



Foundations of Informatics: a Bridging Course

Week 3: Formal Languages and Processes

Part A: Regular Languages

March 31 – April 4, 2025

Thomas Noll

Software Modeling and Verification Group

RWTH Aachen University

<https://moves.rwth-aachen.de/teaching/ws-2024-25/foi/>

Overview of Week 3

1. Regular Languages

- Formal Languages
- Finite Automata
- Regular Expressions
- Minimisation of Finite Automata

2. Context-Free Languages

- Context-Free Grammars and Languages
- Context-Free vs. Regular Languages
- The Word Problem for Context-Free Languages
- The Emptiness Problem for Context-Free Languages
- Closure Properties of Context-Free Languages
- Pushdown Automata

Resources

- J.E. Hopcroft, R. Motwani, J.D. Ullmann: *Introduction to Automata Theory, Languages, and Computation*, 2nd ed., Addison-Wesley, 2001
- A. Asteroth, C. Baier: *Theoretische Informatik*, Pearson Studium, 2002 [in German]
- <http://www.jflap.org/>
(software for experimenting with formal languages and automata)

Outline of Part A

Formal Languages

Finite Automata

- Deterministic Finite Automata

- Operations on Languages and Automata

- Nondeterministic Finite Automata

- More Decidability Results

Regular Expressions

- Definition

- Equivalence of Regular Expressions and Finite Automata

Minimisation of Deterministic Finite Automata

Outlook

Words and Languages

- Computer systems transform data
 - Data encoded as (binary) **words**
- ⇒ Data sets = sets of words = **formal languages**,
data transformations = **functions on words**

Words and Languages

- Computer systems transform data
 - Data encoded as (binary) **words**
- ⇒ Data sets = sets of words = **formal languages**,
data transformations = **functions on words**

Example A.1

- *Java* = {all valid Java programs}
- *Compiler* : *Java* → *Bytecode*

Alphabets

The atomic elements of words are called symbols (or letters).

Definition A.2

An **alphabet** is a finite, non-empty set of symbols (“letters”).

- Σ, Γ, \dots denote alphabets
- a, b, \dots denote letters

Alphabets

The atomic elements of words are called symbols (or letters).

Definition A.2

An **alphabet** is a finite, non-empty set of symbols (“letters”).

- Σ, Γ, \dots denote alphabets
- a, b, \dots denote letters

Example A.3

1. Boolean alphabet $\mathbb{B} := \{0, 1\}$

Alphabets

The atomic elements of words are called symbols (or letters).

Definition A.2

An **alphabet** is a finite, non-empty set of symbols (“letters”).

- Σ, Γ, \dots denote alphabets
- a, b, \dots denote letters

Example A.3

1. Boolean alphabet $\mathbb{B} := \{0, 1\}$
2. Latin alphabet $\Sigma_{\text{latin}} := \{a, b, c, \dots, z\}$

Alphabets

The atomic elements of words are called symbols (or letters).

Definition A.2

An **alphabet** is a finite, non-empty set of symbols (“letters”).

- Σ, Γ, \dots denote alphabets
- a, b, \dots denote letters

Example A.3

1. Boolean alphabet $\mathbb{B} := \{0, 1\}$
2. Latin alphabet $\Sigma_{\text{latin}} := \{a, b, c, \dots, z\}$
3. Keyboard alphabet Σ_{key}

Alphabets

The atomic elements of words are called symbols (or letters).

Definition A.2

An **alphabet** is a finite, non-empty set of symbols (“letters”).

- Σ, Γ, \dots denote alphabets
- a, b, \dots denote letters

Example A.3

1. Boolean alphabet $\mathbb{B} := \{0, 1\}$
2. Latin alphabet $\Sigma_{\text{latin}} := \{a, b, c, \dots, z\}$
3. Keyboard alphabet Σ_{key}
4. Morse alphabet $\Sigma_{\text{morse}} := \{., -, \sqcup\}$

Definition A.4

- A **word** is a finite sequence of letters from a given alphabet Σ .
- Σ^* is the set of all words over Σ .

Definition A.4

- A **word** is a finite sequence of letters from a given alphabet Σ .
- Σ^* is the set of all words over Σ .
- $|w|$ denotes the **length** of a word $w \in \Sigma^*$, i.e., $|a_1 \dots a_n| := n$.
- The **empty word** is denoted by ε , i.e., $|\varepsilon| = 0$.

Definition A.4

- A **word** is a finite sequence of letters from a given alphabet Σ .
- Σ^* is the set of all words over Σ .
- $|w|$ denotes the **length** of a word $w \in \Sigma^*$, i.e., $|a_1 \dots a_n| := n$.
- The **empty word** is denoted by ε , i.e., $|\varepsilon| = 0$.
- The **concatenation** of two words $v = a_1 \dots a_m$ ($m \in \mathbb{N}$) and $w = b_1 \dots b_n$ ($n \in \mathbb{N}$) is the word

$$v \cdot w := a_1 \dots a_m b_1 \dots b_n$$

(often written as vw).

- Thus: $w \cdot \varepsilon = \varepsilon \cdot w = w$.

Definition A.4

- A **word** is a finite sequence of letters from a given alphabet Σ .
- Σ^* is the set of all words over Σ .
- $|w|$ denotes the **length** of a word $w \in \Sigma^*$, i.e., $|a_1 \dots a_n| := n$.
- The **empty word** is denoted by ε , i.e., $|\varepsilon| = 0$.
- The **concatenation** of two words $v = a_1 \dots a_m$ ($m \in \mathbb{N}$) and $w = b_1 \dots b_n$ ($n \in \mathbb{N}$) is the word

$$v \cdot w := a_1 \dots a_m b_1 \dots b_n$$

(often written as vw).

- Thus: $w \cdot \varepsilon = \varepsilon \cdot w = w$.
- A **prefix/suffix** v of a word w is an initial/trailing part of w , i.e., $w = vv'/w = v'v$ for some $v' \in \Sigma^*$.

Definition A.4

- A **word** is a finite sequence of letters from a given alphabet Σ .
- Σ^* is the set of all words over Σ .
- $|w|$ denotes the **length** of a word $w \in \Sigma^*$, i.e., $|a_1 \dots a_n| := n$.
- The **empty word** is denoted by ε , i.e., $|\varepsilon| = 0$.
- The **concatenation** of two words $v = a_1 \dots a_m$ ($m \in \mathbb{N}$) and $w = b_1 \dots b_n$ ($n \in \mathbb{N}$) is the word

$$v \cdot w := a_1 \dots a_m b_1 \dots b_n$$

(often written as vw).

- Thus: $w \cdot \varepsilon = \varepsilon \cdot w = w$.
- A **prefix/suffix** v of a word w is an initial/trailing part of w , i.e., $w = vv'/w = v'v$ for some $v' \in \Sigma^*$.
- If $w = a_1 \dots a_n$, then $w^R := a_n \dots a_1$.

Definition A.5

A set of words $L \subseteq \Sigma^*$ is called a (formal) language over Σ .

Definition A.5

A set of words $L \subseteq \Sigma^*$ is called a (formal) language over Σ .

Example A.6

1. over $\mathbb{B} = \{0, 1\}$: set of all bit strings containing 1101

Definition A.5

A set of words $L \subseteq \Sigma^*$ is called a (formal) language over Σ .

Example A.6

1. over $\mathbb{B} = \{0, 1\}$: set of all bit strings containing 1101
2. over $\Sigma = \{I, V, X, L, C, D, M\}$: set of all valid roman numbers

Definition A.5

A set of words $L \subseteq \Sigma^*$ is called a (formal) language over Σ .

Example A.6

1. over $\mathbb{B} = \{0, 1\}$: set of all bit strings containing 1101
2. over $\Sigma = \{I, V, X, L, C, D, M\}$: set of all valid roman numbers
3. over Σ_{key} : set of all valid Java programs

Seen:

- Basic notions: alphabets, words
- Formal languages as sets of words

Formal Languages II

Seen:

- Basic notions: alphabets, words
- Formal languages as sets of words

Next:

- Description of computations on words

Outline of Part A

Formal Languages

Finite Automata

- Deterministic Finite Automata

- Operations on Languages and Automata

- Nondeterministic Finite Automata

- More Decidability Results

Regular Expressions

- Definition

- Equivalence of Regular Expressions and Finite Automata

Minimisation of Deterministic Finite Automata

Outlook

Outline of Part A

Formal Languages

Finite Automata

- Deterministic Finite Automata

- Operations on Languages and Automata

- Nondeterministic Finite Automata

- More Decidability Results

Regular Expressions

- Definition

- Equivalence of Regular Expressions and Finite Automata

Minimisation of Deterministic Finite Automata

Outlook

Example: Pattern Matching

Example A.7 (Pattern 1101)

1. Read Boolean string bit-by-bit
2. Test whether it contains 1101
3. Idea: remember which (initial) part of 1101 has been recognised
4. Five prefixes: ϵ , 1, 11, 110, 1101
5. Diagram: on the board

Example: Pattern Matching

Example A.7 (Pattern 1101)

1. Read Boolean string bit-by-bit
2. Test whether it contains 1101
3. Idea: remember which (initial) part of 1101 has been recognised
4. Five prefixes: ϵ , 1, 11, 110, 1101
5. Diagram: on the board

What we used:

- finitely many (storage) states
- an initial state
- for every current state and every input symbol: a new state
- a successful state

Deterministic Finite Automata I

Definition A.8

A **deterministic finite automaton (DFA)** is of the form

$$\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$$

where

- Q is a finite set of **states**
- Σ denotes the **input alphabet**
- $\delta : Q \times \Sigma \rightarrow Q$ is the **transition function**
- $q_0 \in Q$ is the **initial state**
- $F \subseteq Q$ is the set of **final** (or: **accepting**) **states**

Example A.9

Pattern matching (Example A.7):

- $Q = \{q_0, \dots, q_4\}$
- $\Sigma = \mathbb{B} = \{0, 1\}$
- $\delta : Q \times \Sigma \rightarrow Q$ on the board
- $F = \{q_4\}$

Example A.9

Pattern matching (Example A.7):

- $Q = \{q_0, \dots, q_4\}$
- $\Sigma = \mathbb{B} = \{0, 1\}$
- $\delta : Q \times \Sigma \rightarrow Q$ on the board
- $F = \{q_4\}$

Graphical Representation of DFA:

- states \mapsto nodes
- $\delta(q, a) = q' \mapsto q \xrightarrow{a} q'$
- initial state: incoming edge without source state
- final state(s): additional circle

Acceptance by DFA I

Definition A.10

Let $\langle Q, \Sigma, \delta, q_0, F \rangle$ be a DFA. The **extension** of $\delta : Q \times \Sigma \rightarrow Q$,
$$\delta^* : Q \times \Sigma^* \rightarrow Q,$$

is defined by

$\delta^*(q, w) :=$ state after reading w starting from q .

Formally:

$$\delta^*(q, w) := \begin{cases} q & \text{if } w = \varepsilon \\ \delta^*(\delta(q, a), v) & \text{if } w = av \end{cases}$$

Thus: if $w = a_1 \dots a_n$ and $q \xrightarrow{a_1} q_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} q_n$, then $\delta^*(q, w) = q_n$

Acceptance by DFA I

Definition A.10

Let $\langle Q, \Sigma, \delta, q_0, F \rangle$ be a DFA. The **extension** of $\delta : Q \times \Sigma \rightarrow Q$,
$$\delta^* : Q \times \Sigma^* \rightarrow Q,$$

is defined by

$\delta^*(q, w) :=$ state after reading w starting from q .

Formally:

$$\delta^*(q, w) := \begin{cases} q & \text{if } w = \varepsilon \\ \delta^*(\delta(q, a), v) & \text{if } w = av \end{cases}$$

Thus: if $w = a_1 \dots a_n$ and $q \xrightarrow{a_1} q_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} q_n$, then $\delta^*(q, w) = q_n$

Example A.11

Pattern matching (Example A.9): on the board

Acceptance by DFA II

Definition A.12

- \mathcal{A} **accepts** $w \in \Sigma^*$ if $\delta^*(q_0, w) \in F$.
- The **language recognised (or: accepted)** by \mathcal{A} is

$$L(\mathcal{A}) := \{w \in \Sigma^* \mid \delta^*(q_0, w) \in F\}.$$

- A language $L \subseteq \Sigma^*$ is called **DFA-recognisable** if there exists some DFA \mathcal{A} such that $L(\mathcal{A}) = L$.
- Two DFA $\mathcal{A}_1, \mathcal{A}_2$ are called **equivalent** if

$$L(\mathcal{A}_1) = L(\mathcal{A}_2).$$

Example A.13

1. The set of all bit strings containing **1101** is recognised by the automaton from Example A.9.

Example A.13

1. The set of all bit strings containing **1101** is recognised by the automaton from Example A.9.
2. Two (equivalent) automata recognising the language

$$\{w \in \mathbb{B}^* \mid w \text{ contains } 1\} :$$

on the board

Acceptance by DFA III

Example A.13

1. The set of all bit strings containing **1101** is recognised by the automaton from Example A.9.
2. Two (equivalent) automata recognising the language

$$\{w \in \mathbb{B}^* \mid w \text{ contains } 1\} :$$

on the board

3. An automaton which recognises

$$\{w \in \{0, \dots, 9\}^* \mid \text{value of } w \text{ divisible by } 3\}$$

Idea: test whether sum of digits is divisible by 3 – one state for each residue class (on the board)

Deterministic Finite Automata

Seen:

- Deterministic finite automata as a model of simple sequential computations
- Recognisability of formal languages by automata

Deterministic Finite Automata

Seen:

- Deterministic finite automata as a model of simple sequential computations
- Recognisability of formal languages by automata

Next:

- Composition and transformation of automata
- Which languages are recognisable, which are not (alternative characterisation)
- Language definition \mapsto automaton and vice versa

Outline of Part A

Formal Languages

Finite Automata

Deterministic Finite Automata

Operations on Languages and Automata

Nondeterministic Finite Automata

More Decidability Results

Regular Expressions

Definition

Equivalence of Regular Expressions and Finite Automata

Minimisation of Deterministic Finite Automata

Outlook

Simplest case: Boolean operations (complement, intersection, union)

Question

Let $\mathcal{A}_1, \mathcal{A}_2$ be two DFA with $L(\mathcal{A}_1) = L_1$ and $L(\mathcal{A}_2) = L_2$.

Can we construct automata which recognise

- $\overline{L_1}$ ($:= \Sigma^* \setminus L_1$),
- $L_1 \cap L_2$, and
- $L_1 \cup L_2$?

Language Complement

Theorem A.14

If $L \subseteq \Sigma^$ is DFA-recognisable, then so is \bar{L} .*

Language Complement

Theorem A.14

If $L \subseteq \Sigma^*$ is DFA-recognisable, then so is \bar{L} .

Proof.

Let $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$ be a DFA such that $L(\mathfrak{A}) = L$. Then:

$$w \in \bar{L} \iff w \notin L \iff \delta^*(q_0, w) \notin F \iff \delta^*(q_0, w) \in Q \setminus F.$$

Thus, \bar{L} is recognised by the DFA $\langle Q, \Sigma, \delta, q_0, Q \setminus F \rangle$. □

Language Complement

Theorem A.14

If $L \subseteq \Sigma^*$ is DFA-recognisable, then so is \bar{L} .

Proof.

Let $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$ be a DFA such that $L(\mathfrak{A}) = L$. Then:

$$w \in \bar{L} \iff w \notin L \iff \delta^*(q_0, w) \notin F \iff \delta^*(q_0, w) \in Q \setminus F.$$

Thus, \bar{L} is recognised by the DFA $\langle Q, \Sigma, \delta, q_0, Q \setminus F \rangle$. □

Example A.15

on the board

Language Intersection I

Theorem A.16

If $L_1, L_2 \subseteq \Sigma^$ are DFA-recognisable, then so is $L_1 \cap L_2$.*

Language Intersection I

Theorem A.16

If $L_1, L_2 \subseteq \Sigma^*$ are DFA-recognisable, then so is $L_1 \cap L_2$.

Proof.

Let $\mathcal{A}_i = \langle Q_i, \Sigma, \delta_i, q_0^i, F_i \rangle$ be DFA such that $L(\mathcal{A}_i) = L_i$ ($i = 1, 2$). The new automaton \mathcal{A} has to accept w iff \mathcal{A}_1 and \mathcal{A}_2 accept w

Idea: let \mathcal{A}_1 and \mathcal{A}_2 run in parallel

- use pairs of states $(q_1, q_2) \in Q_1 \times Q_2$
- start with both components in initial state
- a transition updates both components independently
- for acceptance both components need to be in a final state



Language Intersection II

Proof (continued).

Formally: let the **product automaton**

$$\mathfrak{A} := \langle Q_1 \times Q_2, \Sigma, \delta, (q_0^1, q_0^2), F_1 \times F_2 \rangle$$

be defined by

$$\delta((q_1, q_2), a) := (\delta_1(q_1, a), \delta_2(q_2, a)) \text{ for every } a \in \Sigma.$$

Language Intersection II

Proof (continued).

Formally: let the **product automaton**

$$\mathfrak{A} := \langle Q_1 \times Q_2, \Sigma, \delta, (q_0^1, q_0^2), F_1 \times F_2 \rangle$$

be defined by

$$\delta((q_1, q_2), a) := (\delta_1(q_1, a), \delta_2(q_2, a)) \text{ for every } a \in \Sigma.$$

This definition yields (for every $w \in \Sigma^*$):

$$\delta^*((q_1, q_2), w) = (\delta_1^*(q_1, w), \delta_2^*(q_2, w)) \quad (*)$$

Language Intersection II

Proof (continued).

Formally: let the **product automaton**

$$\mathfrak{A} := \langle Q_1 \times Q_2, \Sigma, \delta, (q_0^1, q_0^2), F_1 \times F_2 \rangle$$

be defined by

$$\delta((q_1, q_2), a) := (\delta_1(q_1, a), \delta_2(q_2, a)) \text{ for every } a \in \Sigma.$$

This definition yields (for every $w \in \Sigma^*$):

$$\delta^*((q_1, q_2), w) = (\delta_1^*(q_1, w), \delta_2^*(q_2, w)) \quad (*)$$

Thus: \mathfrak{A} accepts $w \iff \delta^*((q_0^1, q_0^2), w) \in F_1 \times F_2$ □

$$\begin{aligned} &\stackrel{(*)}{\iff} (\delta_1^*(q_0^1, w), \delta_2^*(q_0^2, w)) \in F_1 \times F_2 \\ &\iff \delta_1^*(q_0^1, w) \in F_1 \text{ and } \delta_2^*(q_0^2, w) \in F_2 \\ &\iff \mathfrak{A}_1 \text{ accepts } w \text{ and } \mathfrak{A}_2 \text{ accepts } w \end{aligned}$$

Example A.17

on the board

Theorem A.18

If $L_1, L_2 \subseteq \Sigma^$ are DFA-recognisable, then so is $L_1 \cup L_2$.*

Language Union

Theorem A.18

If $L_1, L_2 \subseteq \Sigma^$ are DFA-recognisable, then so is $L_1 \cup L_2$.*

Proof.

Let $\mathfrak{A}_i = \langle Q_i, \Sigma, \delta_i, q_0^i, F_i \rangle$ be DFA such that $L(\mathfrak{A}_i) = L_i$ ($i = 1, 2$). The new automaton \mathfrak{A} has to accept w iff \mathfrak{A}_1 or \mathfrak{A}_2 accepts w .

Language Union

Theorem A.18

If $L_1, L_2 \subseteq \Sigma^*$ are DFA-recognisable, then so is $L_1 \cup L_2$.

Proof.

Let $\mathcal{A}_i = \langle Q_i, \Sigma, \delta_i, q_0^i, F_i \rangle$ be DFA such that $L(\mathcal{A}_i) = L_i$ ($i = 1, 2$). The new automaton \mathcal{A} has to accept w iff \mathcal{A}_1 or \mathcal{A}_2 accepts w .

Idea: reuse product construction

Construct \mathcal{A} as before but choose as final states those pairs $(q_1, q_2) \in Q_1 \times Q_2$ with $q_1 \in F_1$ or $q_2 \in F_2$. Thus the set of final states is given by

$$F := (F_1 \times Q_2) \cup (Q_1 \times F_2).$$



Language Concatenation

Definition A.19

The **concatenation** of two languages $L_1, L_2 \subseteq \Sigma^*$ is given by

$$L_1 \cdot L_2 := \{v \cdot w \in \Sigma^* \mid v \in L_1, w \in L_2\}.$$

Abbreviations: $w \cdot L := \{w\} \cdot L$, $L \cdot w := L \cdot \{w\}$

Language Concatenation

Definition A.19

The **concatenation** of two languages $L_1, L_2 \subseteq \Sigma^*$ is given by

$$L_1 \cdot L_2 := \{v \cdot w \in \Sigma^* \mid v \in L_1, w \in L_2\}.$$

Abbreviations: $w \cdot L := \{w\} \cdot L$, $L \cdot w := L \cdot \{w\}$

Example A.20

1. If $L_1 = \{101, 1\}$ and $L_2 = \{011, 1\}$, then
$$L_1 \cdot L_2 = \{101011, 1011, 11\}.$$

Language Concatenation

Definition A.19

The **concatenation** of two languages $L_1, L_2 \subseteq \Sigma^*$ is given by

$$L_1 \cdot L_2 := \{v \cdot w \in \Sigma^* \mid v \in L_1, w \in L_2\}.$$

Abbreviations: $w \cdot L := \{w\} \cdot L$, $L \cdot w := L \cdot \{w\}$

Example A.20

1. If $L_1 = \{101, 1\}$ and $L_2 = \{011, 1\}$, then

$$L_1 \cdot L_2 = \{101011, 1011, 11\}.$$

2. If $L_1 = 00 \cdot \mathbb{B}^*$ and $L_2 = 11 \cdot \mathbb{B}^*$, then

$$L_1 \cdot L_2 = \{w \in \mathbb{B}^* \mid w \text{ has prefix } 00 \text{ and contains } 11\}.$$

DFA-Recognisability of Concatenation

Conjecture

If $L_1, L_2 \subseteq \Sigma^*$ are DFA-recognisable, then so is $L_1 \cdot L_2$.

DFA-Recognisability of Concatenation

Conjecture

If $L_1, L_2 \subseteq \Sigma^*$ are DFA-recognisable, then so is $L_1 \cdot L_2$.

Proof (attempt).

Let $\mathcal{A}_i = \langle Q_i, \Sigma, \delta_i, q_0^i, F_i \rangle$ be DFA such that $L(\mathcal{A}_i) = L_i$ ($i = 1, 2$). The new automaton \mathcal{A} has to accept w iff a prefix of w is recognised by \mathcal{A}_1 , and if \mathcal{A}_2 accepts the remaining suffix.

Idea: choose $Q := Q_1 \cup Q_2$ where each $q \in F_1$ is identified with q_0^2

But: on the board



DFA-Recognisability of Concatenation

Conjecture

If $L_1, L_2 \subseteq \Sigma^*$ are DFA-recognisable, then so is $L_1 \cdot L_2$.

Proof (attempt).

Let $\mathcal{A}_i = \langle Q_i, \Sigma, \delta_i, q_0^i, F_i \rangle$ be DFA such that $L(\mathcal{A}_i) = L_i$ ($i = 1, 2$). The new automaton \mathcal{A} has to accept w iff a prefix of w is recognised by \mathcal{A}_1 , and if \mathcal{A}_2 accepts the remaining suffix.

Idea: choose $Q := Q_1 \cup Q_2$ where each $q \in F_1$ is identified with q_0^2

But: on the board



Conclusion

Required: automata model where the successor state (for a given state and input symbol) is **not unique**

Definition A.21

- The **n th power** of a language $L \subseteq \Sigma^*$ is the n -fold concatenation of L with itself ($n \in \mathbb{N}$):

$$L^n := \underbrace{L \cdot \dots \cdot L}_{n \text{ times}} = \{w_1 \dots w_n \mid \forall i \in \{1, \dots, n\} : w_i \in L\}.$$

Inductively: $L^0 := \{\varepsilon\}$, $L^{n+1} := L^n \cdot L$

- The **iteration** (or: **Kleene star**) of L is

$$L^* := \bigcup_{n \in \mathbb{N}} L^n = \{w_1 \dots w_n \mid n \in \mathbb{N}, \forall i \in \{1, \dots, n\} : w_i \in L\}.$$

Definition A.21

- The **n th power** of a language $L \subseteq \Sigma^*$ is the n -fold concatenation of L with itself ($n \in \mathbb{N}$):

$$L^n := \underbrace{L \cdot \dots \cdot L}_{n \text{ times}} = \{w_1 \dots w_n \mid \forall i \in \{1, \dots, n\} : w_i \in L\}.$$

Inductively: $L^0 := \{\varepsilon\}$, $L^{n+1} := L^n \cdot L$

- The **iteration** (or: **Kleene star**) of L is

$$L^* := \bigcup_{n \in \mathbb{N}} L^n = \{w_1 \dots w_n \mid n \in \mathbb{N}, \forall i \in \{1, \dots, n\} : w_i \in L\}.$$

Remarks:

- we always have $\varepsilon \in L^*$ (since $L^0 \subseteq L^*$ and $L^0 = \{\varepsilon\}$)
- $w \in L^*$ iff $w = \varepsilon$ or if w can be decomposed into $n \geq 1$ subwords v_1, \dots, v_n (i.e., $w = v_1 \cdot \dots \cdot v_n$) such that $v_i \in L$ for every $1 \leq i \leq n$
- again we would suspect that the iteration of a DFA-recognisable language is DFA-recognisable, but there is no simple (deterministic) construction

Seen:

- Operations on languages:
 - complement
 - intersection
 - union
 - concatenation
 - iteration
- DFA constructions for:
 - complement
 - intersection
 - union

Seen:

- Operations on languages:
 - complement
 - intersection
 - union
 - concatenation
 - iteration
- DFA constructions for:
 - complement
 - intersection
 - union

Next:

- Automata model for (direct implementation of) concatenation and iteration

Outline of Part A

Formal Languages

Finite Automata

Deterministic Finite Automata

Operations on Languages and Automata

Nondeterministic Finite Automata

More Decidability Results

Regular Expressions

Definition

Equivalence of Regular Expressions and Finite Automata

Minimisation of Deterministic Finite Automata

Outlook

Nondeterministic Finite Automata I

Idea:

- for a given state and a given input symbol, **several transitions** (or none at all) are possible
- an input word generally induces **several state sequences** (“runs”)
- the word is accepted if **at least one** accepting run exists

Nondeterministic Finite Automata I

Idea:

- for a given state and a given input symbol, **several transitions** (or none at all) are possible
- an input word generally induces **several state sequences** (“runs”)
- the word is accepted if **at least one** accepting run exists

Advantages:

- simplifies representation of languages
 - example: $\mathbb{B}^* \cdot 1101 \cdot \mathbb{B}^*$ (on the board)
- yields direct constructions for concatenation and iteration of languages
- more adequate modelling of systems with nondeterministic behaviour
 - communication protocols, multi-agent systems, ...

Nondeterministic Finite Automata II

Definition A.22

A **nondeterministic finite automaton (NFA)** is of the form

$$\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$$

where

- Q is a finite set of **states**
- Σ denotes the **input alphabet**
- $\Delta \subseteq Q \times \Sigma \times Q$ is the **transition relation**
- $q_0 \in Q$ is the **initial state**
- $F \subseteq Q$ is the set of **final states**

Nondeterministic Finite Automata II

Definition A.22

A **nondeterministic finite automaton (NFA)** is of the form

$$\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$$

where

- Q is a finite set of **states**
- Σ denotes the **input alphabet**
- $\Delta \subseteq Q \times \Sigma \times Q$ is the **transition relation**
- $q_0 \in Q$ is the **initial state**
- $F \subseteq Q$ is the set of **final states**

Remarks:

- $(q, a, q') \in \Delta$ usually written as $q \xrightarrow{a} q'$
- every DFA can be considered as an NFA $((q, a, q') \in \Delta \iff \delta(q, a) = q')$

Acceptance by NFA

Definition A.23

- Let $w = a_1 \dots a_n \in \Sigma^*$.
- A w -labelled \mathcal{A} -run from q_1 to q_2 is a sequence

$$p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} \dots p_{n-1} \xrightarrow{a_n} p_n$$

such that $p_0 = q_1$, $p_n = q_2$, and $(p_{i-1}, a_i, p_i) \in \Delta$ for every $1 \leq i \leq n$ (we also write: $q_1 \xrightarrow{w} q_2$).

- \mathcal{A} **accepts** w if there is a w -labelled \mathcal{A} -run from q_0 to some $q \in F$
- The **language recognised by** \mathcal{A} is

$$L(\mathcal{A}) := \{w \in \Sigma^* \mid \mathcal{A} \text{ accepts } w\}.$$

- A language $L \subseteq \Sigma^*$ is called **NFA-recognisable** if there exists a NFA \mathcal{A} such that $L(\mathcal{A}) = L$.
- Two NFA $\mathcal{A}_1, \mathcal{A}_2$ are called **equivalent** if $L(\mathcal{A}_1) = L(\mathcal{A}_2)$.

Acceptance Test for NFA

Algorithm A.24 (Acceptance Test for NFA)

Input: NFA $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$, $w \in \Sigma^*$

Question: $w \in L(\mathfrak{A})$?

Procedure: Computation of the **reachability set**

$$R_{\mathfrak{A}}(w) := \{q \in Q \mid q_0 \xrightarrow{w} q\}$$

Iterative procedure for $w = a_1 \dots a_n$:

1. let $R_{\mathfrak{A}}(\varepsilon) := \{q_0\}$

2. for $i := 1, \dots, n$: let

$$R_{\mathfrak{A}}(a_1 \dots a_i) := \{q \in Q \mid \exists p \in R_{\mathfrak{A}}(a_1 \dots a_{i-1}) : p \xrightarrow{a_i} q\}$$

Output: “yes” if $R_{\mathfrak{A}}(w) \cap F \neq \emptyset$, otherwise “no”

Remark: this algorithm solves the **word problem** for NFA

Acceptance Test for NFA

Algorithm A.24 (Acceptance Test for NFA)

Input: NFA $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$, $w \in \Sigma^*$

Question: $w \in L(\mathfrak{A})$?

Procedure: Computation of the **reachability set**

$$R_{\mathfrak{A}}(w) := \{q \in Q \mid q_0 \xrightarrow{w} q\}$$

Iterative procedure for $w = a_1 \dots a_n$:

1. let $R_{\mathfrak{A}}(\varepsilon) := \{q_0\}$

2. for $i := 1, \dots, n$: let

$$R_{\mathfrak{A}}(a_1 \dots a_i) := \{q \in Q \mid \exists p \in R_{\mathfrak{A}}(a_1 \dots a_{i-1}) : p \xrightarrow{a_i} q\}$$

Output: “yes” if $R_{\mathfrak{A}}(w) \cap F \neq \emptyset$, otherwise “no”

Remark: this algorithm solves the **word problem** for NFA

Example A.25

on the board

NFA-Recognisability of Concatenation

Definition of NFA looks promising, but... (on the board)

NFA-Recognisability of Concatenation

Definition of NFA looks promising, but... (on the board)

Solution: admit empty word ε as transition label

Definition A.26

A **nondeterministic finite automaton with ε -transitions (ε -NFA)** is of the form

$\mathcal{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$ where

- Q is a finite set of **states**
- Σ denotes the **input alphabet**
- $\Delta \subseteq Q \times \Sigma_\varepsilon \times Q$ is the **transition relation** where $\Sigma_\varepsilon := \Sigma \cup \{\varepsilon\}$
- $q_0 \in Q$ is the **initial state**
- $F \subseteq Q$ is the set of **final states**

Remarks:

- every NFA is an ε -NFA
- definitions of runs and acceptance: in analogy to NFA

Definition A.26

A **nondeterministic finite automaton with ε -transitions (ε -NFA)** is of the form

$\mathcal{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$ where

- Q is a finite set of **states**
- Σ denotes the **input alphabet**
- $\Delta \subseteq Q \times \Sigma_\varepsilon \times Q$ is the **transition relation** where $\Sigma_\varepsilon := \Sigma \cup \{\varepsilon\}$
- $q_0 \in Q$ is the **initial state**
- $F \subseteq Q$ is the set of **final states**

Remarks:

- every NFA is an ε -NFA
- definitions of runs and acceptance: in analogy to NFA

Example A.27

on the board

Concatenation and Iteration via ε -NFA

Theorem A.28

If $L_1, L_2 \subseteq \Sigma^$ are ε -NFA-recognisable, then so is $L_1 \cdot L_2$.*

Concatenation and Iteration via ε -NFA

Theorem A.28

If $L_1, L_2 \subseteq \Sigma^$ are ε -NFA-recognisable, then so is $L_1 \cdot L_2$.*

Proof (idea).

on the board



Concatenation and Iteration via ε -NFA

Theorem A.28

If $L_1, L_2 \subseteq \Sigma^$ are ε -NFA-recognisable, then so is $L_1 \cdot L_2$.*

Proof (idea).

on the board □

Theorem A.29

If $L \subseteq \Sigma^$ is ε -NFA-recognisable, then so is L^* .*

Proof.

see Theorem A.46 □

Types of Finite Automata

1. DFA (Definition A.8)
2. NFA (Definition A.22)
3. ε -NFA (Definition A.26)

Types of Finite Automata

1. DFA (Definition A.8)
2. NFA (Definition A.22)
3. ε -NFA (Definition A.26)

From the definitions we immediately obtain:

Corollary A.30

1. *Every DFA-recognisable language is NFA-recognisable.*
2. *Every NFA-recognisable language is ε -NFA-recognisable.*

Types of Finite Automata

1. DFA (Definition A.8)
2. NFA (Definition A.22)
3. ε -NFA (Definition A.26)

From the definitions we immediately obtain:

Corollary A.30

1. *Every DFA-recognisable language is NFA-recognisable.*
2. *Every NFA-recognisable language is ε -NFA-recognisable.*

Goal: establish reverse inclusions

Theorem A.31

Every NFA can be transformed into an equivalent DFA.

From NFA to DFA I

Theorem A.31

Every NFA can be transformed into an equivalent DFA.

Proof.

Idea: let the DFA operate on **sets of states** (“powerset construction”)

- Initial state of DFA := {initial state of NFA}
- $P \xrightarrow{a} P'$ in DFA iff there exist $q \in P, q' \in P'$ such that $q \xrightarrow{a} q'$ in NFA
- P final state in DFA iff it contains some final state of NFA



From NFA to DFA II

Proof (continued).

Let $\mathcal{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$ a NFA. **Powerset construction** of $\mathcal{A}' = \langle Q', \Sigma, \delta', q'_0, F' \rangle$:

- $Q' := 2^Q := \{P \mid P \subseteq Q\}$
- $\delta' : Q' \times \Sigma \rightarrow Q'$ with $q \in \delta'(P, a) \iff$ there exists $p \in P$ such that $(p, a, q) \in \Delta$
- $q'_0 := \{q_0\}$
- $F' := \{P \subseteq Q \mid P \cap F \neq \emptyset\}$

From NFA to DFA II

Proof (continued).

Let $\mathcal{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$ a NFA. **Powerset construction** of $\mathcal{A}' = \langle Q', \Sigma, \delta', q'_0, F' \rangle$:

- $Q' := 2^Q := \{P \mid P \subseteq Q\}$
- $\delta' : Q' \times \Sigma \rightarrow Q'$ with $q \in \delta'(P, a) \iff$ there exists $p \in P$ such that $(p, a, q) \in \Delta$
- $q'_0 := \{q_0\}$
- $F' := \{P \subseteq Q \mid P \cap F \neq \emptyset\}$

This yields

$$q_0 \xrightarrow{w} q \text{ in } \mathcal{A} \iff q \in \delta'^*(\{q_0\}, w) \text{ in } \mathcal{A}'$$

and thus

$$\mathcal{A} \text{ accepts } w \iff \mathcal{A}' \text{ accepts } w$$

From NFA to DFA II

Proof (continued).

Let $\mathcal{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$ a NFA. **Powerset construction** of $\mathcal{A}' = \langle Q', \Sigma, \delta', q'_0, F' \rangle$:

- $Q' := 2^Q := \{P \mid P \subseteq Q\}$
- $\delta' : Q' \times \Sigma \rightarrow Q'$ with $q \in \delta'(P, a) \iff$ there exists $p \in P$ such that $(p, a, q) \in \Delta$
- $q'_0 := \{q_0\}$
- $F' := \{P \subseteq Q \mid P \cap F \neq \emptyset\}$

This yields

$$q_0 \xrightarrow{w} q \text{ in } \mathcal{A} \iff q \in \delta'^*(\{q_0\}, w) \text{ in } \mathcal{A}'$$

and thus

$$\mathcal{A} \text{ accepts } w \iff \mathcal{A}' \text{ accepts } w$$

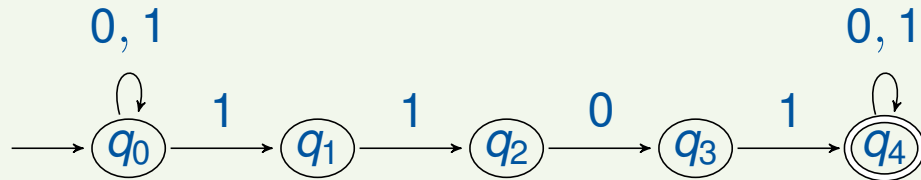
(**Remark:** only **reachable** subsets of Q need to be considered.)



From NFA to DFA III

Example A.32

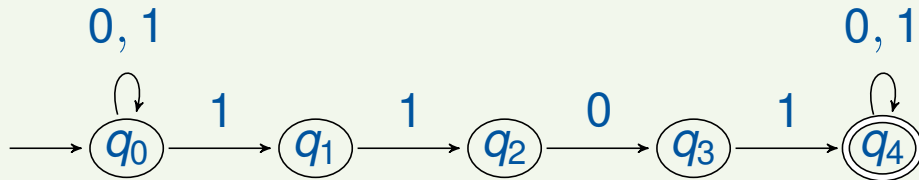
NFA:



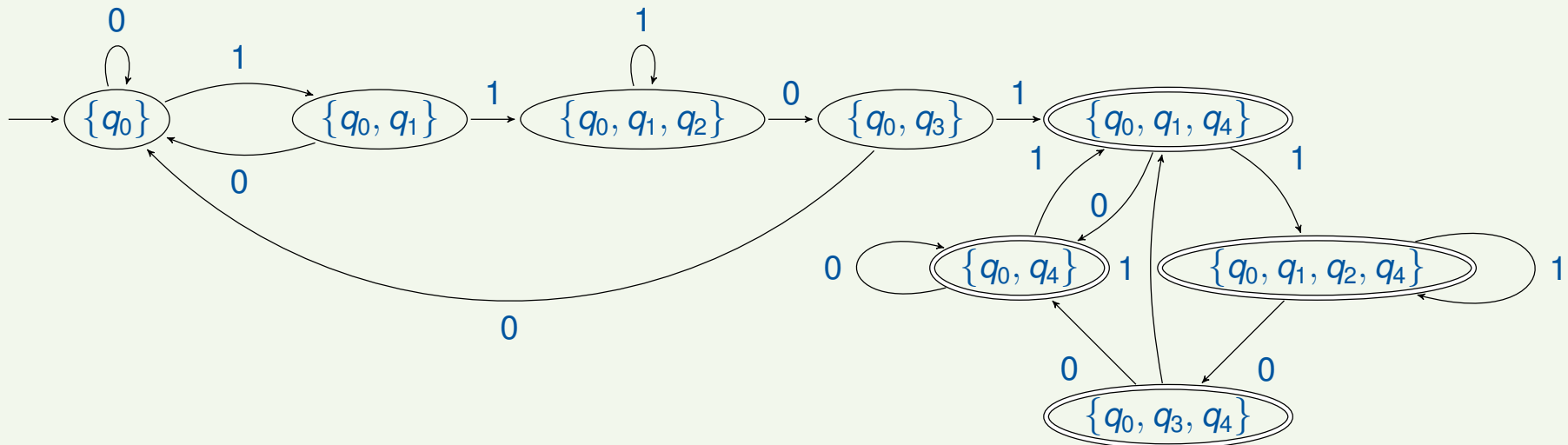
From NFA to DFA III

Example A.32

NFA:



Corresponding DFA:



Theorem A.33

Every ε -NFA can be transformed into an equivalent NFA.

From ε -NFA to NFA I

Theorem A.33

Every ε -NFA can be transformed into an equivalent NFA.

Proof (idea).

Let $\mathcal{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$ be a ε -NFA. We construct the NFA \mathcal{A}' by eliminating all ε -transitions, adding appropriate direct transitions: if $p \xrightarrow{\varepsilon}^* q$, $q \xrightarrow{a} q'$, and $q' \xrightarrow{\varepsilon}^* r$ in \mathcal{A} , then $p \xrightarrow{a} r$ in \mathcal{A}' . Moreover $F' := F \cup \{q_0\}$ if $q_0 \xrightarrow{\varepsilon}^* q \in F$ in \mathcal{A} , and $F' := F$ otherwise. □

From ε -NFA to NFA I

Theorem A.33

Every ε -NFA can be transformed into an equivalent NFA.

Proof (idea).

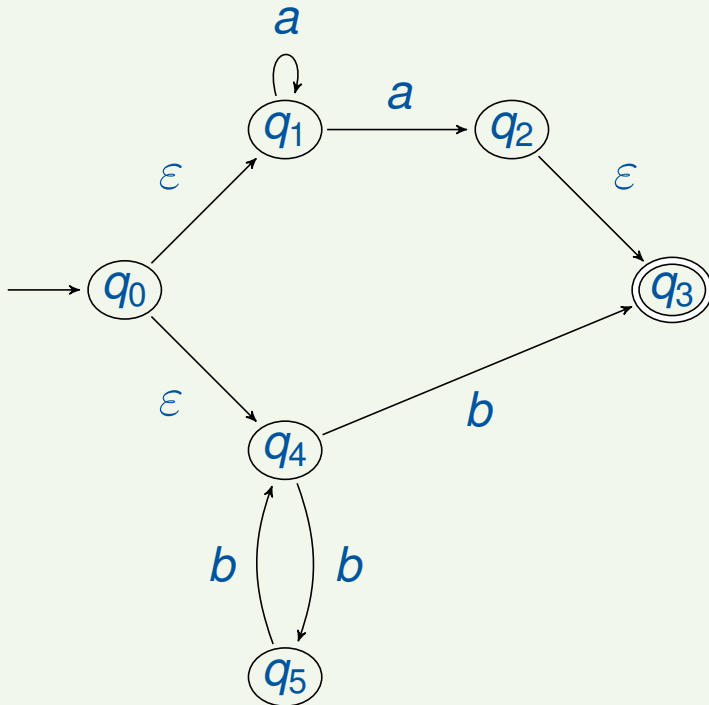
Let $\mathcal{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$ be a ε -NFA. We construct the NFA \mathcal{A}' by eliminating all ε -transitions, adding appropriate direct transitions: if $p \xrightarrow{\varepsilon^*} q$, $q \xrightarrow{a} q'$, and $q' \xrightarrow{\varepsilon^*} r$ in \mathcal{A} , then $p \xrightarrow{a} r$ in \mathcal{A}' . Moreover $F' := F \cup \{q_0\}$ if $q_0 \xrightarrow{\varepsilon^*} q \in F$ in \mathcal{A} , and $F' := F$ otherwise. □

Corollary A.34

All types of finite automata recognise the same class of languages.

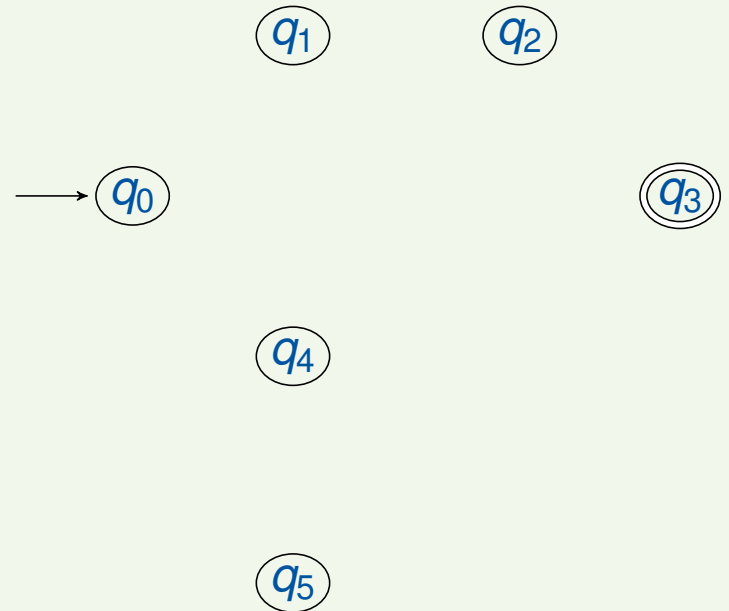
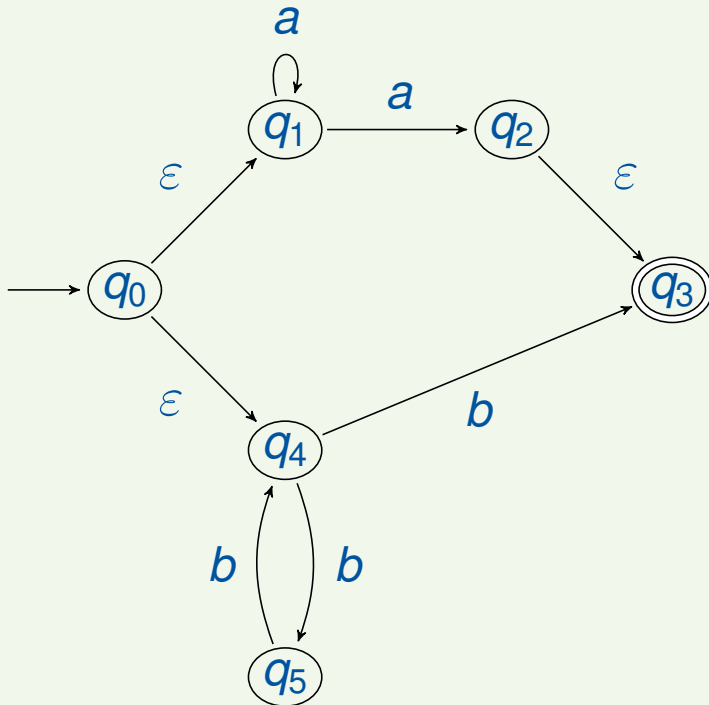
From ε -NFA to NFA II

Example A.35



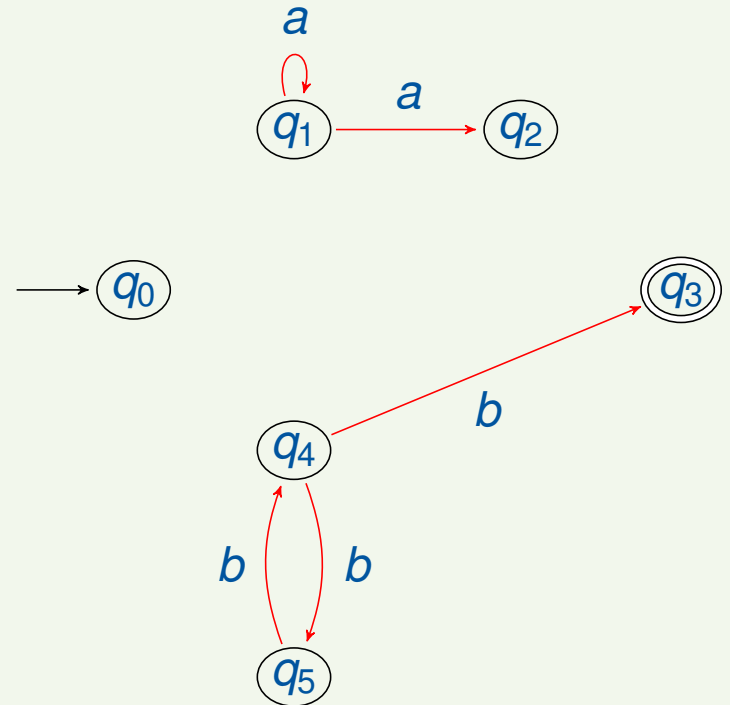
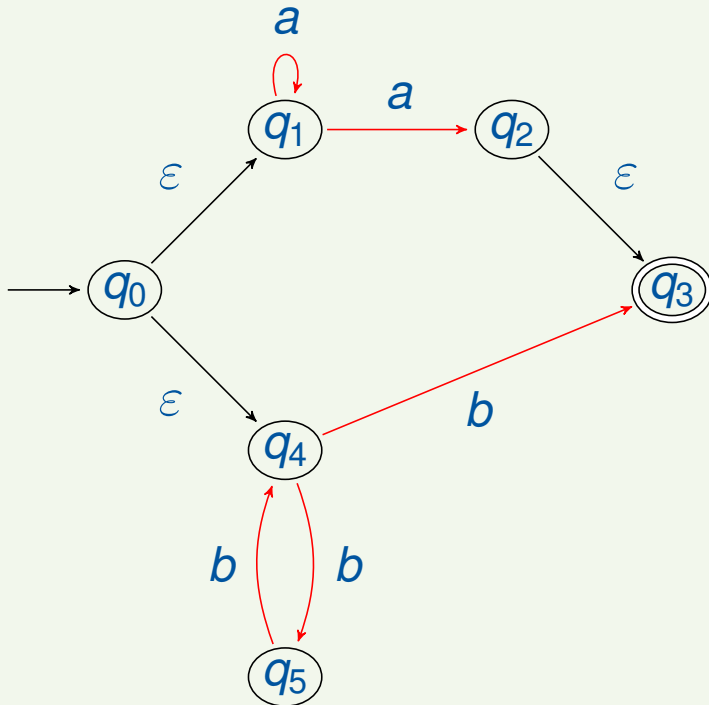
From ε -NFA to NFA II

Example A.35



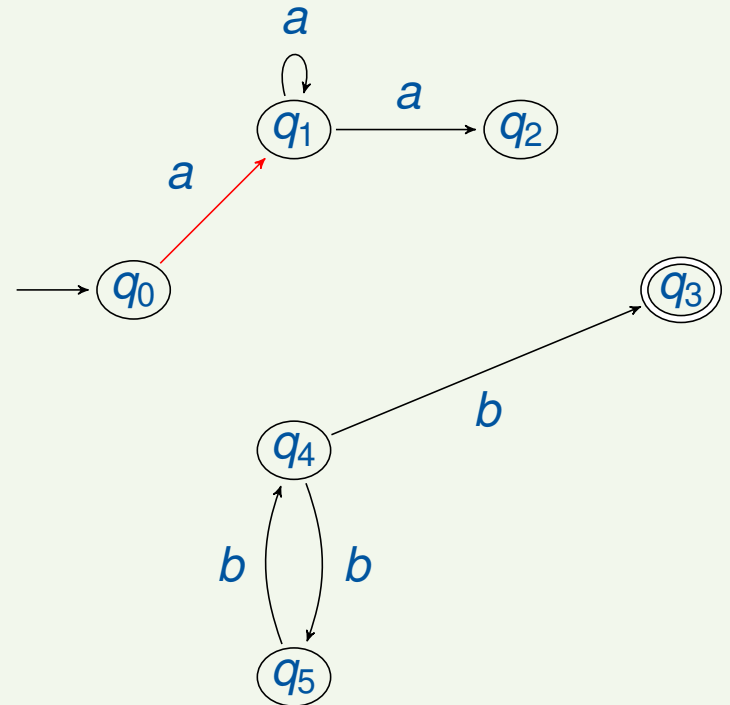
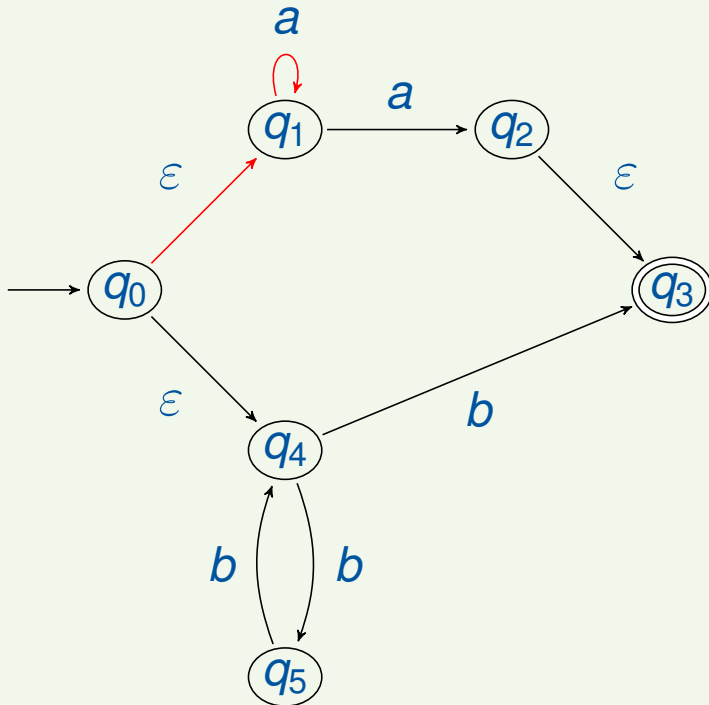
From ε -NFA to NFA II

Example A.35



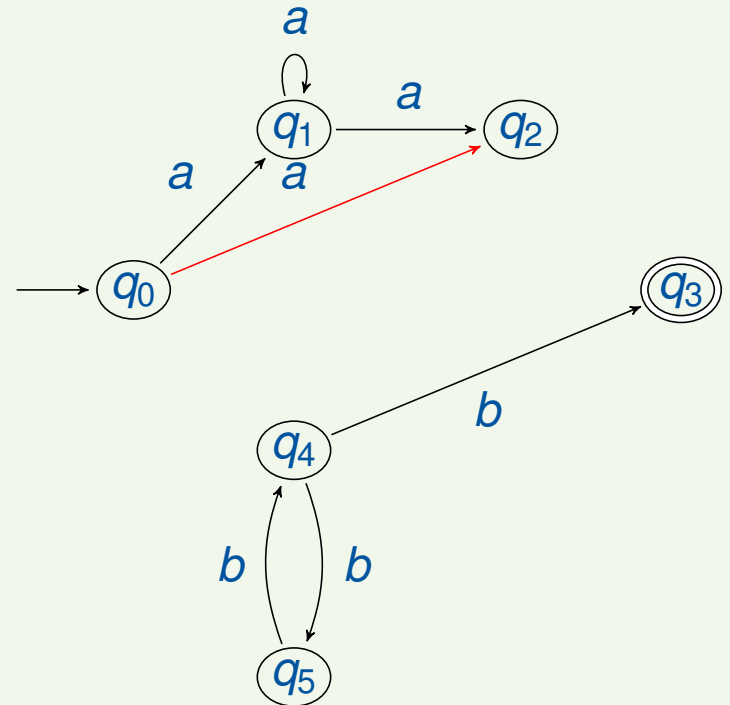
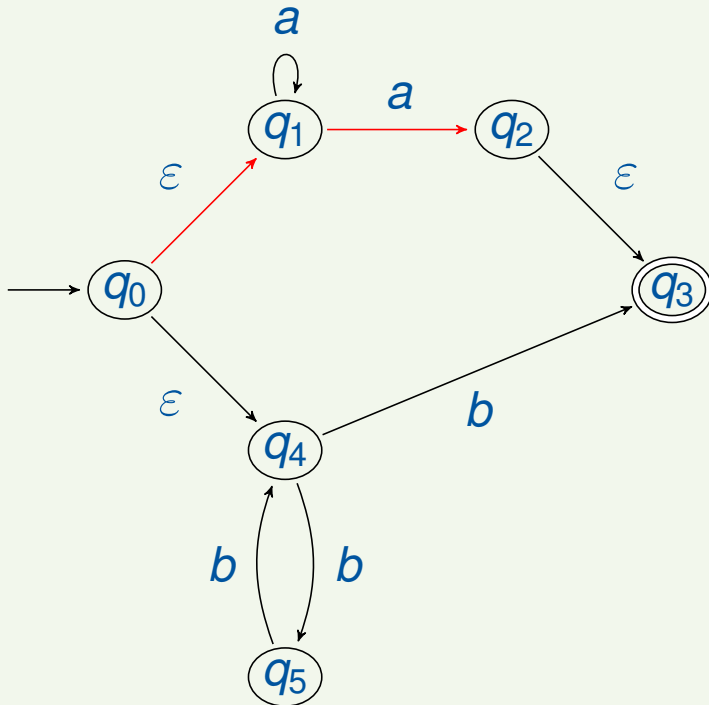
From ε -NFA to NFA II

Example A.35



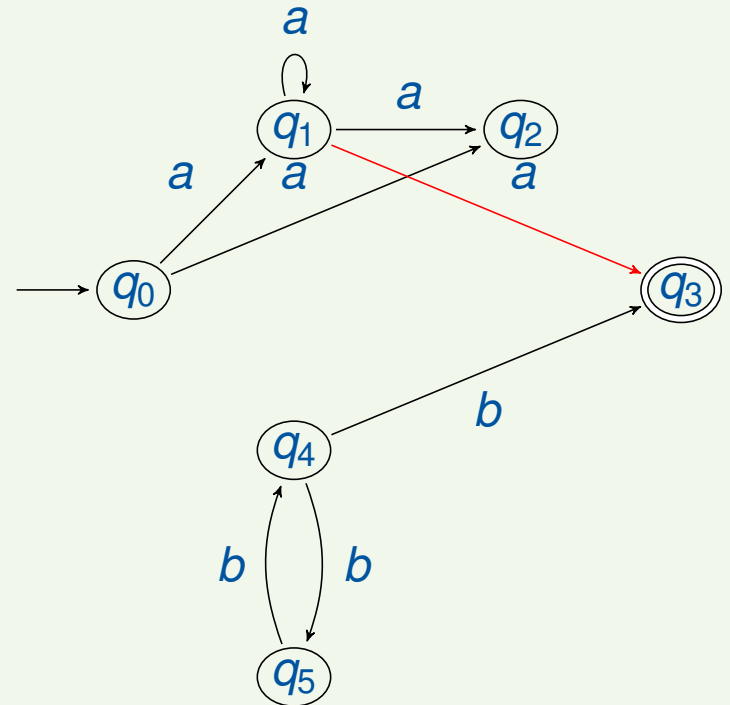
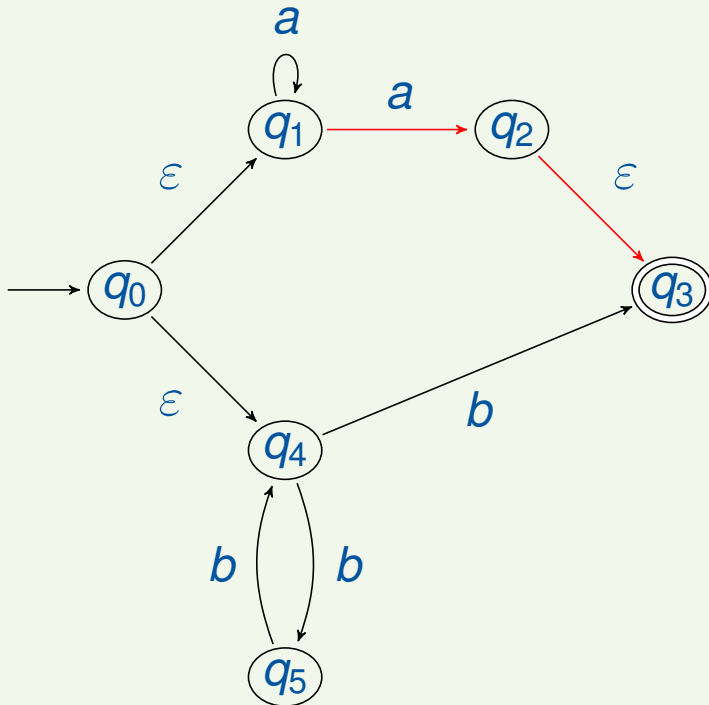
From ε -NFA to NFA II

Example A.35



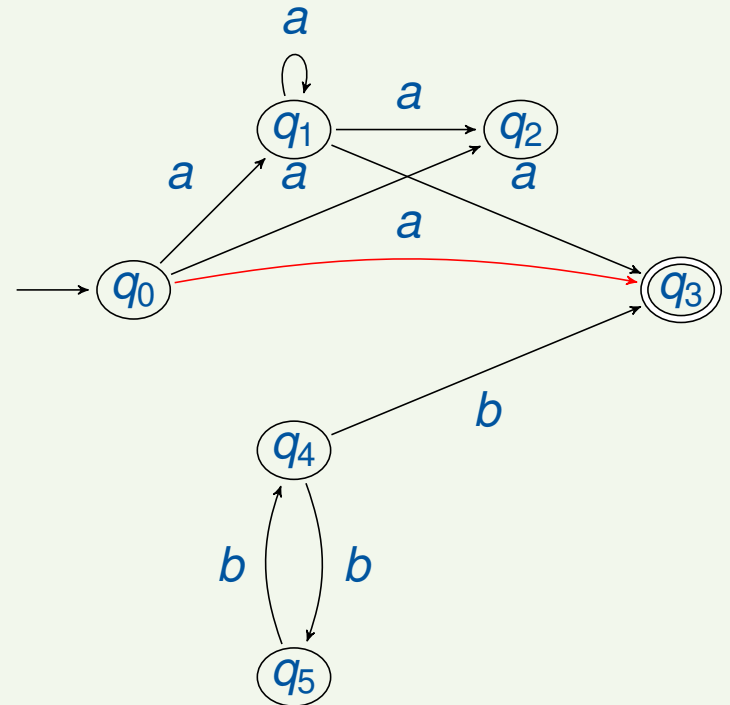
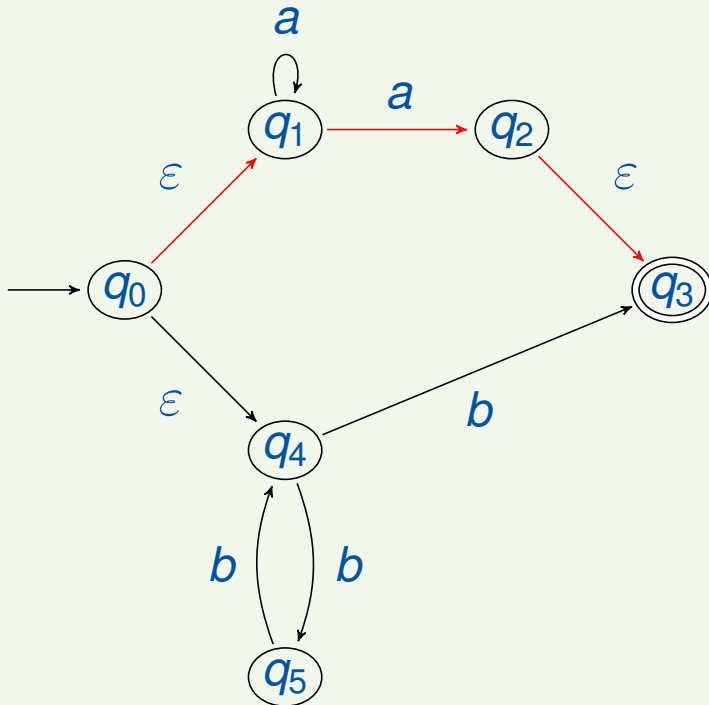
From ε -NFA to NFA II

Example A.35



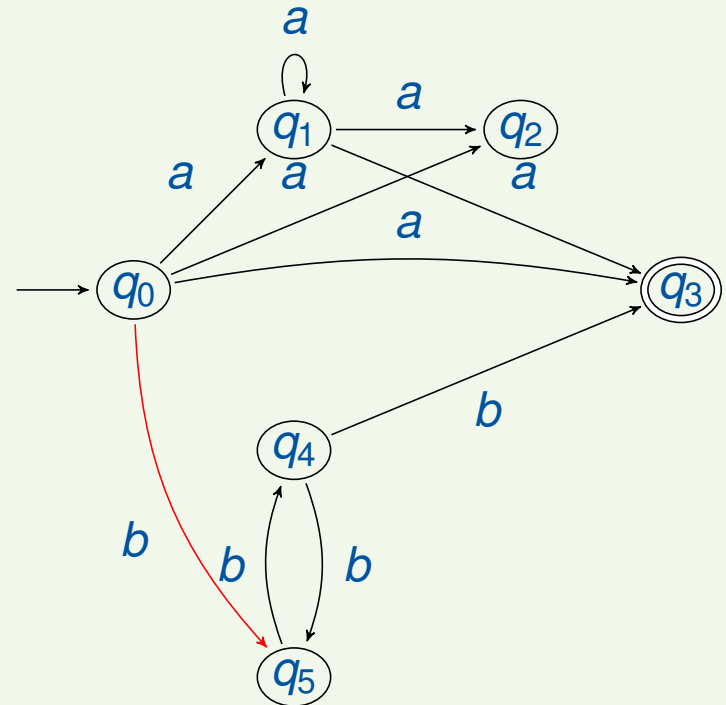
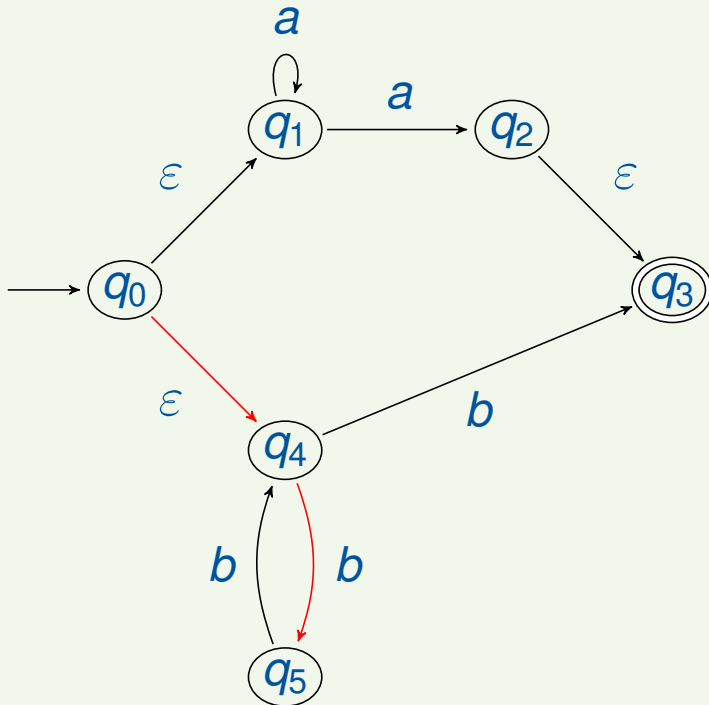
From ε -NFA to NFA II

Example A.35



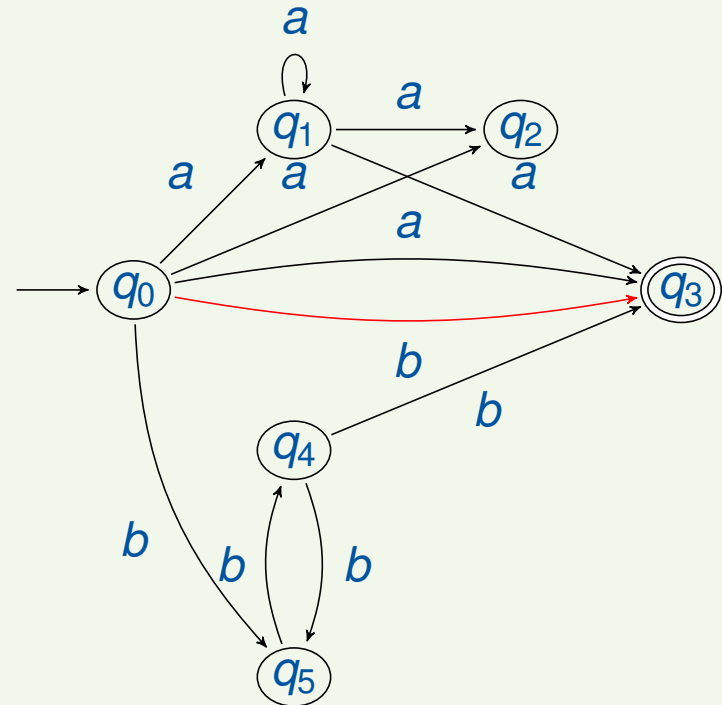
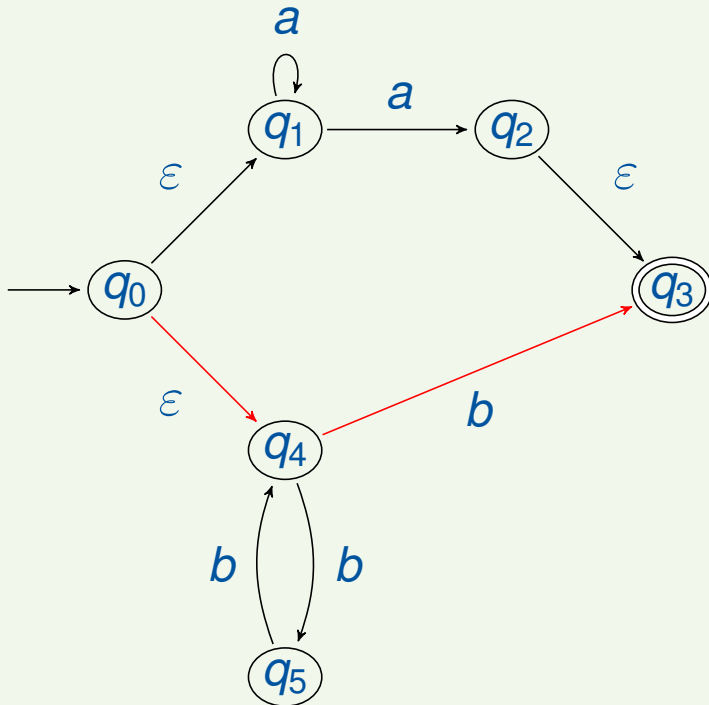
From ε -NFA to NFA II

Example A.35



From ε -NFA to NFA II

Example A.35



Seen:

- Definition of ε -NFA
- Determinisation of (ε -)NFA

Nondeterministic Finite Automata

Seen:

- Definition of ε -NFA
- Determinisation of (ε -)NFA

Next:

- More decidability results

Outline of Part A

Formal Languages

Finite Automata

Deterministic Finite Automata

Operations on Languages and Automata

Nondeterministic Finite Automata

More Decidability Results

Regular Expressions

Definition

Equivalence of Regular Expressions and Finite Automata

Minimisation of Deterministic Finite Automata

Outlook

The Word Problem Revisited

Definition A.36

The **word problem for DFA** is specified as follows:

Given a DFA \mathcal{A} and a word $w \in \Sigma^*$, decide whether

$$w \in L(\mathcal{A}).$$

The Word Problem Revisited

Definition A.36

The **word problem for DFA** is specified as follows:

Given a DFA \mathcal{A} and a word $w \in \Sigma^*$, decide whether

$$w \in L(\mathcal{A}).$$

As we have seen (Def. A.10, Alg. A.24, Thm. A.33):

Theorem A.37

*The word problem for DFA (NFA, ε -NFA) is **decidable**.*

The Emptiness Problem

Definition A.38

The **emptiness problem for DFA** is specified as follows:
Given a DFA \mathcal{A} , decide whether $L(\mathcal{A}) = \emptyset$.

The Emptiness Problem

Definition A.38

The **emptiness problem for DFA** is specified as follows:
Given a DFA \mathcal{A} , decide whether $L(\mathcal{A}) = \emptyset$.

Remark: important result for formal verification (unreachability of bad [= final] states)

The Emptiness Problem

Definition A.38

The **emptiness problem for DFA** is specified as follows:
Given a DFA \mathcal{A} , decide whether $L(\mathcal{A}) = \emptyset$.

Remark: important result for formal verification (unreachability of bad [= final] states)

Theorem A.39

*The emptiness problem for DFA (NFA, ε -NFA) is **decidable**.*

The Emptiness Problem

Definition A.38

The **emptiness problem for DFA** is specified as follows:
Given a DFA \mathcal{A} , decide whether $L(\mathcal{A}) = \emptyset$.

Remark: important result for formal verification (unreachability of bad [= final] states)

Theorem A.39

*The emptiness problem for DFA (NFA, ε -NFA) is **decidable**.*

Proof.

It holds that $L(\mathcal{A}) \neq \emptyset$ iff in \mathcal{A} some final state is reachable from the initial state (simple graph-theoretic problem). □

The Equivalence Problem

Definition A.40

The **equivalence problem for DFA** is specified as follows:
Given two DFA $\mathcal{A}_1, \mathcal{A}_2$, decide whether $L(\mathcal{A}_1) = L(\mathcal{A}_2)$.

The Equivalence Problem

Definition A.40

The **equivalence problem for DFA** is specified as follows:
Given two DFA $\mathcal{A}_1, \mathcal{A}_2$, decide whether $L(\mathcal{A}_1) = L(\mathcal{A}_2)$.

Theorem A.41

*The equivalence problem for DFA (NFA, ε -NFA) is **decidable**.*

The Equivalence Problem

Definition A.40

The **equivalence problem for DFA** is specified as follows:
Given two DFA $\mathfrak{A}_1, \mathfrak{A}_2$, decide whether $L(\mathfrak{A}_1) = L(\mathfrak{A}_2)$.

Theorem A.41

*The equivalence problem for DFA (NFA, ε -NFA) is **decidable**.*

Proof.

$$L(\mathfrak{A}_1) = L(\mathfrak{A}_2)$$

The Equivalence Problem

Definition A.40

The **equivalence problem for DFA** is specified as follows:
Given two DFA $\mathfrak{A}_1, \mathfrak{A}_2$, decide whether $L(\mathfrak{A}_1) = L(\mathfrak{A}_2)$.

Theorem A.41

*The equivalence problem for DFA (NFA, ε -NFA) is **decidable**.*

Proof.

$$\begin{aligned} & L(\mathfrak{A}_1) = L(\mathfrak{A}_2) \\ \iff & L(\mathfrak{A}_1) \subseteq L(\mathfrak{A}_2) \text{ and } L(\mathfrak{A}_2) \subseteq L(\mathfrak{A}_1) \end{aligned}$$

The Equivalence Problem

Definition A.40

The **equivalence problem for DFA** is specified as follows:
Given two DFA $\mathfrak{A}_1, \mathfrak{A}_2$, decide whether $L(\mathfrak{A}_1) = L(\mathfrak{A}_2)$.

Theorem A.41

*The equivalence problem for DFA (NFA, ε -NFA) is **decidable**.*

Proof.

$$\begin{aligned} & L(\mathfrak{A}_1) = L(\mathfrak{A}_2) \\ \iff & L(\mathfrak{A}_1) \subseteq L(\mathfrak{A}_2) \text{ and } L(\mathfrak{A}_2) \subseteq L(\mathfrak{A}_1) \\ \iff & (L(\mathfrak{A}_1) \setminus L(\mathfrak{A}_2)) = \emptyset \text{ and } (L(\mathfrak{A}_2) \setminus L(\mathfrak{A}_1)) = \emptyset \end{aligned}$$

The Equivalence Problem

Definition A.40

The **equivalence problem for DFA** is specified as follows:
Given two DFA $\mathfrak{A}_1, \mathfrak{A}_2$, decide whether $L(\mathfrak{A}_1) = L(\mathfrak{A}_2)$.

Theorem A.41

*The equivalence problem for DFA (NFA, ε -NFA) is **decidable**.*

Proof.

$$\begin{aligned} & L(\mathfrak{A}_1) = L(\mathfrak{A}_2) \\ \iff & L(\mathfrak{A}_1) \subseteq L(\mathfrak{A}_2) \text{ and } L(\mathfrak{A}_2) \subseteq L(\mathfrak{A}_1) \\ \iff & (L(\mathfrak{A}_1) \setminus L(\mathfrak{A}_2)) = \emptyset \text{ and } (L(\mathfrak{A}_2) \setminus L(\mathfrak{A}_1)) = \emptyset \\ \iff & (L(\mathfrak{A}_1) \setminus L(\mathfrak{A}_2)) \cup (L(\mathfrak{A}_2) \setminus L(\mathfrak{A}_1)) = \emptyset \end{aligned}$$

The Equivalence Problem

Definition A.40

The **equivalence problem for DFA** is specified as follows:
Given two DFA $\mathfrak{A}_1, \mathfrak{A}_2$, decide whether $L(\mathfrak{A}_1) = L(\mathfrak{A}_2)$.

Theorem A.41

*The equivalence problem for DFA (NFA, ε -NFA) is **decidable**.*

Proof.

$$\begin{aligned} & L(\mathfrak{A}_1) = L(\mathfrak{A}_2) \\ \iff & L(\mathfrak{A}_1) \subseteq L(\mathfrak{A}_2) \text{ and } L(\mathfrak{A}_2) \subseteq L(\mathfrak{A}_1) \\ \iff & (L(\mathfrak{A}_1) \setminus L(\mathfrak{A}_2)) = \emptyset \text{ and } (L(\mathfrak{A}_2) \setminus L(\mathfrak{A}_1)) = \emptyset \\ \iff & (L(\mathfrak{A}_1) \setminus L(\mathfrak{A}_2)) \cup (L(\mathfrak{A}_2) \setminus L(\mathfrak{A}_1)) = \emptyset \\ \iff & \underbrace{(L(\mathfrak{A}_1) \cap \underbrace{\overline{L(\mathfrak{A}_2)}}_{\text{DFA-recognisable (Thm. A.14)}})}_{\text{DFA-recognisable (Thm. A.16)}} \cup \underbrace{(L(\mathfrak{A}_2) \cap \underbrace{\overline{L(\mathfrak{A}_1)}}_{\text{DFA-recognisable (Thm. A.14)}})}_{\text{DFA-recognisable (Thm. A.16)}} = \emptyset \\ & \underbrace{\hspace{15em}}_{\text{DFA-recognisable (Thm. A.18)}} \\ & \underbrace{\hspace{15em}}_{\text{decidable (Thm. A.39)}} \end{aligned}$$

Seen:

- Decidability of word problem
- Decidability of emptiness problem
- Decidability of equivalence problem

Seen:

- Decidability of word problem
- Decidability of emptiness problem
- Decidability of equivalence problem

Next:

- Non-algorithmic description of languages

Outline of Part A

Formal Languages

Finite Automata

- Deterministic Finite Automata

- Operations on Languages and Automata

- Nondeterministic Finite Automata

- More Decidability Results

Regular Expressions

- Definition

- Equivalence of Regular Expressions and Finite Automata

Minimisation of Deterministic Finite Automata

Outlook

Outline of Part A

Formal Languages

Finite Automata

Deterministic Finite Automata

Operations on Languages and Automata

Nondeterministic Finite Automata

More Decidability Results

Regular Expressions

Definition

Equivalence of Regular Expressions and Finite Automata

Minimisation of Deterministic Finite Automata

Outlook

An Example

Example A.42

Consider the set of all words over $\Sigma := \{a, b\}$ which

1. start with one or three a symbols
2. continue with a (potentially empty) sequence of blocks, each containing at least one b and exactly two a 's
3. conclude with a (potentially empty) sequence of b 's

An Example

Example A.42

Consider the set of all words over $\Sigma := \{a, b\}$ which

1. start with one or three a symbols
2. continue with a (potentially empty) sequence of blocks, each containing at least one b and exactly two a 's
3. conclude with a (potentially empty) sequence of b 's

Corresponding **regular expression**:

$$\underbrace{(a \mid aaa)}_{(1)} \underbrace{(\underbrace{bb^*ab^*ab^*}_{b \text{ before } a\text{'s}} \mid \underbrace{b^*abb^*ab^*}_{b \text{ between } a\text{'s}} \mid \underbrace{b^*ab^*abb^*}_{b \text{ after } a\text{'s}})^*}_{(2)} \underbrace{b^*}_{(3)}$$

Syntax of Regular Expressions

Definition A.43

The set of **regular expressions** over Σ is inductively defined by:

- \emptyset and ε are regular expressions
- every $a \in \Sigma$ is a regular expression
- if α and β are regular expressions, then so are
 - $\alpha \mid \beta$
 - $\alpha \cdot \beta$
 - α^*

Syntax of Regular Expressions

Definition A.43

The set of **regular expressions** over Σ is inductively defined by:

- \emptyset and ε are regular expressions
- every $a \in \Sigma$ is a regular expression
- if α and β are regular expressions, then so are
 - $\alpha \mid \beta$
 - $\alpha \cdot \beta$
 - α^*

Notation:

- \cdot can be omitted
- $*$ binds stronger than \cdot , \cdot binds stronger than \mid
 - thus: $a \mid bc^* := a \mid (b \cdot (c^*))$
- α^+ abbreviates $\alpha \cdot \alpha^*$

Definition A.44

Every regular expression α defines a language $L(\alpha)$:

$$L(\emptyset) := \emptyset$$

$$L(\varepsilon) := \{\varepsilon\}$$

$$L(a) := \{a\}$$

$$L(\alpha \mid \beta) := L(\alpha) \cup L(\beta)$$

$$L(\alpha \cdot \beta) := L(\alpha) \cdot L(\beta)$$

$$L(\alpha^*) := (L(\alpha))^*$$

Semantics of Regular Expressions

Definition A.44

Every regular expression α defines a language $L(\alpha)$:

$$L(\emptyset) := \emptyset$$

$$L(\varepsilon) := \{\varepsilon\}$$

$$L(a) := \{a\}$$

$$L(\alpha \mid \beta) := L(\alpha) \cup L(\beta)$$

$$L(\alpha \cdot \beta) := L(\alpha) \cdot L(\beta)$$

$$L(\alpha^*) := (L(\alpha))^*$$

A language L is called **regular** if it is definable by a regular expression, i.e., if $L = L(\alpha)$ for some regular expression α .

Example A.45

1. $\{aa\}$ is regular since

$$L(a \cdot a) = L(a) \cdot L(a) = \{a\} \cdot \{a\} = \{aa\}$$

Example A.45

1. $\{aa\}$ is regular since

$$L(a \cdot a) = L(a) \cdot L(a) = \{a\} \cdot \{a\} = \{aa\}$$

2. $\{a, b\}^*$ is regular since

$$L((a \mid b)^*) = (L(a \mid b))^* = (L(a) \cup L(b))^* = (\{a\} \cup \{b\})^* = \{a, b\}^*$$

Example A.45

1. $\{aa\}$ is regular since

$$L(a \cdot a) = L(a) \cdot L(a) = \{a\} \cdot \{a\} = \{aa\}$$

2. $\{a, b\}^*$ is regular since

$$L((a \mid b)^*) = (L(a \mid b))^* = (L(a) \cup L(b))^* = (\{a\} \cup \{b\})^* = \{a, b\}^*$$

3. The set of all words over $\{a, b\}$ containing abb is regular since

$$L((a \mid b)^* \cdot a \cdot b \cdot b \cdot (a \mid b)^*) = \{a, b\}^* \cdot \{abb\} \cdot \{a, b\}^*$$

Outline of Part A

Formal Languages

Finite Automata

Deterministic Finite Automata

Operations on Languages and Automata

Nondeterministic Finite Automata

More Decidability Results

Regular Expressions

Definition

Equivalence of Regular Expressions and Finite Automata

Minimisation of Deterministic Finite Automata

Outlook

Regular Languages and Finite Automata I

Theorem A.46 (Kleene's Theorem)

To each regular expression there corresponds an ε -NFA, and vice versa.

Regular Languages and Finite Automata I

Theorem A.46 (Kleene's Theorem)

To each regular expression there corresponds an ε -NFA, and vice versa.

Proof.

\Rightarrow : by induction over the given regular expression α , we construct an ε -NFA \mathcal{A}_α with exactly one final state q_f and without transitions into the initial/leaving the final state:

Regular Languages and Finite Automata I

Theorem A.46 (Kleene's Theorem)

To each regular expression there corresponds an ε -NFA, and vice versa.

Proof.

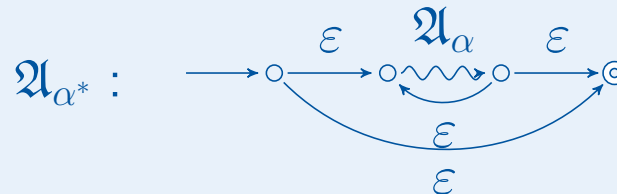
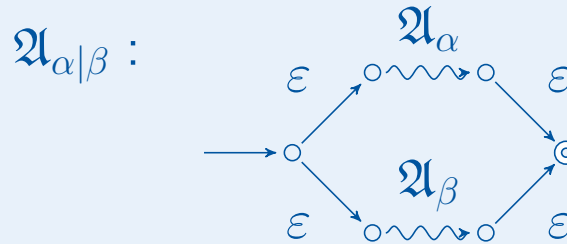
\Rightarrow : by induction over the given regular expression α , we construct an ε -NFA \mathcal{A}_α with exactly one final state q_f and without transitions into the initial/leaving the final state:

$$\mathcal{A}_\emptyset : \quad \longrightarrow \circ \quad \odot$$

$$\mathcal{A}_\varepsilon : \quad \longrightarrow \odot$$

$$\mathcal{A}_a : \quad \longrightarrow \circ \xrightarrow{a} \odot$$

$$\mathcal{A}_{\alpha \cdot \beta} : \quad \longrightarrow \circ \xrightarrow{\mathcal{A}_\alpha} \circ \xrightarrow{\varepsilon} \circ \xrightarrow{\mathcal{A}_\beta} \odot$$



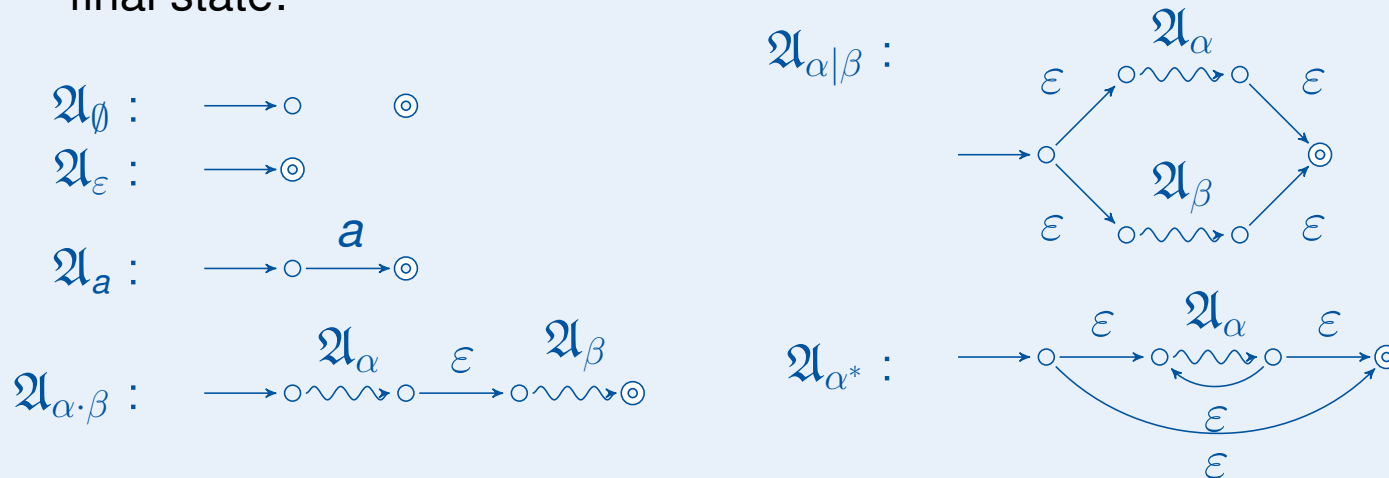
Regular Languages and Finite Automata I

Theorem A.46 (Kleene's Theorem)

To each regular expression there corresponds an ε -NFA, and vice versa.

Proof.

\Rightarrow : by induction over the given regular expression α , we construct an ε -NFA \mathcal{A}_α with exactly one final state q_f and without transitions into the initial/leaving the final state:



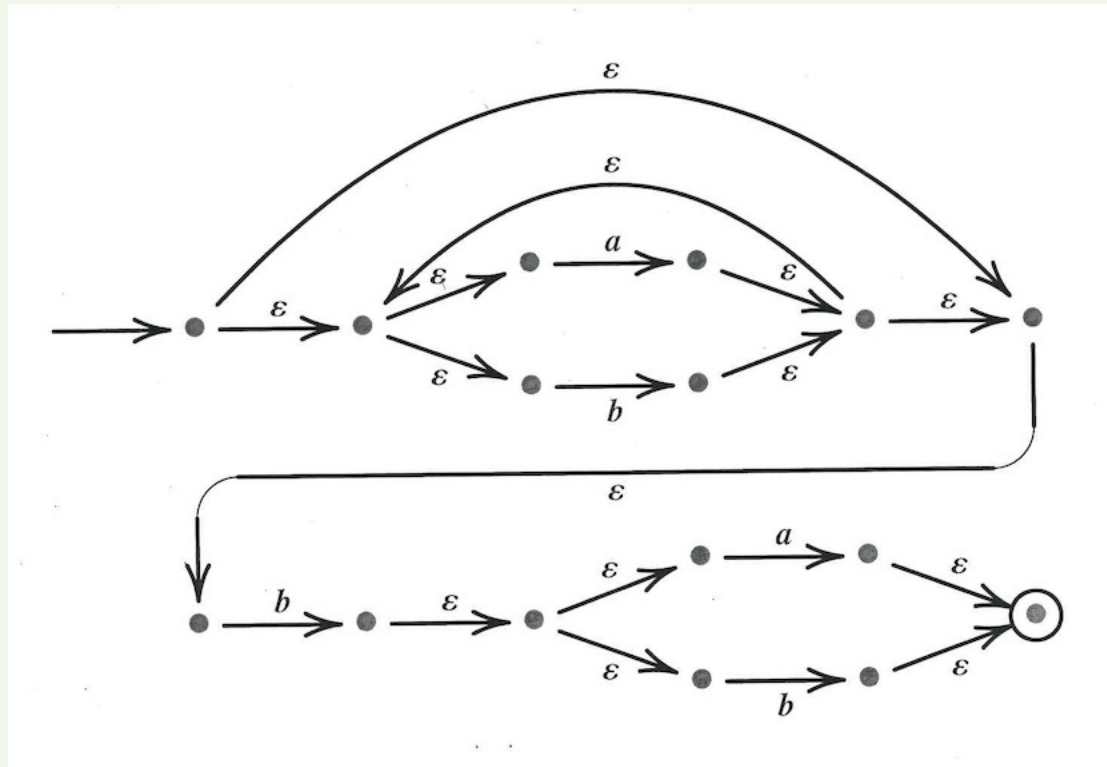
\Leftarrow : by solving a regular equation system (details omitted)



Regular Languages and Finite Automata II

Example A.47

For the regular expression $(a \mid b)^* \cdot b \cdot (a \mid b)$, we obtain the following ε -NFA:



Corollary A.48

The following properties are equivalent:

- L is regular
- L is DFA-recognisable
- L is NFA-recognisable
- L is ε -NFA-recognisable

Implementation of Pattern Matching

Algorithm A.49 (Pattern Matching)

Input: regular expression α and $w \in \Sigma^*$

Question: does w contain some $v \in L(\alpha)$?

Procedure:

1. let $\beta := (a_1 \mid \dots \mid a_n)^* \cdot \alpha$ (for $\Sigma = \{a_1, \dots, a_n\}$)
2. determine ε -NFA \mathfrak{A}_β for β
3. eliminate ε -transitions
4. apply powerset construction to obtain DFA \mathfrak{A}
5. let \mathfrak{A} run on w

Output: “yes” if \mathfrak{A} passes through some final state, otherwise “no”

Remark: in UNIX/LINUX implemented by `grep` and `lex`

Regular Expressions in UNIX (grep, flex, ...)

Syntax	Meaning
printable character	this character
\n, \t, \123, etc.	newline, tab, octal representation, etc.
.	any character except \n
[<i>Chars</i>]	one of <i>Chars</i> ; ranges possible ("0–9")
[^ <i>Chars</i>]	none of <i>Chars</i>
\\, \., \[, etc.	\, ., [, etc.
" <i>Text</i> "	<i>Text</i> without interpretation of ., [, \, etc.
^ α	α at beginning of line
α \$	α at end of line
α ?	zero or one α
α *	zero or more α
α +	one or more α
$\alpha\{n, m\}$	between n and m times α (" , m " optional)
(α)	α
$\alpha_1\alpha_2$	concatenation
$\alpha_1 \mid \alpha_2$	alternative

Regular Expressions

Seen:

- Definition of regular expressions
- Equivalence of regular and DFA-recognisable languages

Regular Expressions

Seen:

- Definition of regular expressions
- Equivalence of regular and DFA-recognisable languages

Next:

- “Optimisation” of finite automata

Outline of Part A

Formal Languages

Finite Automata

Deterministic Finite Automata

Operations on Languages and Automata

Nondeterministic Finite Automata

More Decidability Results

Regular Expressions

Definition

Equivalence of Regular Expressions and Finite Automata

Minimisation of Deterministic Finite Automata

Outlook

Motivation

Goal: space-efficient implementation of regular languages

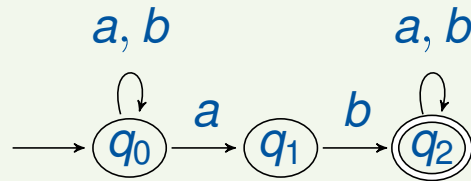
Given: DFA $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$

Wanted: DFA $\mathcal{A}_{min} = \langle Q', \Sigma, \delta', q'_0, F' \rangle$ such that $L(\mathcal{A}_{min}) = L(\mathcal{A})$ and $|Q'|$ **minimal**

State Equivalence

Example A.50

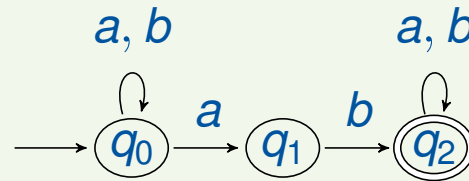
NFA for accepting $(a \mid b)^* ab(a \mid b)^*$:



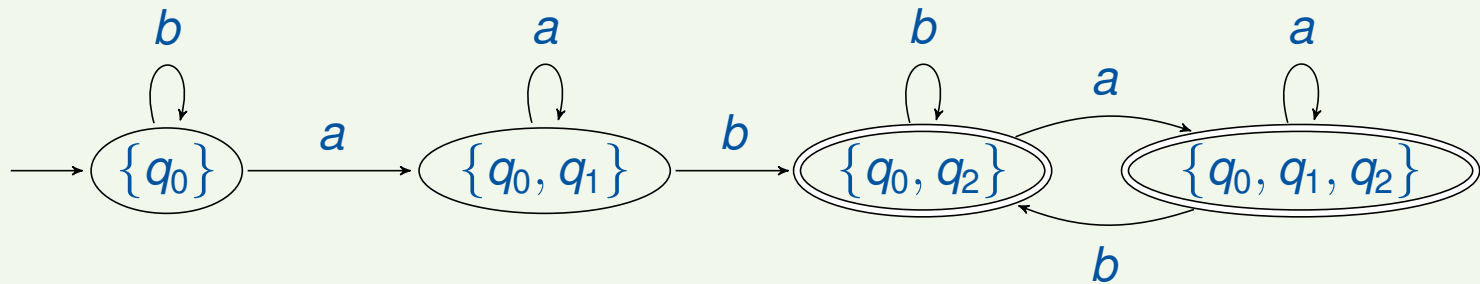
State Equivalence

Example A.50

NFA for accepting $(a \mid b)^* ab(a \mid b)^*$:



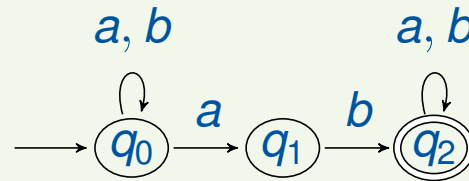
Powerset construction yields DFA \mathcal{A} :



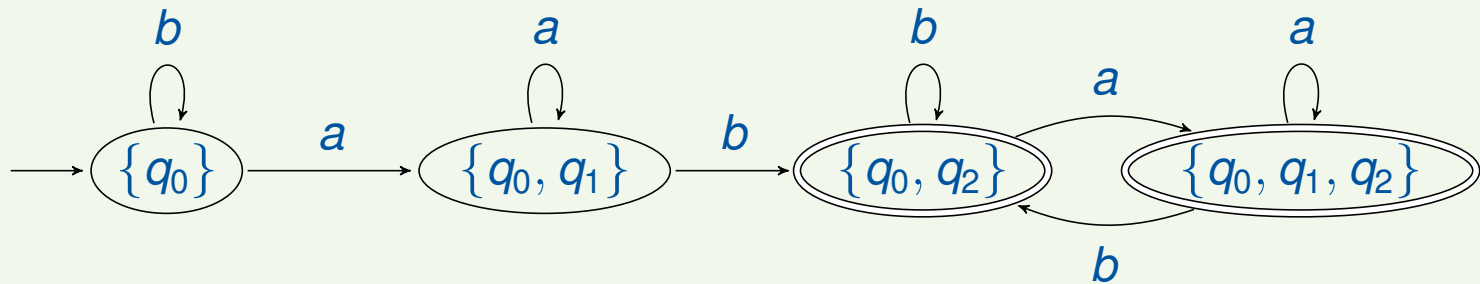
State Equivalence

Example A.50

NFA for accepting $(a \mid b)^* ab(a \mid b)^*$:



Powerset construction yields DFA \mathfrak{A} :

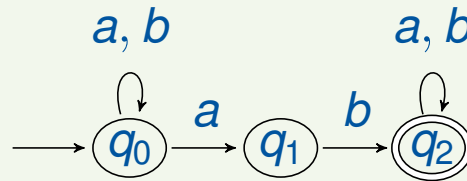


Observation: $\{q_0, q_2\}$ and $\{q_0, q_1, q_2\}$ are **equivalent** (every suffix accepted)

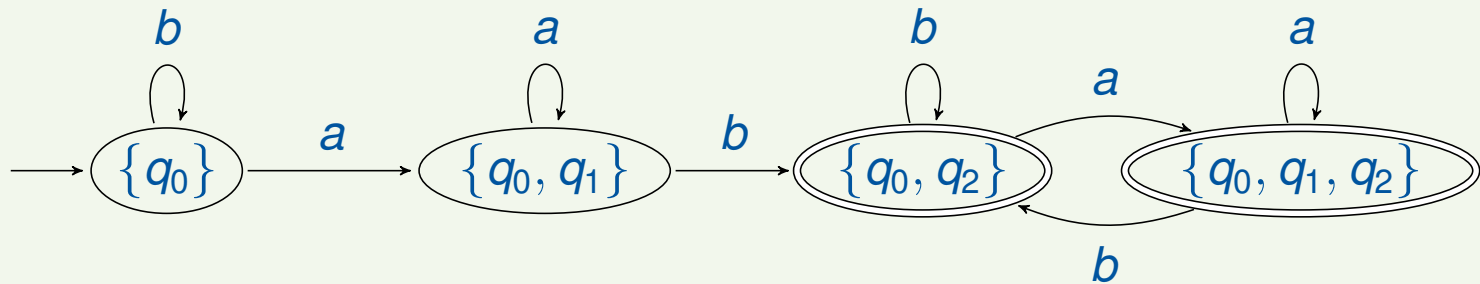
State Equivalence

Example A.50

NFA for accepting $(a \mid b)^* ab(a \mid b)^*$:



Powerset construction yields DFA \mathfrak{A} :



Observation: $\{q_0, q_2\}$ and $\{q_0, q_1, q_2\}$ are **equivalent** (every suffix accepted)

Definition A.51

Given DFA $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$, states $p, q \in Q$ are **equivalent** if

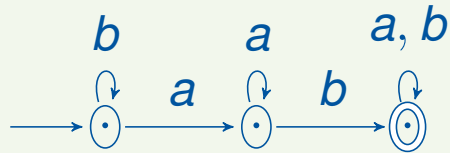
$$\forall w \in \Sigma^* : \delta^*(p, w) \in F \iff \delta^*(q, w) \in F.$$

State Merging

Minimisation: **merging** of equivalent states

Example A.52 (cf. Example A.50)

DFA after merging of $\{q_0, q_2\}$ and $\{q_0, q_1, q_2\}$:

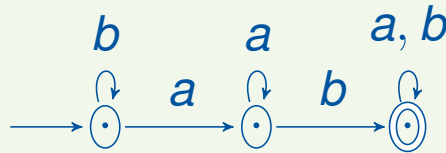


State Merging

Minimisation: **merging** of equivalent states

Example A.52 (cf. Example A.50)

DFA after merging of $\{q_0, q_2\}$ and $\{q_0, q_1, q_2\}$:



Problem: **identification** of equivalent states

Approach: iterative computation of **inequivalent** states by refinement

Corollary A.53

$p, q \in Q$ are **inequivalent** if there exists $w \in \Sigma^*$ such that
 $\delta^*(p, w) \in F$ and $\delta^*(q, w) \notin F$
(or vice versa, i.e., p and q can be distinguished by w)

Computing State (In-)Equivalence

Lemma A.54

Inductive characterisation of state inequivalence:

- $w = \varepsilon: p \in F, q \notin F \implies p, q \text{ inequivalent (by } \varepsilon)$
- $w = av: p', q' \text{ inequivalent (by } v), p \xrightarrow{a} p', q \xrightarrow{a} q' \implies p, q \text{ inequivalent (by } w)$

Computing State (In-)Equivalence

Lemma A.54

Inductive characterisation of state inequivalence:

- $w = \varepsilon: p \in F, q \notin F \implies p, q$ inequivalent (by ε)
- $w = av: p', q'$ inequivalent (by v), $p \xrightarrow{a} p', q \xrightarrow{a} q' \implies p, q$ inequivalent (by w)

Algorithm A.55 (State Equivalence for DFA)

Input: DFA $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$

Procedure: Computation of “**equivalence matrix**” over $Q \times Q$

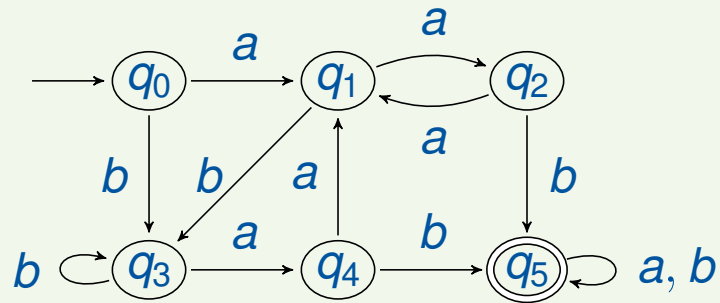
1. mark every pair (p, q) with $p \in F, q \notin F$ by ε
2. for every unmarked pair (p, q) and every $a \in \Sigma$:
if $(\delta(p, a), \delta(q, a))$ marked by v , then mark (p, q) by av
3. repeat until no change

Output: all equivalent (= unmarked) pairs of states

Minimisation Example

Example A.56

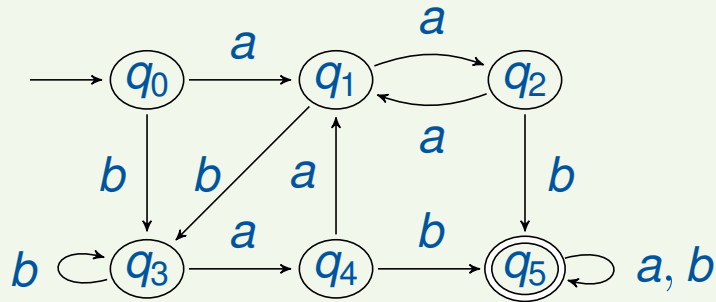
Given DFA:



Minimisation Example

Example A.56

Given DFA:



Equivalence matrix:

	q_0	q_1	q_2	q_3	q_4	q_5
q_0	X					
q_1	X	X				
q_2	X	X	X			
q_3	X	X	X	X		
q_4	X	X	X	X	X	
q_5	X	X	X	X	X	X

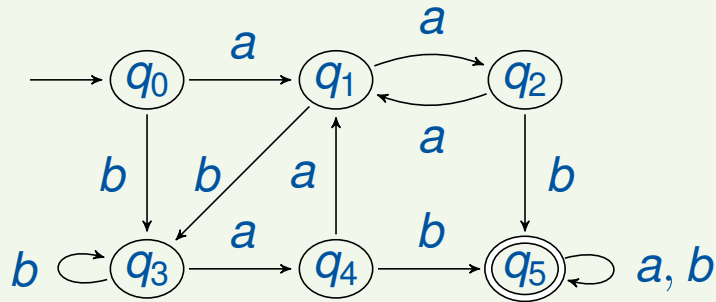
Remarks:

- entries (q_i, q_i) not needed as always equivalent
- entries (q_i, q_j) with $i > j$ not needed due to symmetry

Minimisation Example

Example A.56

Given DFA:



Equivalence matrix:

	q_0	q_1	q_2	q_3	q_4	q_5
q_0	X					ε
q_1	X	X				ε
q_2	X	X	X			ε
q_3	X	X	X	X		ε
q_4	X	X	X	X	X	ε
q_5	X	X	X	X	X	X

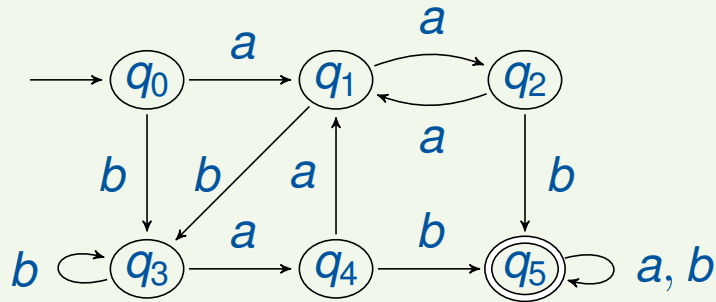
Algorithm A.55:

1. Mark every pair (p, q) with $p \in F, q \notin F$ by ε

Minimisation Example

Example A.56

Given DFA:



Equivalence matrix:

	q_0	q_1	q_2	q_3	q_4	q_5
q_0	X					ε
q_1	X	X				ε
q_2	X	X	X			ε
q_3	X	X	X	X		ε
q_4	X	X	X	X	X	ε
q_5	X	X	X	X	X	X

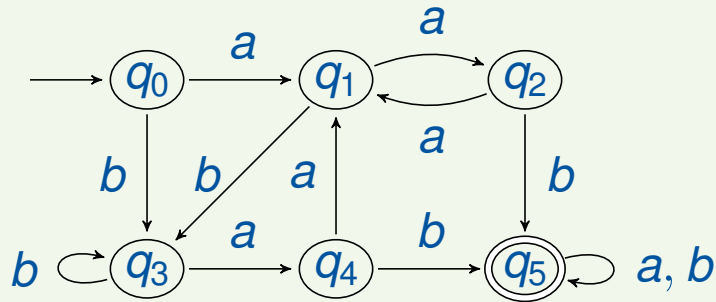
Algorithm A.55:

2. If $(\delta(p, a), \delta(q, a))$ marked by ε , then mark (p, q) by a (not applicable)

Minimisation Example

Example A.56

Given DFA:



Equivalence matrix:

	q_0	q_1	q_2	q_3	q_4	q_5
q_0	X		<i>b</i>		<i>b</i>	ε
q_1	X	X	<i>b</i>		<i>b</i>	ε
q_2	X	X	X	<i>b</i>		ε
q_3	X	X	X	X	<i>b</i>	ε
q_4	X	X	X	X	X	ε
q_5	X	X	X	X	X	X

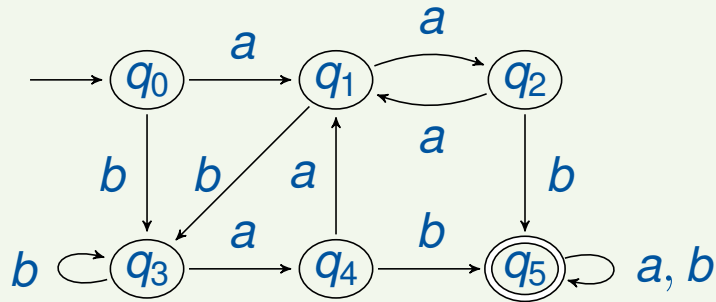
Algorithm A.55:

2. If $(\delta(p, b), \delta(q, b))$ marked by ε , then mark (p, q) by b

Minimisation Example

Example A.56

Given DFA:



Equivalence matrix:

	q_0	q_1	q_2	q_3	q_4	q_5
q_0	X	ab	b	ab	b	ε
q_1	X	X	b		b	ε
q_2	X	X	X	b		ε
q_3	X	X	X	X	b	ε
q_4	X	X	X	X	X	ε
q_5	X	X	X	X	X	X

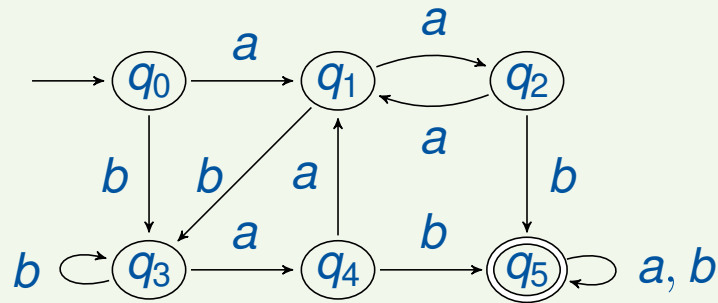
Algorithm A.55:

2. If $(\delta(p, a), \delta(q, a))$ marked by $c \in \{a, b\}$, then mark (p, q) by ac

Minimisation Example

Example A.56

Given DFA:



Equivalence matrix:

	q_0	q_1	q_2	q_3	q_4	q_5
q_0	X	ab	b	ab	b	ε
q_1	X	X	b		b	ε
q_2	X	X	X	b		ε
q_3	X	X	X	X	b	ε
q_4	X	X	X	X	X	ε
q_5	X	X	X	X	X	X

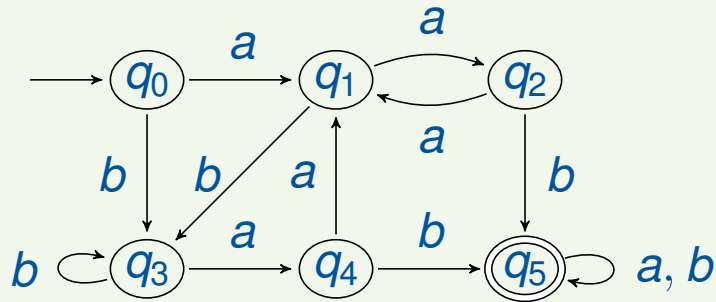
Algorithm A.55:

2. If $(\delta(p, b), \delta(q, b))$ marked by $c \in \{a, b\}$, then mark (p, q) by bc (not applicable)

Minimisation Example

Example A.56

Given DFA:



Equivalence matrix:

	q_0	q_1	q_2	q_3	q_4	q_5
q_0	X	ab	b	ab	b	ε
q_1	X	X	b	✓	b	ε
q_2	X	X	X	b	✓	ε
q_3	X	X	X	X	b	ε
q_4	X	X	X	X	X	ε
q_5	X	X	X	X	X	X

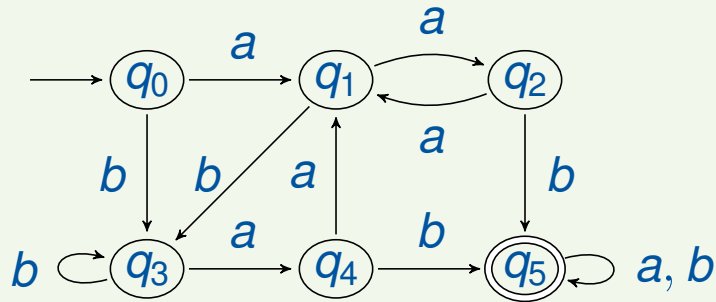
Algorithm A.55:

3. No further changes $\Rightarrow (q_1, q_3), (q_2, q_4)$ equivalent

Minimisation Example

Example A.56

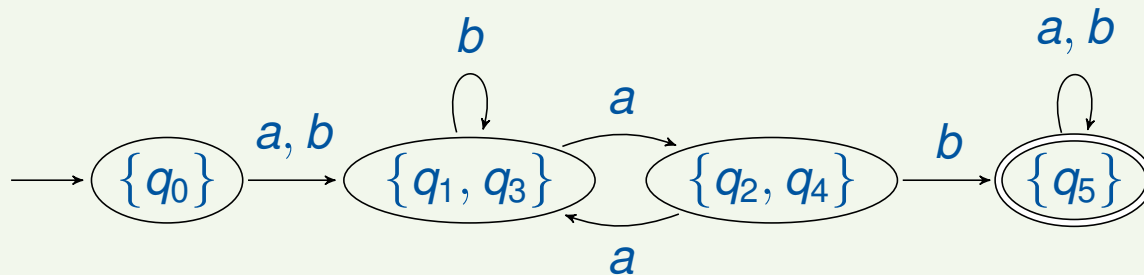
Given DFA:



Equivalence matrix:

	q_0	q_1	q_2	q_3	q_4	q_5
q_0	X	ab	b	ab	b	ε
q_1	X	X	b	✓	b	ε
q_2	X	X	X	b	✓	ε
q_3	X	X	X	X	b	ε
q_4	X	X	X	X	X	ε
q_5	X	X	X	X	X	X

Resulting minimal DFA:



Correctness of Minimisation

Theorem A.57

For every DFA \mathcal{A} ,

$$L(\mathcal{A}) = L(\mathcal{A}_{min})$$

Correctness of Minimisation

Theorem A.57

For every DFA \mathcal{A} ,

$$L(\mathcal{A}) = L(\mathcal{A}_{min})$$

Remark: the minimal DFA is **unique**, in the following sense:

$$\forall \text{DFA } \mathcal{A}, \mathcal{B} : L(\mathcal{A}) = L(\mathcal{B}) \implies \mathcal{A}_{min} \approx \mathcal{B}_{min}$$

where \approx refers to automata isomorphism (= identity up to naming of states)

Outline of Part A

Formal Languages

Finite Automata

- Deterministic Finite Automata

- Operations on Languages and Automata

- Nondeterministic Finite Automata

- More Decidability Results

Regular Expressions

- Definition

- Equivalence of Regular Expressions and Finite Automata

Minimisation of Deterministic Finite Automata

Outlook

Outlook

- **Pumping Lemma** (to prove non-regularity of languages)
 - can be used to show that $\{a^n b^n \mid n \geq 1\}$ is not regular
- More **language operations** (homomorphisms, ...)
- Construction of **scanners** for compilers