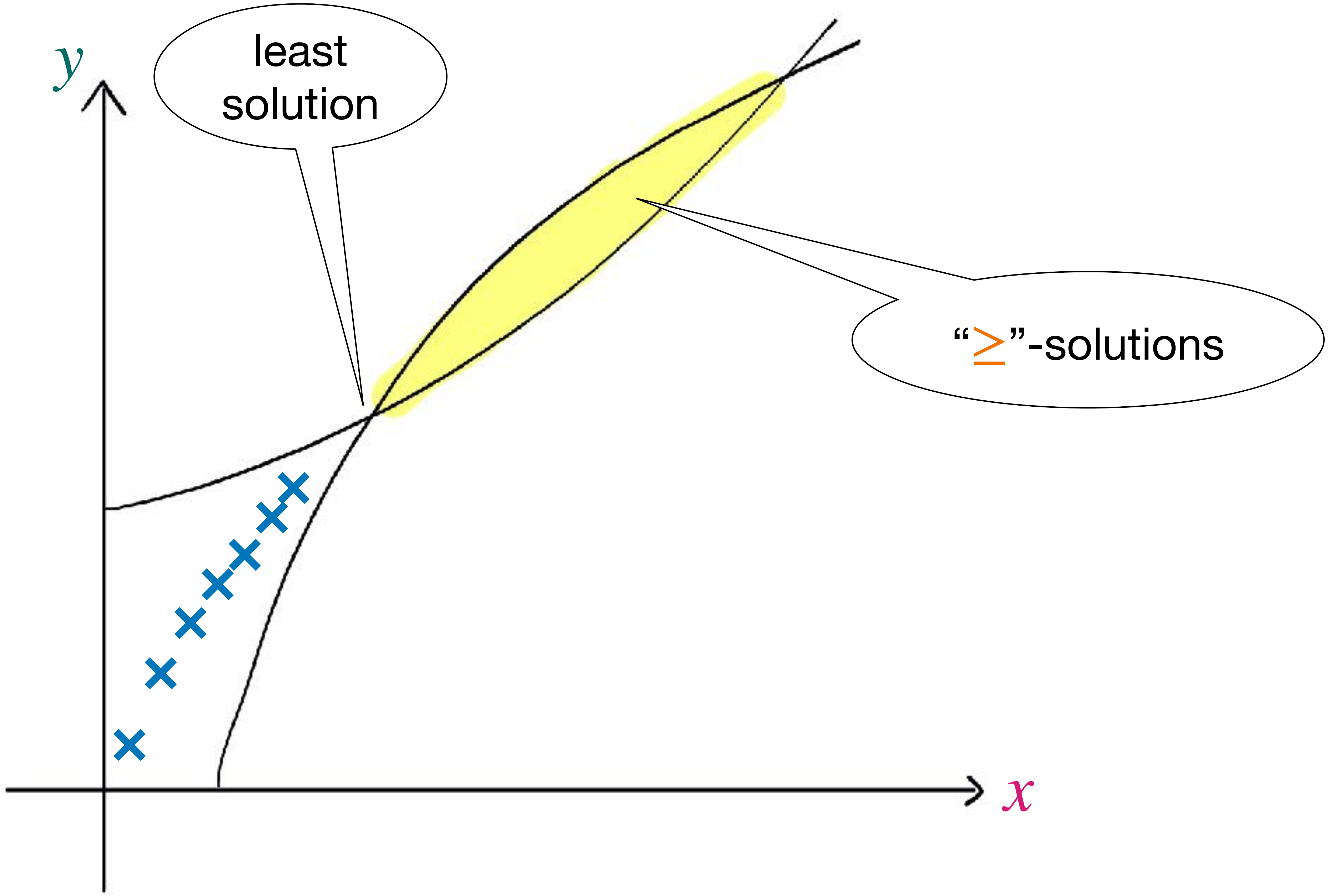

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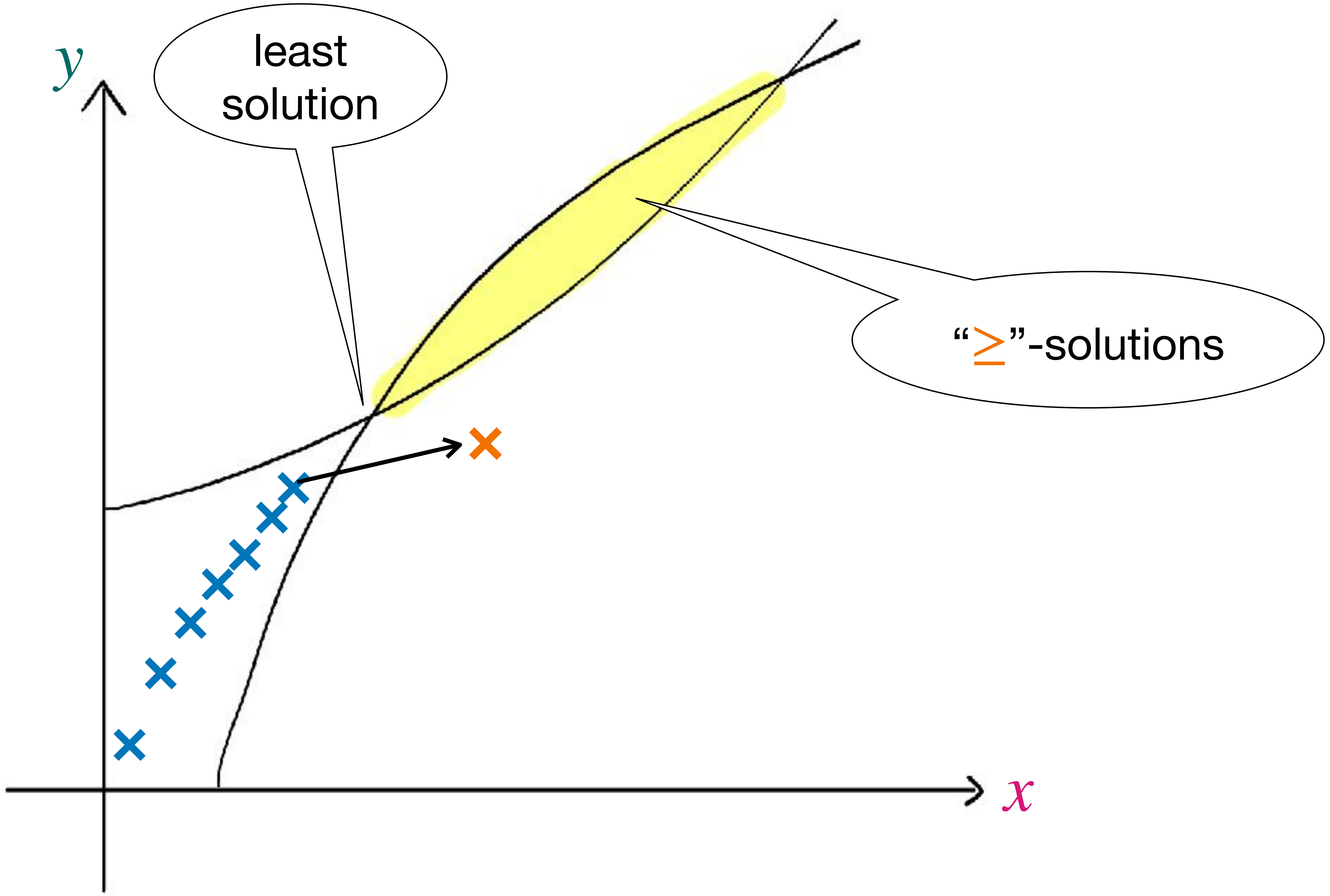

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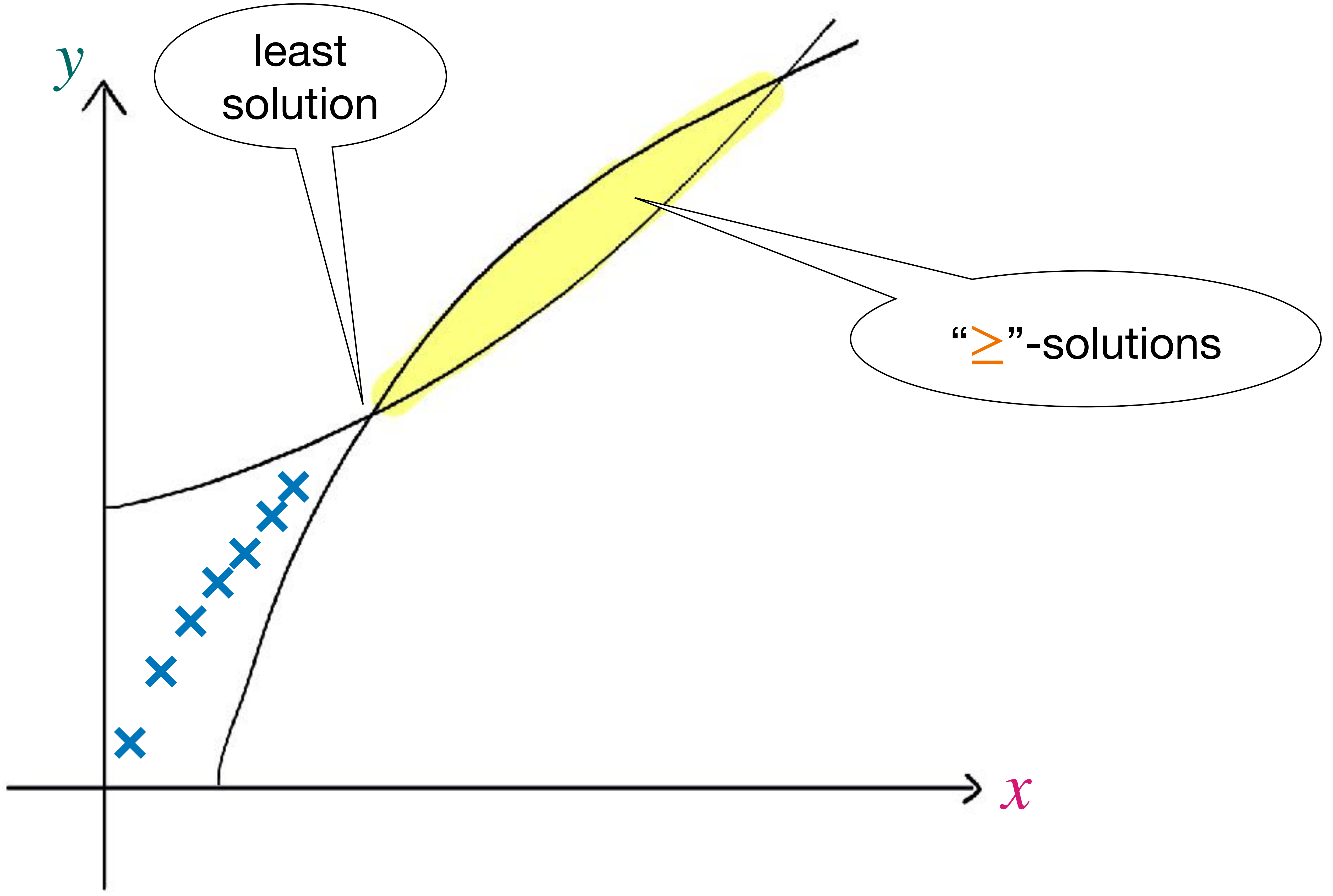

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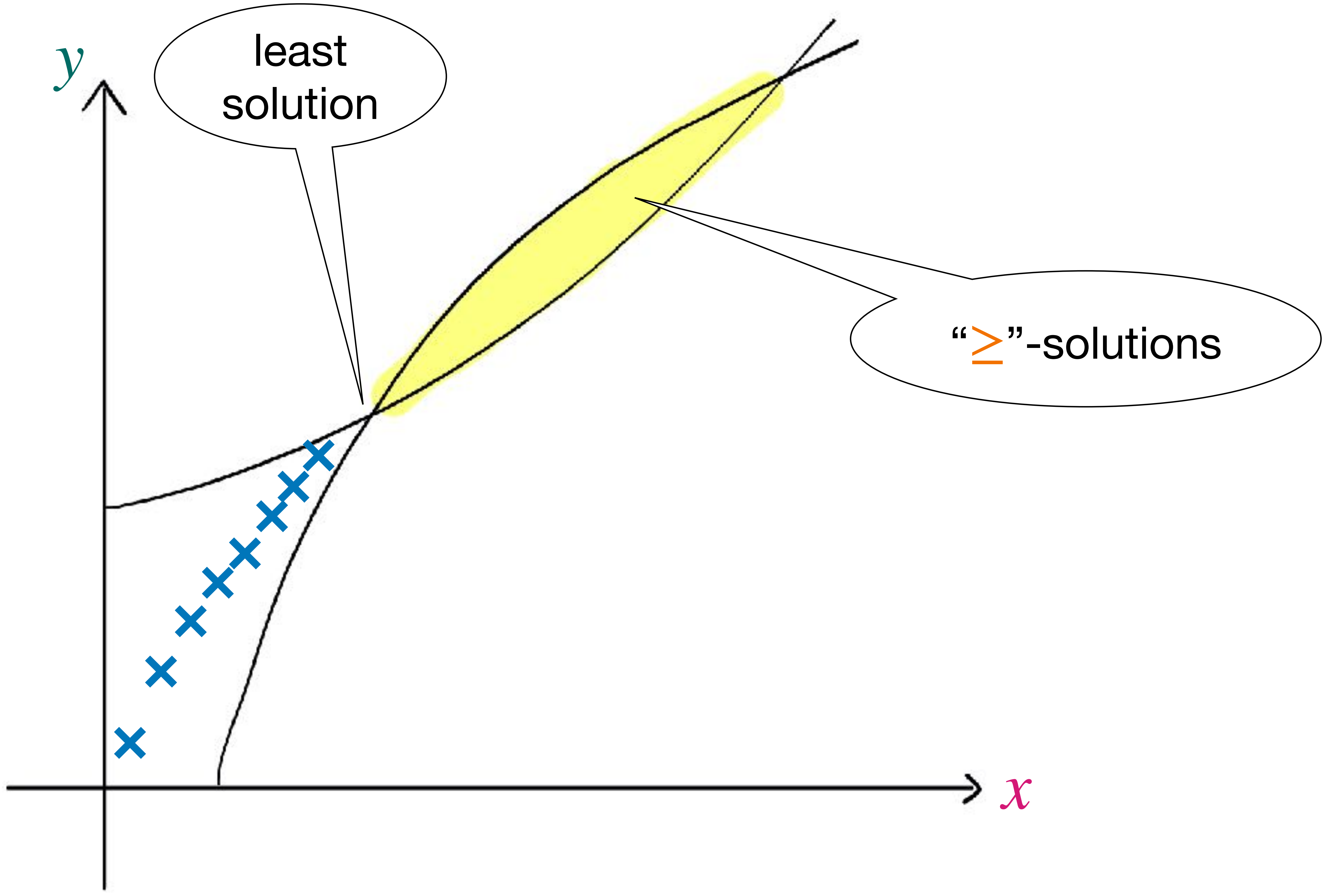

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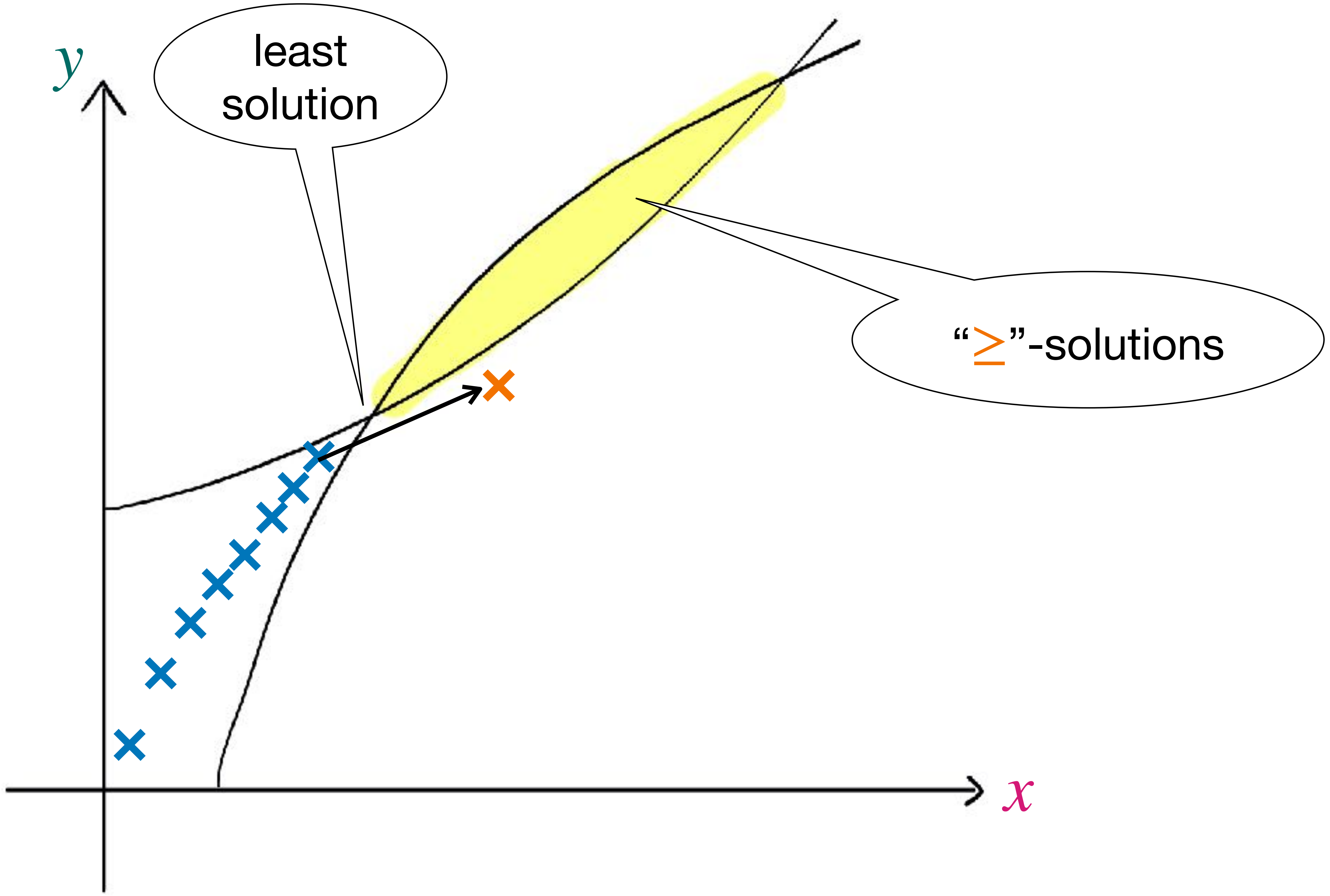

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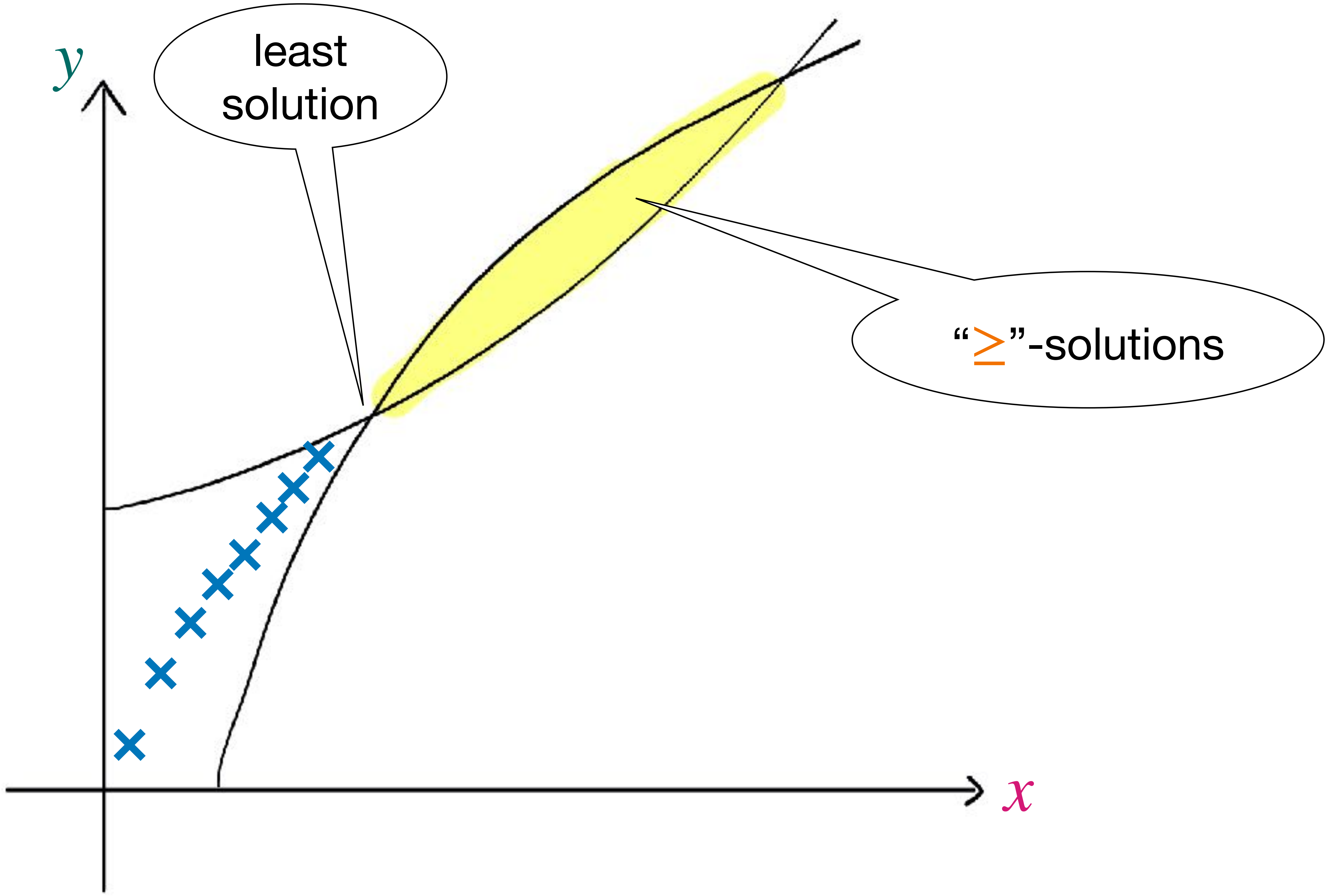

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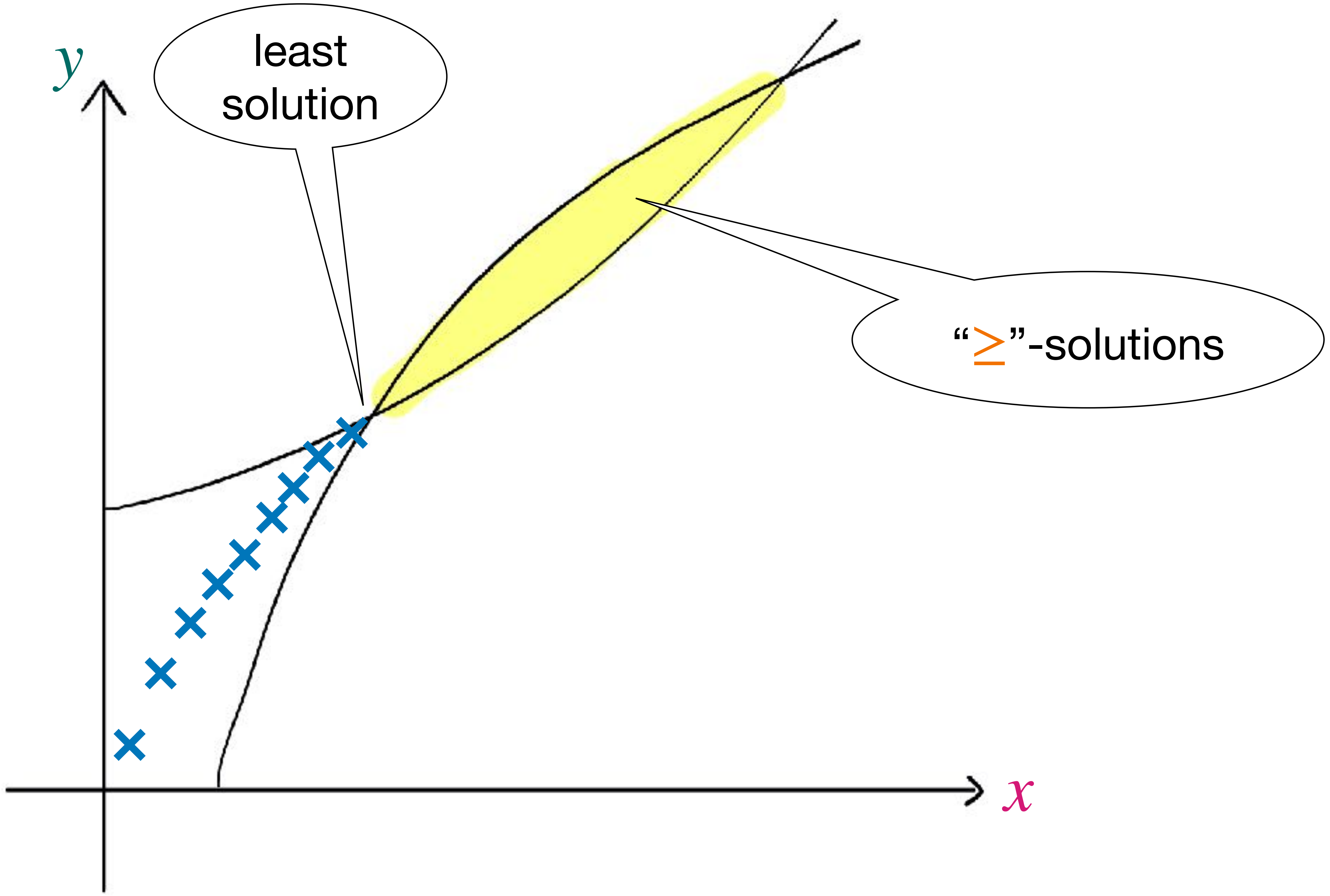

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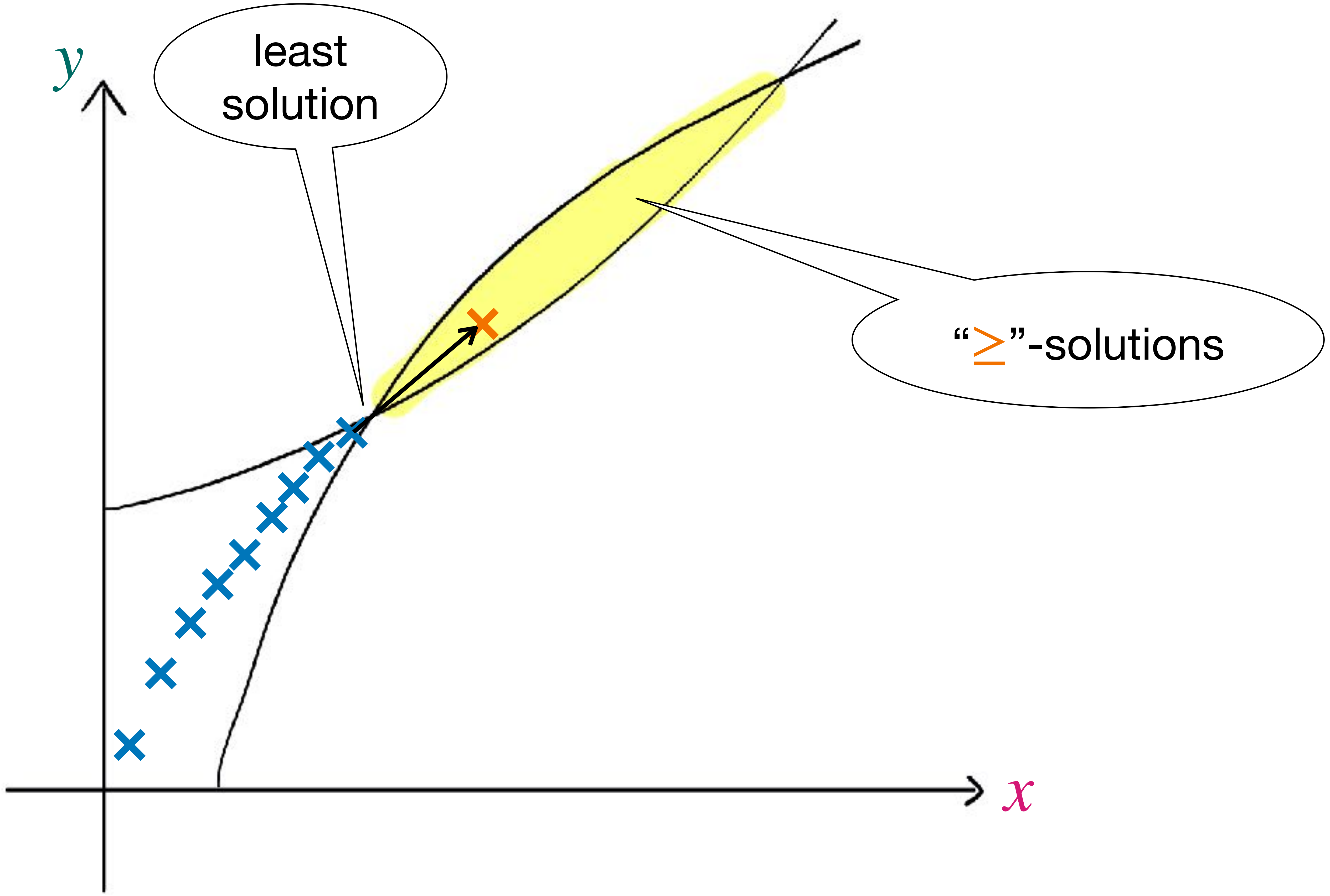

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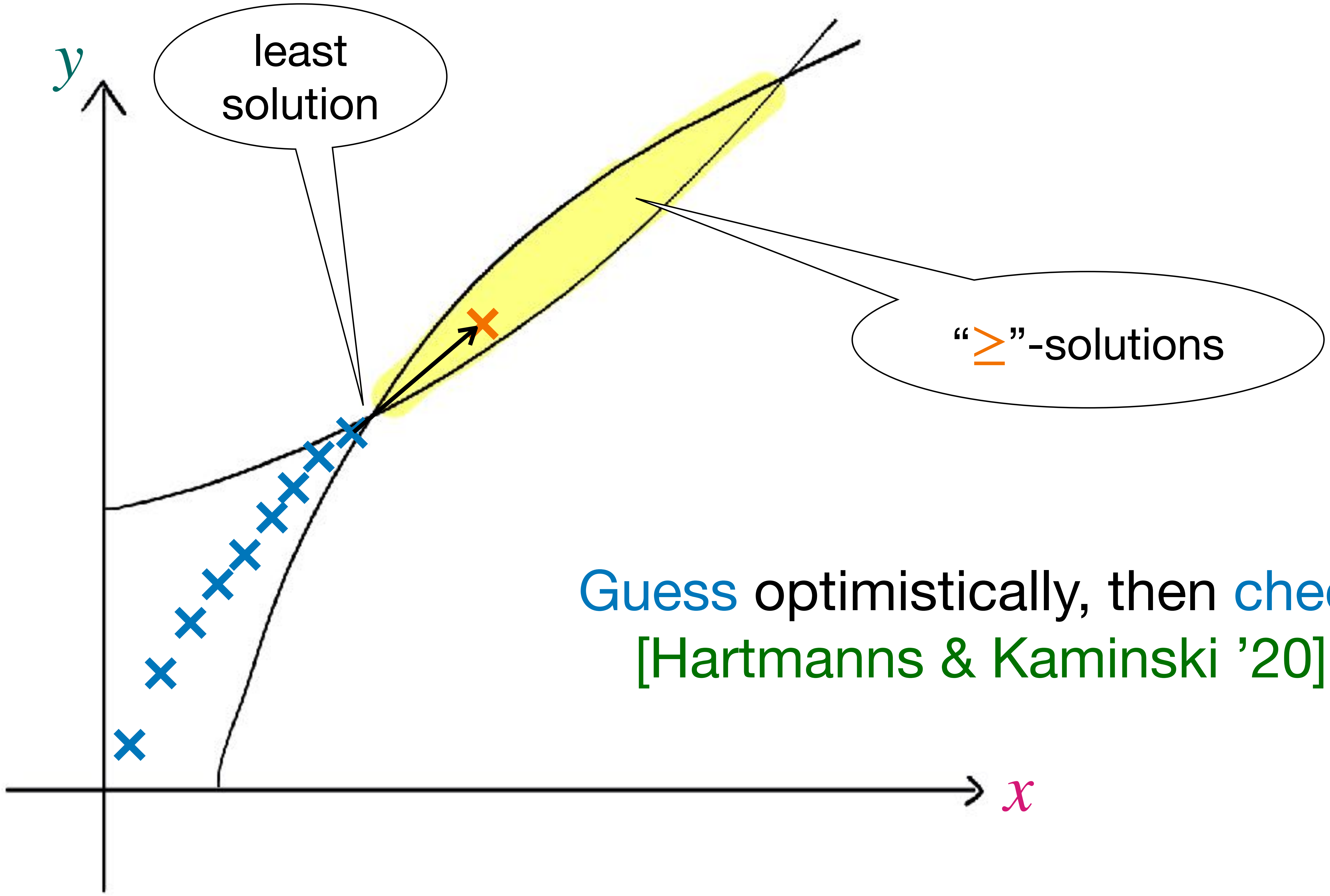
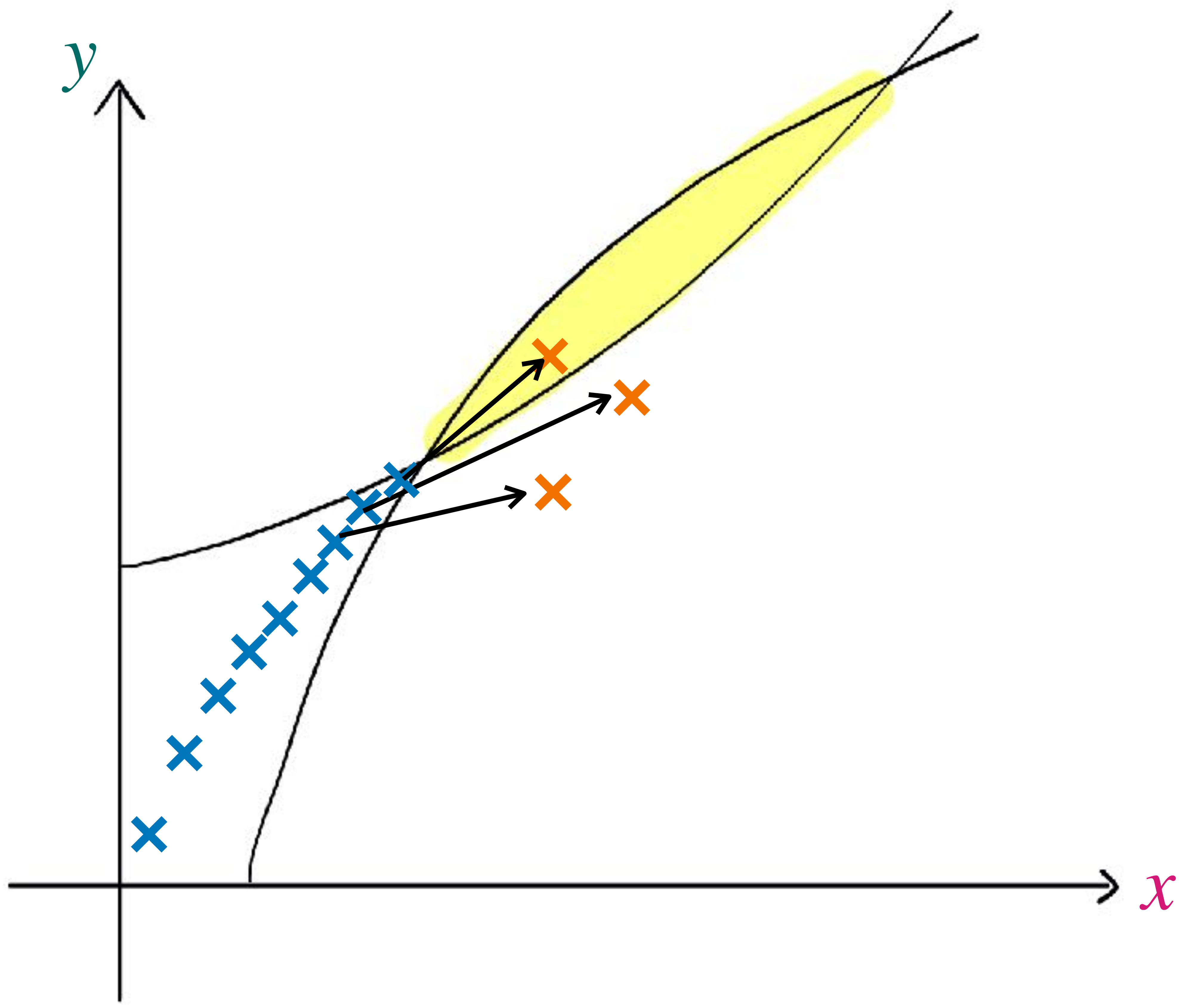


Guess optimistically, then check
[Hartmanns & Kaminski '20]

Termination?

- Algorithm does not terminate if we guess in the wrong direction



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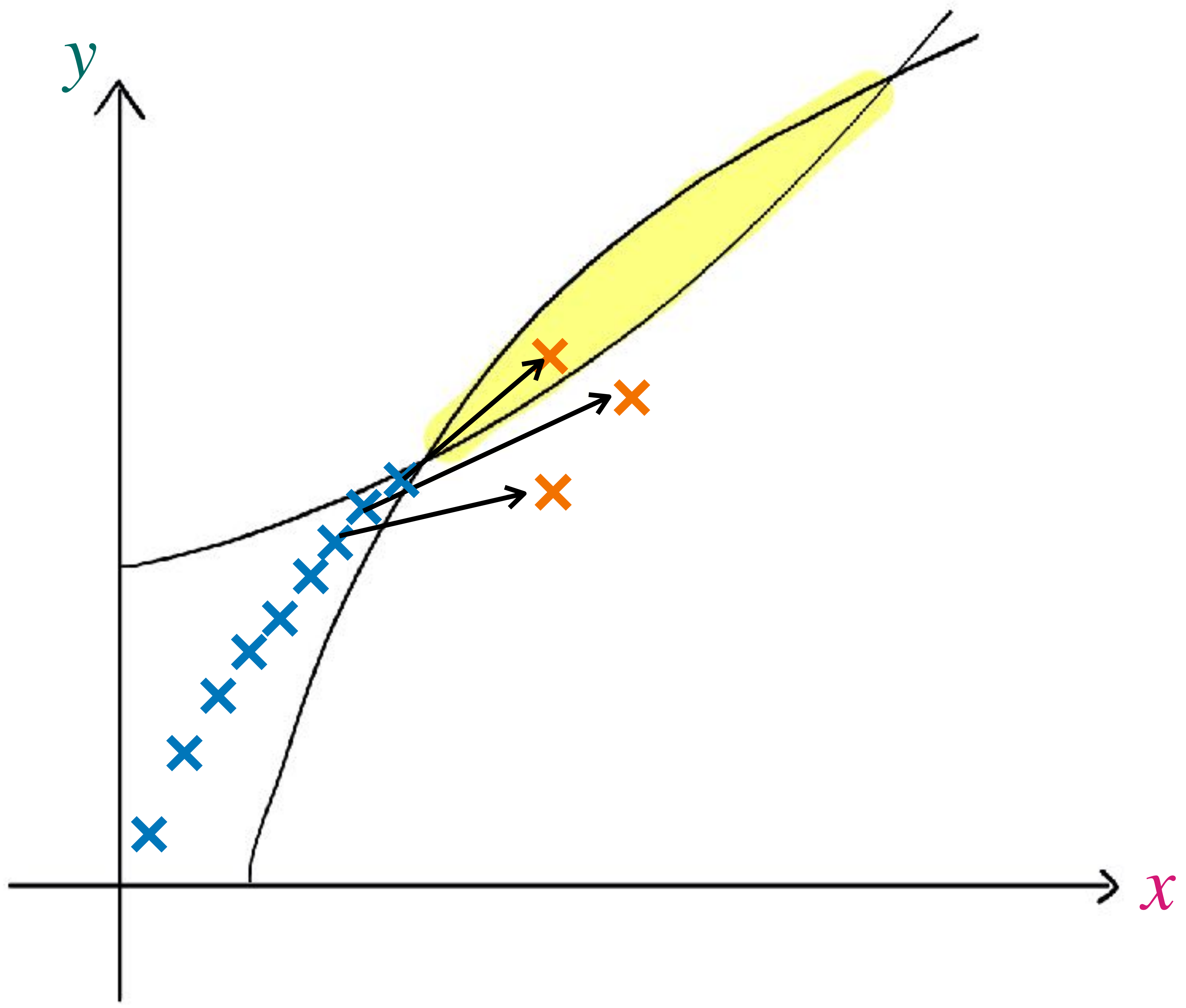
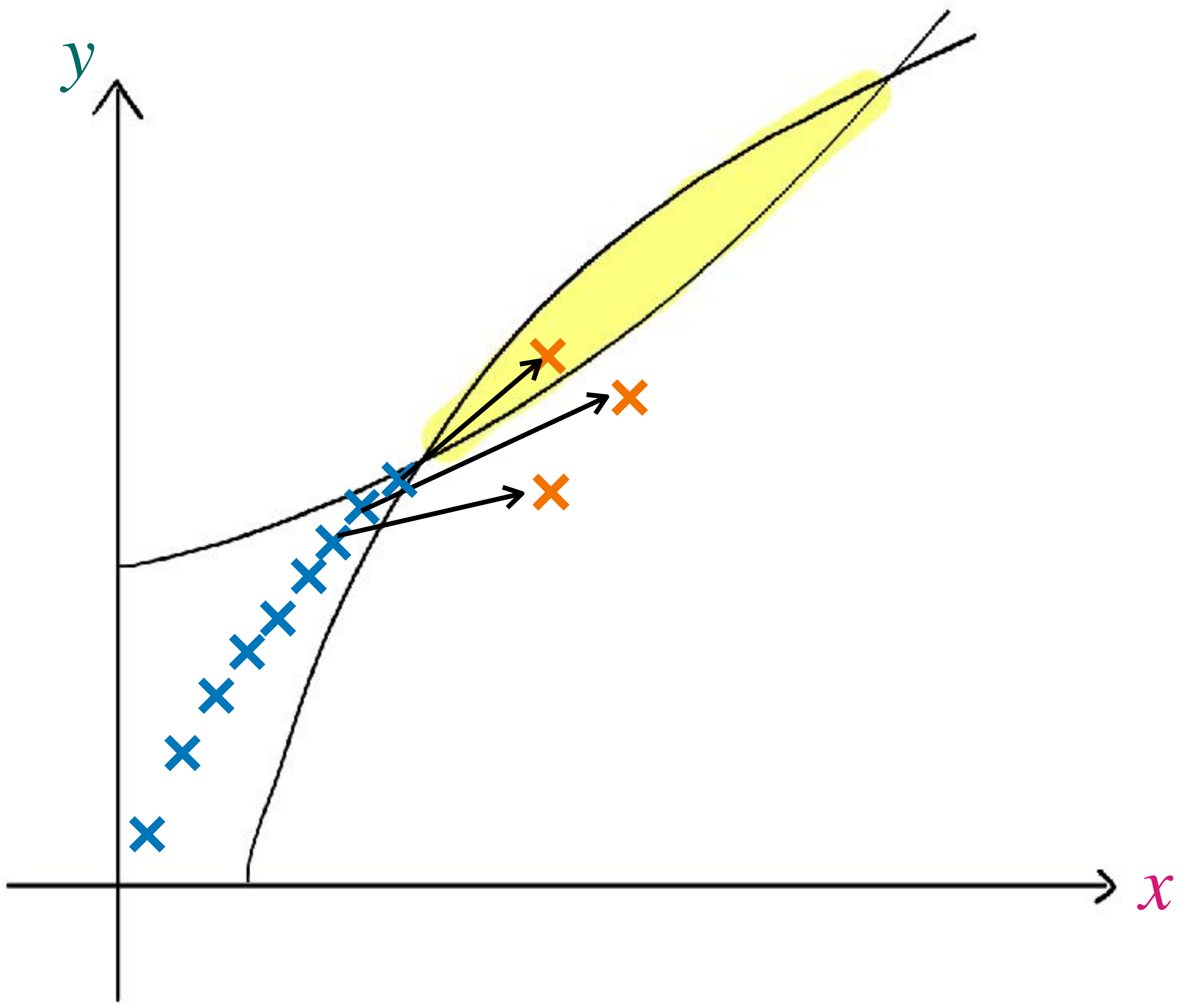


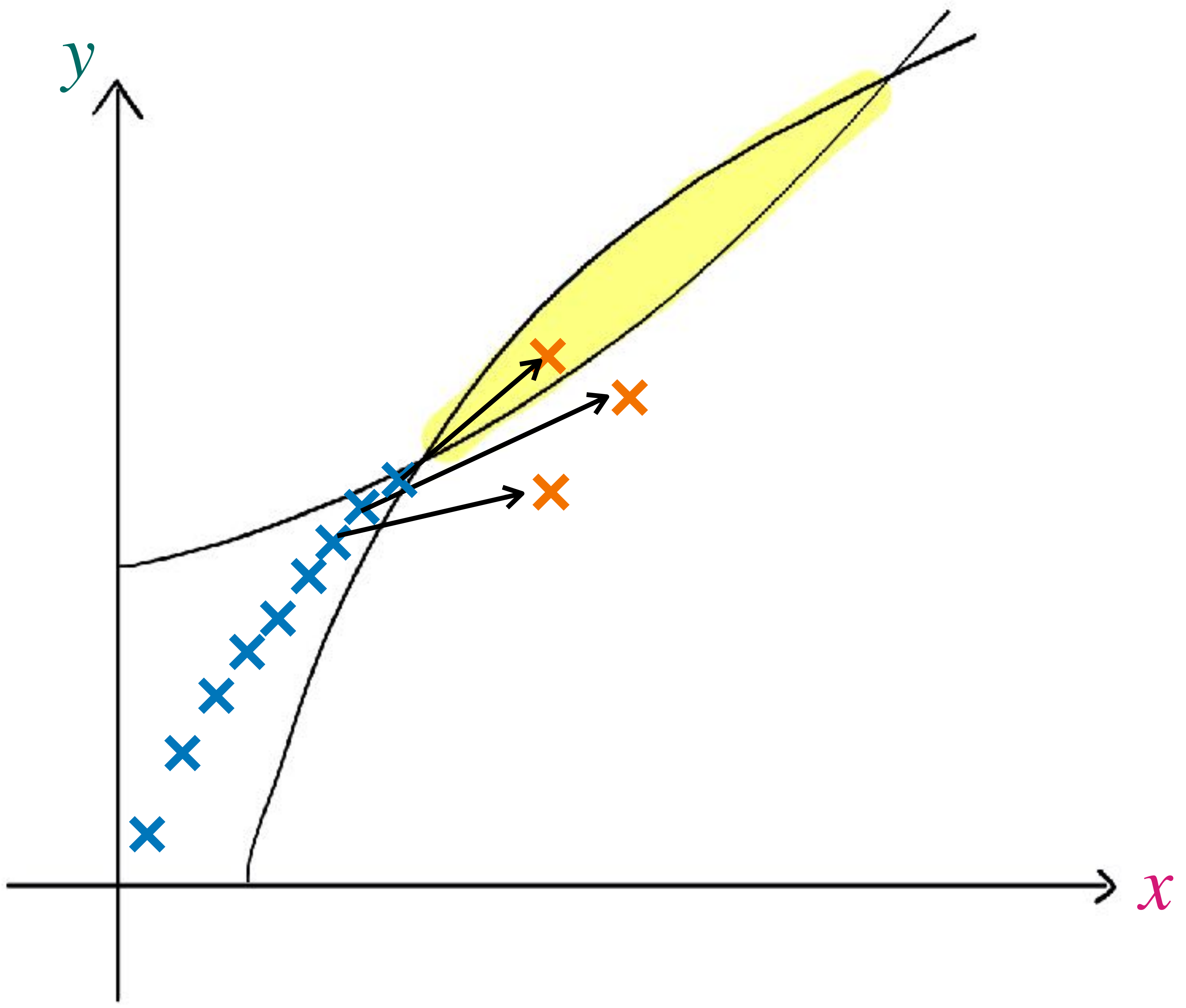
Theorem
Convergence is guaranteed* if guessing direction is approximately an eigenvector of the system's Jacobi matrix evaluated at the current under-approximation.

Float vs Exact Arithmetic



Float vs Exact Arithmetic

- Certificate should be formal proof
  → prefer exact rational numbers



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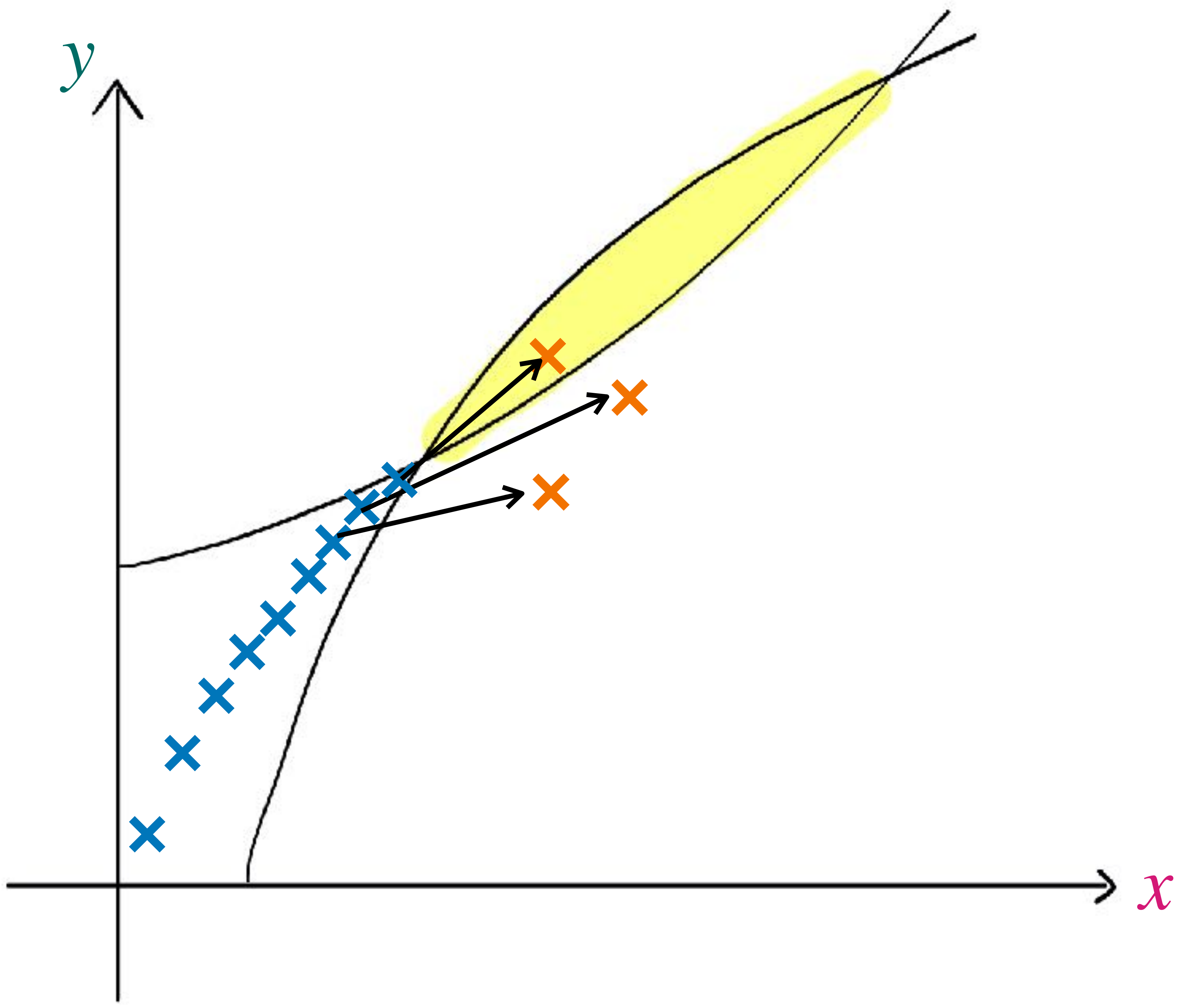

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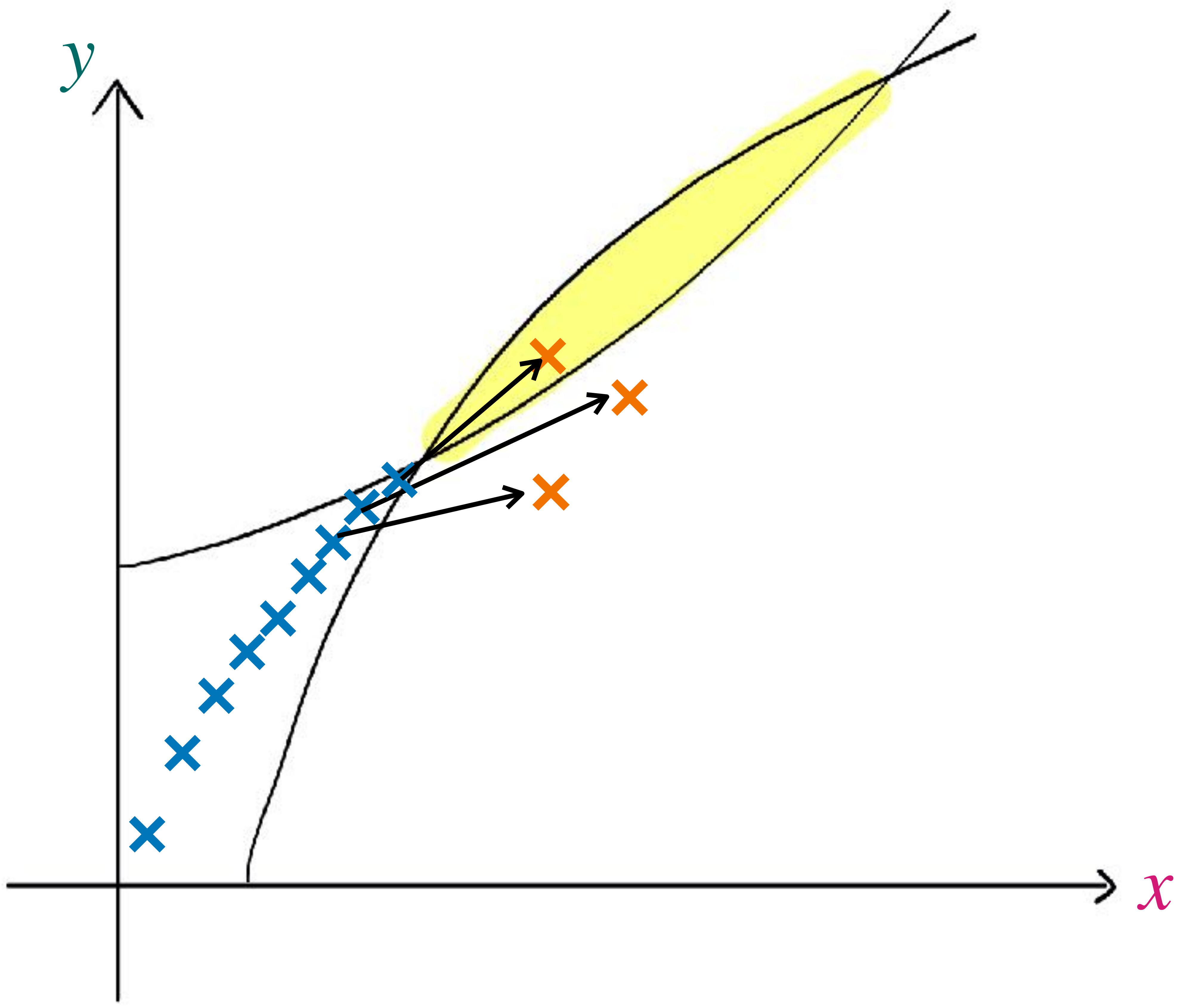

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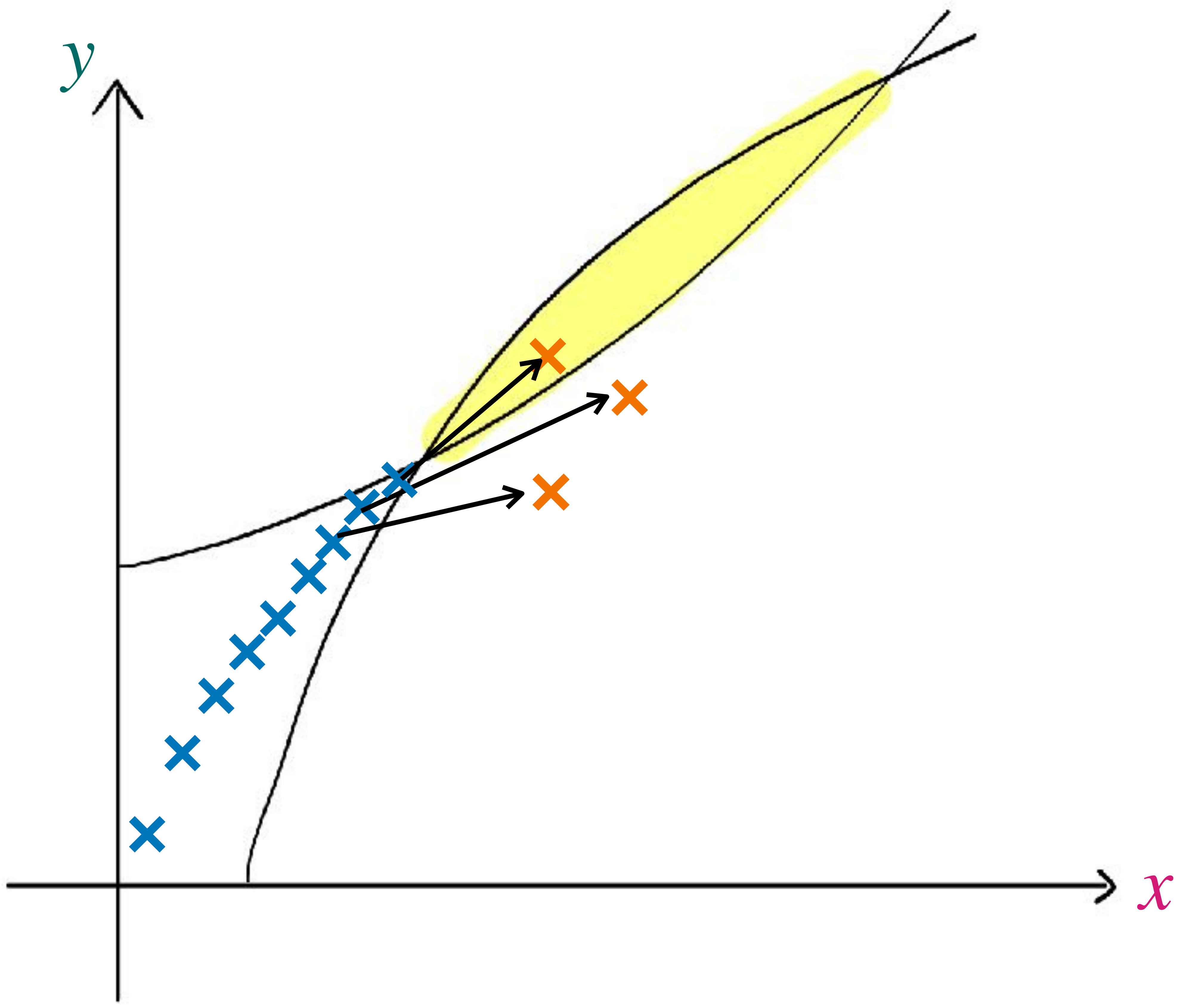

Float vs Exact Arithmetic

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1. Run algorithm with floats
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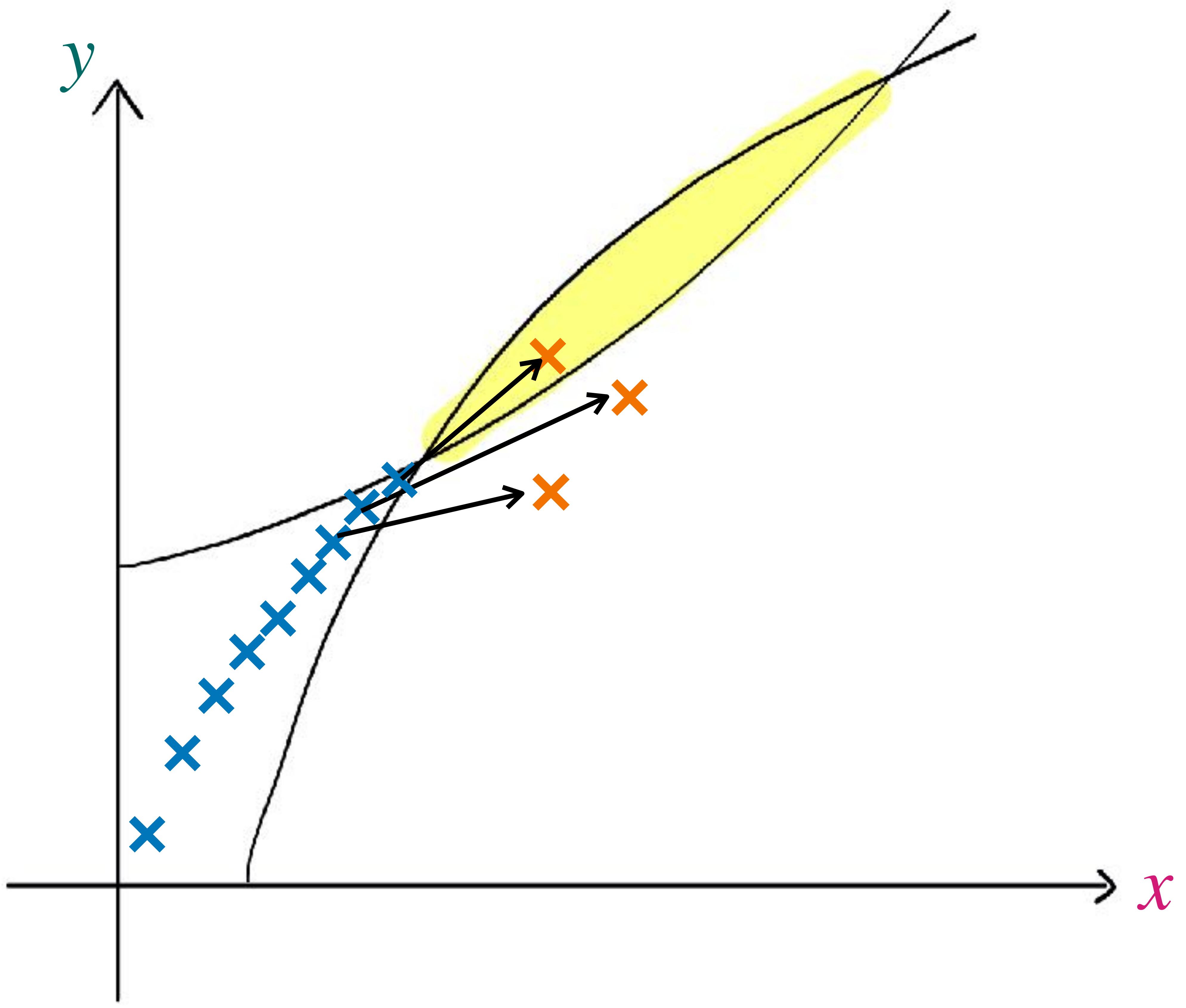
3. Check $\geq$ with exact arithmetic (often fails!)
4. Repair
   → generalized k-induction [Batz et al. ’21]



Application

Consistency of stochastic CFG

$$X \to a \mid XYY$$

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Consistency of stochastic CFG

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$\Rightarrow$

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- Grammar consistent $\Longleftrightarrow$ least solution $(x, y) = (1,1)$
- Our algorithm finds certificates for inconsistency

Stochastic Grammars Benchmark

Certificates for inconsistency

name	non-terminals	rules	time OVI	time SMT (z3)
brown	37	22,866	**3.2s**	TO
lemonde	121	32,885	**40.1s**	TO
negra	256	29,297	**10.2s**	37.2s
swbd	309	47,578	**19.0s**	TO
tiger	318	52,184	94.5s	**17.5s**
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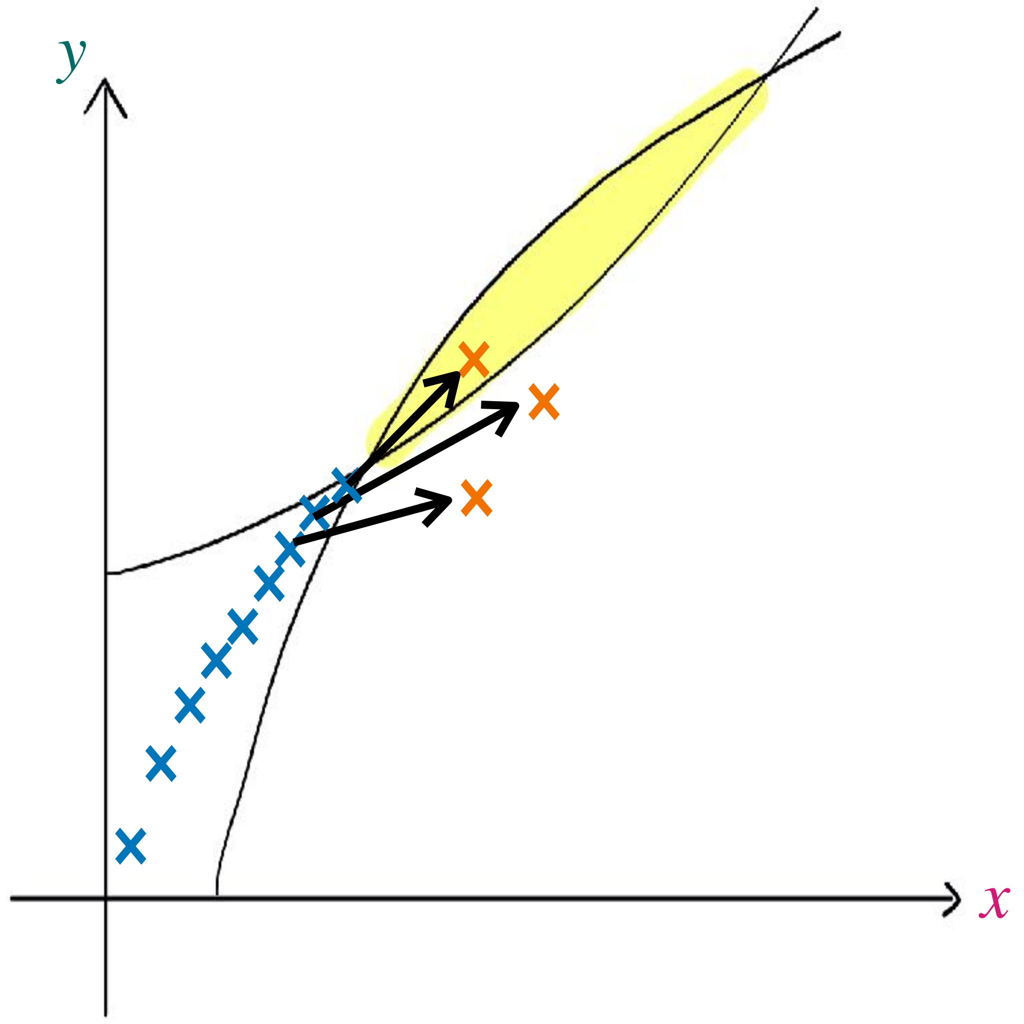
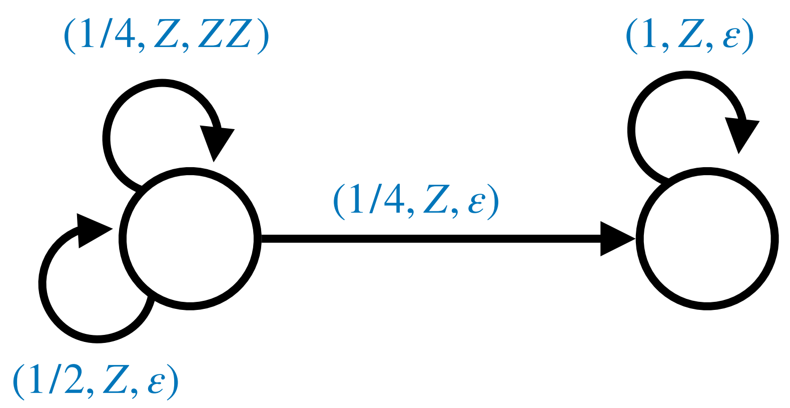
- #decimal digits of numerators/denominators in exact rationals always < 10
- ≈ 90% of runtime for arbitrary precision rational arithmetic

Summary & Outlook



- Optimistic “guess-and-check” algorithm for computing self-certifying upper bounds on least solution of positive polynomial equations
- Certified verification of recursive probabilistic system
- Open: Complexity of algorithm
- Follow-up paper: Certificates for lower bounds & termination [W. & Katoen LICS’23]



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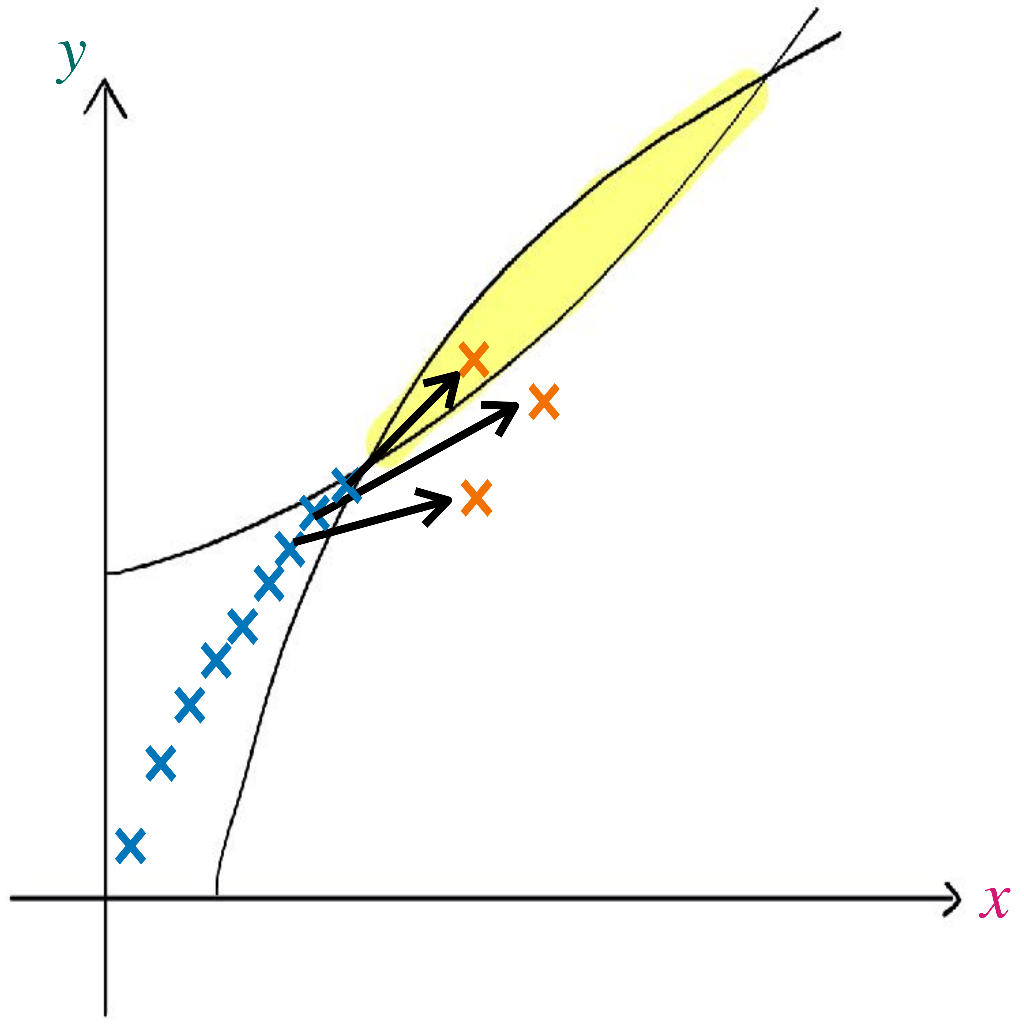

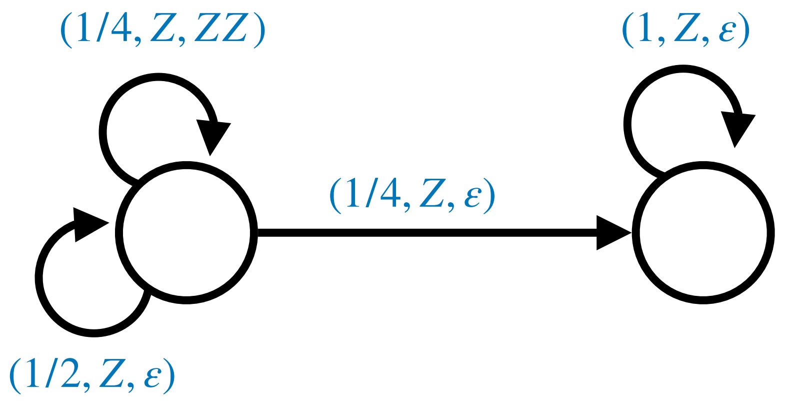

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Thank you for listening!

benchmark	$\lvert Q\rvert$	$\lvert P\rvert$	$\lvert \Gamma\rvert$	vars	terms	sccs	scc$_{max}$	cert	G	D	$t_{\mathbb{Q}}$	t_{tot}	$cert_{z3}$	D_{z3}	t_{z3}
rw-0.499	18	29	5	38	45	1	12	✓	5	5	17%	163	✓	2	11
rw-0.500	18	29	5	38	45	1	12	✗	10	-	-	7327	✓	2	10
rw-0.501	18	29	5	38	45	1	12	✓	5	4	6%	36	✓	13	12
geom-offspring	24	40	5	52	80	4	24	✓	8	6	13%	15	✓	9	16
golden	27	49	6	81	94	1	36	✓	1	5	30%	10	✓	7	14
and-or	50	90	7	149	182	1	48	✓	2	4	26%	19	✓	12	15260
gen-fun	85	219	7	202	327	1	16	✓	2	3	32%	22	✓	15	141
virus	68	149	27	341	551	1	220	✓	1	5	38%	40	✓	7	139
escape10	109	174	23	220	263	1	122	✓	1	4	5%	56	✓	7	48
escape25	258	413	53	518	621	1	300	✓	1	5	17%	245	✓	7	15958
escape50	508	813	103	1018	1221	1	600	✓	1	7	23%	653	✓	7	410
escape75	760	1215	153	1522	1825	1	904	✓	2	9	10%	3803	✗	-	TO
escape100	1009	1614	203	2020	2423	1	1202	✗	5	-	-	29027	✓	6	939
escape200	2008	3213	403	4018	4821	1	2400	✗	6	-	-	83781	✗	-	TO
sequential5	230	490	39	1017	1200	10	12	✓	15	4	26%	103	✓	8	1074
sequential7	572	1354	137	3349	3856	14	12	✓	21	5	27%	1049	✓	8	12822
sequential10	3341	8666	1036	26367	29616	20	12	✓	30	5	2%	100613	✓	8	453718
mod5	44	103	10	296	425	1	86	✓	1	5	39%	28	✓	9	34150
mod7	64	159	14	680	1017	1	222	✓	1	6	69%	172	✓	7	443
mod10	95	244	20	1574	2403	1	557	✗	1	-	-	675	✓	7	1245