Foundations of Informatics: a Bridging Course

Week 3: Formal Languages and Processes
Part A: Regular Languages
March 11–15, 2024

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`https://moves.rwth-aachen.de/teaching/ws-23-24/foi/`

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Overview of Week 3

1. Regular Languages
   – Formal Languages
   – Finite Automata
   – Regular Expressions
   – Minimisation of Finite Automata
2. Context-Free Languages
   – Context-Free Grammars and Languages
   – Context-Free vs. Regular Languages
   – The Word Problem for Context-Free Languages
   – The Emptiness Problem for Context-Free Languages
   – Closure Properties of Context-Free Languages
   – Pushdown Automata

Resources

- J.E. Hopcroft, R. Motwani, J.D. Ullmann: *Introduction to Automata Theory, Languages, and Computation*, 2nd ed., Addison-Wesley, 2001
- A. Asteroth, C. Baier: *Theoretische Informatik*, Pearson Studium, 2002 [in German]
- `http://www.jflap.org/`
  (software for experimenting with formal languages and automata)

Outline of Part A

Formal Languages

Finite Automata

- Deterministic Finite Automata
- Operations on Languages and Automata
- Nondeterministic Finite Automata
- More Decidability Results

Regular Expressions

- Definition
- Equivalence of Regular Expressions and Finite Automata

Minimisation of Deterministic Finite Automata

Outlook

Words and Languages

- Computer systems transform data
- Data encoded as (binary) words

$\Rightarrow$ Data sets = sets of words = formal languages,
data transformations = functions on words

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data transformations = functions on words

Example A.1

- $\mathit{Java} = \{\text{all valid Java programs}\}$
- $\mathit{Compiler} : \mathit{Java} \to \mathit{Bytecode}$

Alphabets

The atomic elements of words are called symbols (or letters).

Definition A.2

An alphabet is a finite, non-empty set of symbols (“letters”).

- $\Sigma, \Gamma, \ldots$ denote alphabets
- $a, b, \ldots$ denote letters

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2. Latin alphabet $\Sigma_{\mathrm{latin}} := \{a, b, c, \ldots, z\}$

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3. Keyboard alphabet Σ_{key}
4. Morse alphabet $\Sigma_{\mathrm{morse}} := \{\cdot, -, \sqcup\}$

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Definition A.4

- A word is a finite sequence of letters from a given alphabet Σ.
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- If $w = a_1 \dots a_n$, then $w^R := a_n \dots a_1$.

Formal Languages I

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2. over $\Sigma = \{\mathrm{I}, \mathrm{V}, \mathrm{X}, \mathrm{L}, \mathrm{C}, \mathrm{D}, \mathrm{M}\}$: set of all valid roman numbers

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1. over $\mathbb{B} = \{0, 1\}$: set of all bit strings containing 1101
2. over $\Sigma = \{I, V, X, L, C, D, M\}$: set of all valid roman numbers
3. over Σ_{key}: set of all valid Java programs

Formal Languages II

Seen:

- Basic notions: alphabets, words
- Formal languages as sets of words

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- Basic notions: alphabets, words
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Next:

- Description of computations on words

Outline of Part A

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Example: Pattern Matching

Example A.7 (Pattern 1101)

1. Read Boolean string bit-by-bit
2. Test whether it contains 1101
3. Idea: remember which (initial) part of 1101 has been recognised
4. Five prefixes: ε, 1, 11, 110, 1101
5. Diagram: on the board

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What we used:

- finitely many (storage) states
- an initial state
- for every current state and every input symbol: a new state
- a successful state

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Deterministic Finite Automata I

Definition A.8

A deterministic finite automaton (DFA) is of the form

$$\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$$

where

- Q is a finite set of states
- Σ denotes the input alphabet
- $\delta : Q \times \Sigma \to Q$ is the transition function
- $q_0 \in Q$ is the initial state
- $F \subseteq Q$ is the set of final (or: accepting) states

Deterministic Finite Automata II

Example A.9

Pattern matching (Example A.7):

- $Q = \{q_0, \ldots, q_4\}$
- $\Sigma = \mathbb{B} = \{0, 1\}$
- $\delta : Q \times \Sigma \to Q$ on the board
- $F = \{q_4\}$

Example A.9

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- $\delta : Q \times \Sigma \to Q$ on the board
- $F = \{q_4\}$

Graphical Representation of DFA:

- states $\mapsto$ nodes
- $\delta(q, a) = q' \mapsto q \xrightarrow{a} q'$
- initial state: incoming edge without source state
- final state(s): additional circle

Acceptance by DFA I

Definition A.10

Let $\langle Q, \Sigma, \delta, q_0, F\rangle$ be a DFA. The extension of $\delta : Q \times \Sigma \to Q$,

$$\delta^* : Q \times \Sigma^* \to Q,$$

is defined by

$$\delta^*(q, w) := \text{state after reading } w \text{ starting from } q.$$

Formally:

$$\delta^*(q, w) := \begin{cases} q & \text{if } w = \varepsilon \\ \delta^*(\delta(q, a), v) & \text{if } w = av \end{cases}$$

Thus: if $w = a_1 \dots a_n$ and $q \xrightarrow{a_1} q_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} q_n$, then $\delta^*(q, w) = q_n$

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Example A.11

Pattern matching (Example A.9): on the board

Acceptance by DFA II

Definition A.12

- $\mathfrak{A}$ accepts $w \in \Sigma^*$ if $\delta^*(q_0, w) \in F$.
- The language recognised (or: accepted) by $\mathfrak{A}$ is

$$L(\mathfrak{A}) := \{w \in \Sigma^* \mid \delta^*(q_0, w) \in F\}.$$

- A language $L \subseteq \Sigma^*$ is called DFA-recognisable if there exists some DFA $\mathfrak{A}$ such that $L(\mathfrak{A}) = L$.
- Two DFA $\mathfrak{A}_1, \mathfrak{A}_2$ are called equivalent if

$$L(\mathfrak{A}_1) = L(\mathfrak{A}_2).$$

Acceptance by DFA III

Example A.13

1. The set of all bit strings containing 1101 is recognised by the automaton from Example A.9.

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Acceptance by DFA III

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1. The set of all bit strings containing 1101 is recognised by the automaton from Example A.9.
2. Two (equivalent) automata recognising the language

$$\{w \in \mathbb{B}^* \mid w \text{ contains } 1\} :$$

   on the board
3. An automaton which recognises

$$\{w \in \{0, \ldots, 9\}^* \mid \text{value of } w \text{ divisible by } 3\}$$

   Idea: test whether sum of digits is divisible by 3 – one state for each residue class (on the board)

Deterministic Finite Automata

Seen:

- Deterministic finite automata as a model of simple sequential computations
- Recognisability of formal languages by automata

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- Deterministic finite automata as a model of simple sequential computations
- Recognisability of formal languages by automata

Next:

- Composition and transformation of automata
- Which languages are recognisable, which are not (alternative characterisation)
- Language definition $\mapsto$ automaton and vice versa

Outline of Part A

Operations on Languages

Simplest case: Boolean operations (complement, intersection, union)

Question

Let $\mathfrak{A}_1, \mathfrak{A}_2$ be two DFA with $L(\mathfrak{A}_1) = L_1$ and $L(\mathfrak{A}_2) = L_2$.
Can we construct automata which recognise

- $\overline{L_1}$ $(:= \Sigma^* \setminus L_1)$,
- $L_1 \cap L_2$, and
- $L_1 \cup L_2$?

Language Complement

Theorem A.14

If $L \subseteq \Sigma^*$ is DFA-recognisable, then so is $\overline{L}$.

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Proof.

Let $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$ be a DFA such that $L(\mathfrak{A}) = L$. Then:

$$w \in \overline{L} \iff w \notin L \iff \delta^*(q_0, w) \notin F \iff \delta^*(q_0, w) \in Q \setminus F.$$

Thus, $\overline{L}$ is recognised by the DFA $\langle Q, \Sigma, \delta, q_0, Q \setminus F \rangle$. □

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$$w \in \overline{L} \iff w \notin L \iff \delta^*(q_0, w) \notin F \iff \delta^*(q_0, w) \in Q \setminus F.$$

Thus, $\overline{L}$ is recognised by the DFA $\langle Q, \Sigma, \delta, q_0, Q \setminus F\rangle$. □

Example A.15

on the board

Language Intersection I

Theorem A.16

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Let $\mathfrak{A}_i = \langle Q_i, \Sigma, \delta_i, q_0^i, F_i \rangle$ be DFA such that $L(\mathfrak{A}_i) = L_i$ $(i = 1, 2)$. The new automaton $\mathfrak{A}$ has to accept w iff $\mathfrak{A}_1$ and $\mathfrak{A}_2$ accept w

Idea: let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ run in parallel

- use pairs of states $(q_1, q_2) \in Q_1 \times Q_2$
- start with both components in initial state
- a transition updates both components independently
- for acceptance both components need to be in a final state

□

Language Intersection II

Proof (continued).

Formally: let the product automaton

$$\mathfrak{A} := \langle Q_1 \times Q_2, \Sigma, \delta, (q_0^1, q_0^2), F_1 \times F_2 \rangle$$

be defined by

$$\delta((q_1, q_2), a) := (\delta_1(q_1, a), \delta_2(q_2, a)) \text{ for every } a \in \Sigma.$$

Language Intersection II

Proof (continued).

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This definition yields (for every $w \in \Sigma^*$):

$$\delta^*((q_1, q_2), w) = (\delta_1^*(q_1, w), \delta_2^*(q_2, w)) \qquad (*)$$

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Thus:

$$\begin{aligned}
\mathfrak{A} \text{ accepts } w \quad &\iff \quad \delta^*((q_0^1, q_0^2), w) \in F_1 \times F_2 \\
&\overset{(*)}{\iff} \quad (\delta_1^*(q_0^1, w), \delta_2^*(q_0^2, w)) \in F_1 \times F_2 \\
&\iff \quad \delta_1^*(q_0^1, w) \in F_1 \text{ and } \delta_2^*(q_0^2, w) \in F_2 \\
&\iff \quad \mathfrak{A}_1 \text{ accepts } w \text{ and } \mathfrak{A}_2 \text{ accepts } w
\end{aligned}$$

□

Example A.17

on the board

Language Union

Theorem A.18

If $L_1, L_2 \subseteq \Sigma^*$ *are DFA-recognisable, then so is* $L_1 \cup L_2$.

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Idea: reuse product construction

Construct $\mathfrak{A}$ as before but choose as final states those pairs $(q_1, q_2) \in Q_1 \times Q_2$ with $q_1 \in F_1$ or $q_2 \in F_2$. Thus the set of final states is given by

$$F := (F_1 \times Q_2) \cup (Q_1 \times F_2).$$

□

Language Concatenation

Definition A.19

The concatenation of two languages $L_1, L_2 \subseteq \Sigma^*$ is given by

$$L_1 \cdot L_2 := \{v \cdot w \in \Sigma^* \mid v \in L_1, w \in L_2\}.$$

Abbreviations: $w \cdot L := \{w\} \cdot L$, $L \cdot w := L \cdot \{w\}$

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$$L_1 \cdot L_2 = \{101011, 1011, 11\}.$$

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1. If $L_1 = \{101, 1\}$ and $L_2 = \{011, 1\}$, then
$$L_1 \cdot L_2 = \{101011, 1011, 11\}.$$
2. If $L_1 = 00 \cdot \mathbb{B}^*$ and $L_2 = 11 \cdot \mathbb{B}^*$, then
$$L_1 \cdot L_2 = \{w \in \mathbb{B}^* \mid w \text{ has prefix } 00 \text{ and contains } 11\}.$$

DFA-Recognisability of Concatenation

Conjecture

If $L_1, L_2 \subseteq \Sigma^*$ are DFA-recognisable, then so is $L_1 \cdot L_2$.

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Idea: choose $Q := Q_1 \cup Q_2$ where each $q \in F_1$ is identified with q_0^2

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But: on the board □

Conclusion

Required: automata model where the successor state (for a given state and input symbol) is not unique

Language Iteration

Definition A.21

- The nth power of a language $L \subseteq \Sigma^*$ is the n-fold concatenation of L with itself ($n \in \mathbb{N}$):
$$L^n := \underbrace{L \cdot \ldots \cdot L}_{n \text{ times}} = \{w_1 \ldots w_n \mid \forall i \in \{1, \ldots, n\} : w_i \in L\}.$$
Inductively: $L^0 := \{\varepsilon\}$, $L^{n+1} := L^n \cdot L$
- The iteration (or: Kleene star) of L is
$$L^* := \bigcup_{n \in \mathbb{N}} L^n = \{w_1 \ldots w_n \mid n \in \mathbb{N}, \forall i \in \{1, \ldots, n\} : w_i \in L\}.$$

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Remarks:

- we always have $\varepsilon \in L^*$ (since $L^0 \subseteq L^*$ and $L^0 = \{\varepsilon\}$)
- $w \in L^*$ iff $w = \varepsilon$ or if w can be decomposed into $n \geq 1$ subwords $v_1, \ldots, v_n$ (i.e., $w = v_1 \cdot \ldots \cdot v_n$) such that $v_i \in L$ for every $1 \leq i \leq n$
- again we would suspect that the iteration of a DFA-recognisable language is DFA-recognisable, but there is no simple (deterministic) construction

Operations on Languages and Automata

Seen:

- Operations on languages:
  - complement
  - intersection
  - union
  - concatenation
  - iteration
- DFA constructions for:
  - complement
  - intersection
  - union

Operations on Languages and Automata

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Next:

- Automata model for (direct implementation of) concatenation and iteration

Outline of Part A

Nondeterministic Finite Automata I

Idea:

- for a given state and a given input symbol, several transitions (or none at all) are possible
- an input word generally induces several state sequences (“runs”)
- the word is accepted if at least one accepting run exists

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Advantages:

- simplifies representation of languages
  - example: $\mathbb{B}^* \cdot 1101 \cdot \mathbb{B}^*$ (on the board)
- yields direct constructions for concatenation and iteration of languages
- more adequate modelling of systems with nondeterministic behaviour
  - communication protocols, multi-agent systems, ...

Nondeterministic Finite Automata II

Definition A.22

A nondeterministic finite automaton (NFA) is of the form

$$\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$$

where

- Q is a finite set of states
- Σ denotes the input alphabet
- $\Delta \subseteq Q \times \Sigma \times Q$ is the transition relation
- $q_0 \in Q$ is the initial state
- $F \subseteq Q$ is the set of final states

Nondeterministic Finite Automata II

Definition A.22

A nondeterministic finite automaton (NFA) is of the form

$$\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$$

where

- Q is a finite set of states
- Σ denotes the input alphabet
- $\Delta \subseteq Q \times \Sigma \times Q$ is the transition relation
- $q_0 \in Q$ is the initial state
- $F \subseteq Q$ is the set of final states

Remarks:

- $(q, a, q') \in \Delta$ usually written as $q \xrightarrow{a} q'$
- every DFA can be considered as an NFA ($(q, a, q') \in \Delta \iff \delta(q, a) = q'$)

Acceptance by NFA

Definition A.23

- Let $w = a_1 \dots a_n \in \Sigma^*$.
- A w-labelled $\mathfrak{A}$-run from q_1 to q_2 is a sequence

$$p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} \dots p_{n-1} \xrightarrow{a_n} p_n$$

  such that $p_0 = q_1$, $p_n = q_2$, and $(p_{i-1}, a_i, p_i) \in \Delta$ for every $1 \leq i \leq n$ (we also write: $q_1 \xrightarrow{w} q_2$).
- $\mathfrak{A}$ accepts w if there is a w-labelled $\mathfrak{A}$-run from q_0 to some $q \in F$
- The language recognised by $\mathfrak{A}$ is

$$L(\mathfrak{A}) := \{w \in \Sigma^* \mid \mathfrak{A} \text{ accepts } w\}.$$

- A language $L \subseteq \Sigma^*$ is called NFA-recognisable if there exists a NFA $\mathfrak{A}$ such that $L(\mathfrak{A}) = L$.
- Two NFA $\mathfrak{A}_1, \mathfrak{A}_2$ are called equivalent if $L(\mathfrak{A}_1) = L(\mathfrak{A}_2)$.

Acceptance Test for NFA

Algorithm A.24 (Acceptance Test for NFA)

Input: NFA $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F\rangle$, $w \in \Sigma^*$

Question: $w \in L(\mathfrak{A})$?

Procedure: Computation of the *reachability set*

$$R_{\mathfrak{A}}(w) := \{q \in Q \mid q_0 \xrightarrow{w} q\}$$

Iterative procedure for $w = a_1 \dots a_n$:

1. *let* $R_{\mathfrak{A}}(\varepsilon) := \{q_0\}$
2. *for* $i := 1, \dots, n$: *let*
   $R_{\mathfrak{A}}(a_1 \dots a_i) := \{q \in Q \mid \exists p \in R_{\mathfrak{A}}(a_1 \dots a_{i-1}) \colon p \xrightarrow{a_i} q\}$

Output: *"yes" if* $R_{\mathfrak{A}}(w) \cap F \neq \emptyset$, *otherwise "no"*

Remark: this algorithm solves the word problem for NFA

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Example A.25

on the board

NFA-Recognisability of Concatenation

Definition of NFA looks promising, but... (on the board)

NFA-Recognisability of Concatenation

Definition of NFA looks promising, but... (on the board)

Solution: admit empty word ε as transition label

ε-NFA

Definition A.26

A nondeterministic finite automaton with ε-transitions (ε-NFA) is of the form $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$ where

- Q is a finite set of states
- Σ denotes the input alphabet
- $\Delta \subseteq Q \times \Sigma_\varepsilon \times Q$ is the transition relation where $\Sigma_\varepsilon := \Sigma \cup \{\varepsilon\}$
- $q_0 \in Q$ is the initial state
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Remarks:

- every NFA is an ε-NFA
- definitions of runs and acceptance: in analogy to NFA

ε-NFA

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Example A.27

on the board

Concatenation and Iteration via ε-NFA

Theorem A.28

If $L_1, L_2 \subseteq \Sigma^*$ are ε-NFA-recognisable, then so is $L_1 \cdot L_2$.

Concatenation and Iteration via ε-NFA

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If $L_1, L_2 \subseteq \Sigma^*$ are ε-NFA-recognisable, then so is $L_1 \cdot L_2$.

Proof (idea).

on the board □

Concatenation and Iteration via ε-NFA

Theorem A.28

If $L_1, L_2 \subseteq \Sigma^*$ are ε-NFA-recognisable, then so is $L_1 \cdot L_2$.

Proof (idea).

on the board □

Theorem A.29

If $L \subseteq \Sigma^*$ is ε-NFA-recognisable, then so is L^*.

Proof.

see Theorem A.46 □

Types of Finite Automata

1. DFA (Definition A.8)
2. NFA (Definition A.22)
3. ε-NFA (Definition A.26)

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Corollary A.30

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2. *Every NFA-recognisable language is ε-NFA-recognisable.*

Types of Finite Automata

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Goal: establish reverse inclusions

From NFA to DFA I

Theorem A.31

Every NFA can be transformed into an equivalent DFA.

From NFA to DFA I

Theorem A.31

Every NFA can be transformed into an equivalent DFA.

Proof.

Idea: let the DFA operate on sets of states ("powerset construction")

- Initial state of DFA := {initial state of NFA}
- $P \xrightarrow{a} P'$ in DFA iff there exist $q \in P, q' \in P'$ such that $q \xrightarrow{a} q'$ in NFA
- P final state in DFA iff it contains some final state of NFA

□

From NFA to DFA II

Proof (continued).

Let $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$ a NFA. Powerset construction of $\mathfrak{A}' = \langle Q', \Sigma, \delta', q_0', F' \rangle$:

- $Q' := 2^Q := \{P \mid P \subseteq Q\}$
- $\delta' : Q' \times \Sigma \to Q'$ with $q \in \delta'(P, a) \iff$ there exists $p \in P$ such that $(p, a, q) \in \Delta$
- $q_0' := \{q_0\}$
- $F' := \{P \subseteq Q \mid P \cap F \neq \emptyset\}$

From NFA to DFA II

Proof (continued).

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- $q_0' := \{q_0\}$
- $F' := \{P \subseteq Q \mid P \cap F \neq \emptyset\}$

This yields

$$q_0 \xrightarrow{w} q \text{ in } \mathfrak{A} \iff q \in \delta'^*(\{q_0\}, w) \text{ in } \mathfrak{A}'$$

and thus

$$\mathfrak{A} \text{ accepts } w \iff \mathfrak{A}' \text{ accepts } w$$

From NFA to DFA II

Proof (continued).

Let $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$ a NFA. Powerset construction of $\mathfrak{A}' = \langle Q', \Sigma, \delta', q_0', F' \rangle$:

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and thus

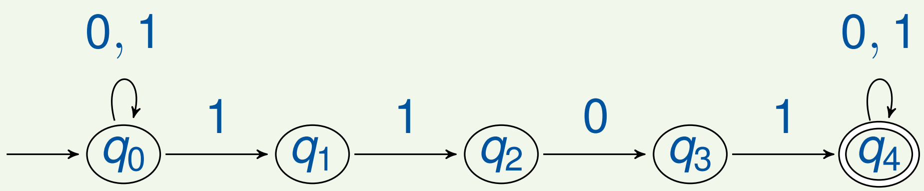
$$\mathfrak{A} \text{ accepts } w \iff \mathfrak{A}' \text{ accepts } w$$

(**Remark:** only reachable subsets of Q need to be considered.) □

From NFA to DFA III

Example A.32

NFA:



Example A.32

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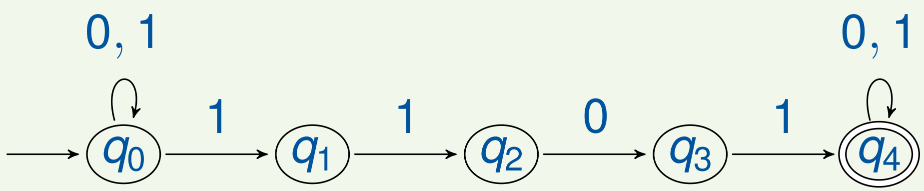
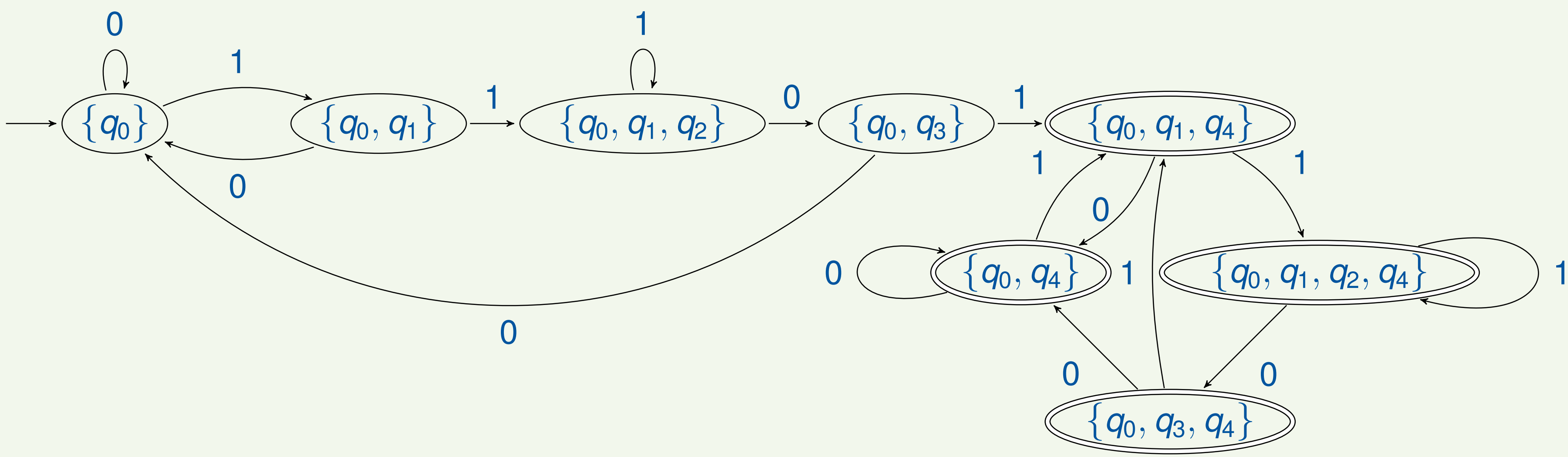


Corresponding DFA:



From ε-NFA to NFA I

Theorem A.33

Every ε-NFA can be transformed into an equivalent NFA.

From ε-NFA to NFA I

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Proof (idea).

Let $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$ be a ε-NFA. We construct the NFA $\mathfrak{A}'$ by eliminating all ε-transitions, adding appropriate direct transitions: if $p \xrightarrow{\varepsilon}^{*} q$, $q \xrightarrow{a} q'$, and $q' \xrightarrow{\varepsilon}^{*} r$ in $\mathfrak{A}$, then $p \xrightarrow{a} r$ in $\mathfrak{A}'$. Moreover $F' := F \cup \{q_0\}$ if $q_0 \xrightarrow{\varepsilon}^{*} q \in F$ in $\mathfrak{A}$, and $F' := F$ otherwise. $\square$

From ε-NFA to NFA I

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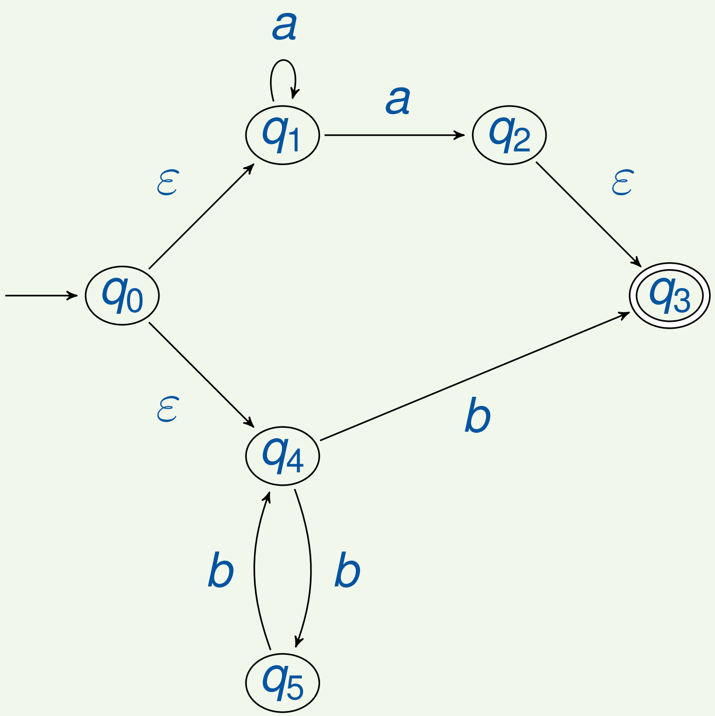
Corollary A.34

All types of finite automata recognise the same class of languages.

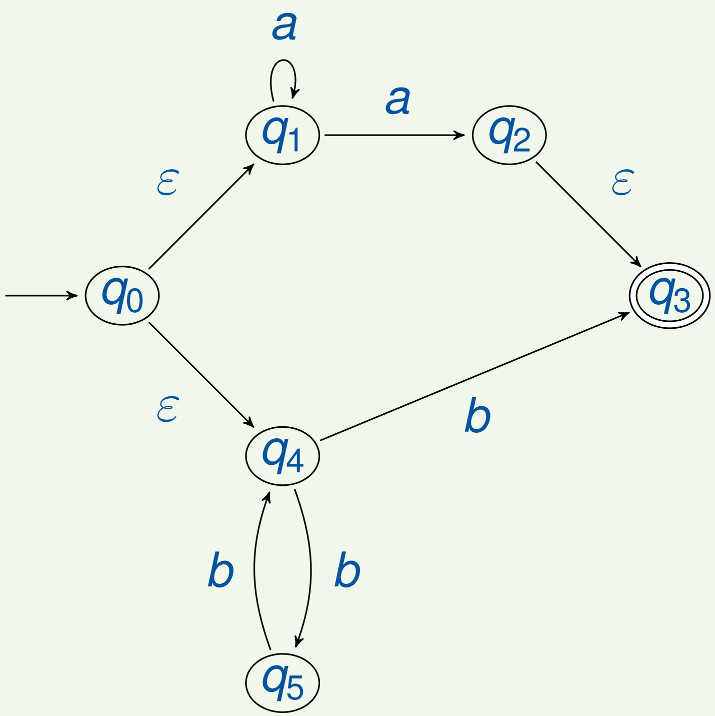
From ε-NFA to NFA II

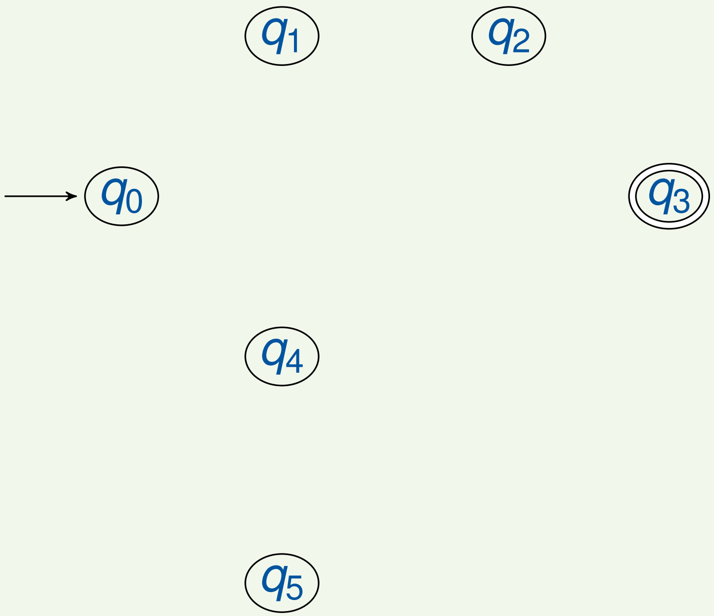
Example A.35



From ε-NFA to NFA II

Example A.35





From ε-NFA to NFA II

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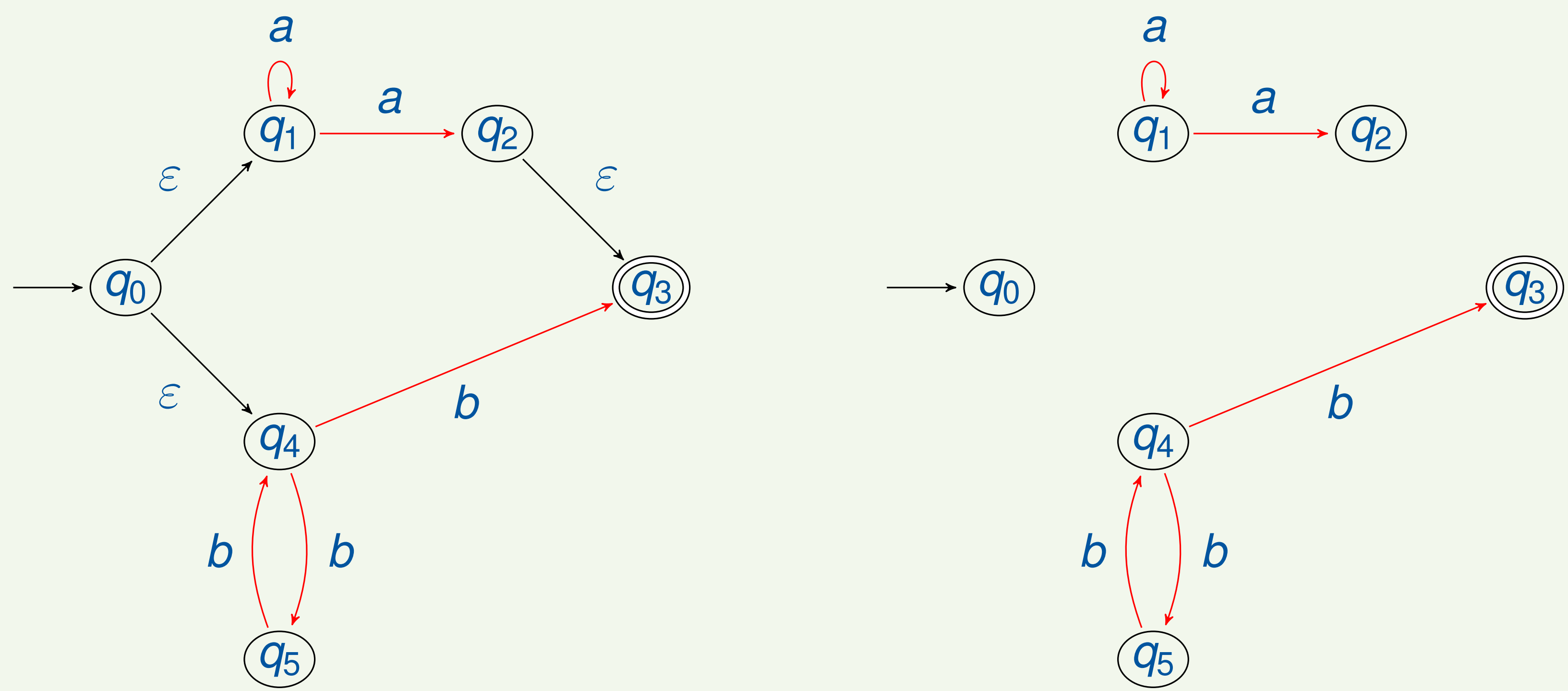

From ε-NFA to NFA II

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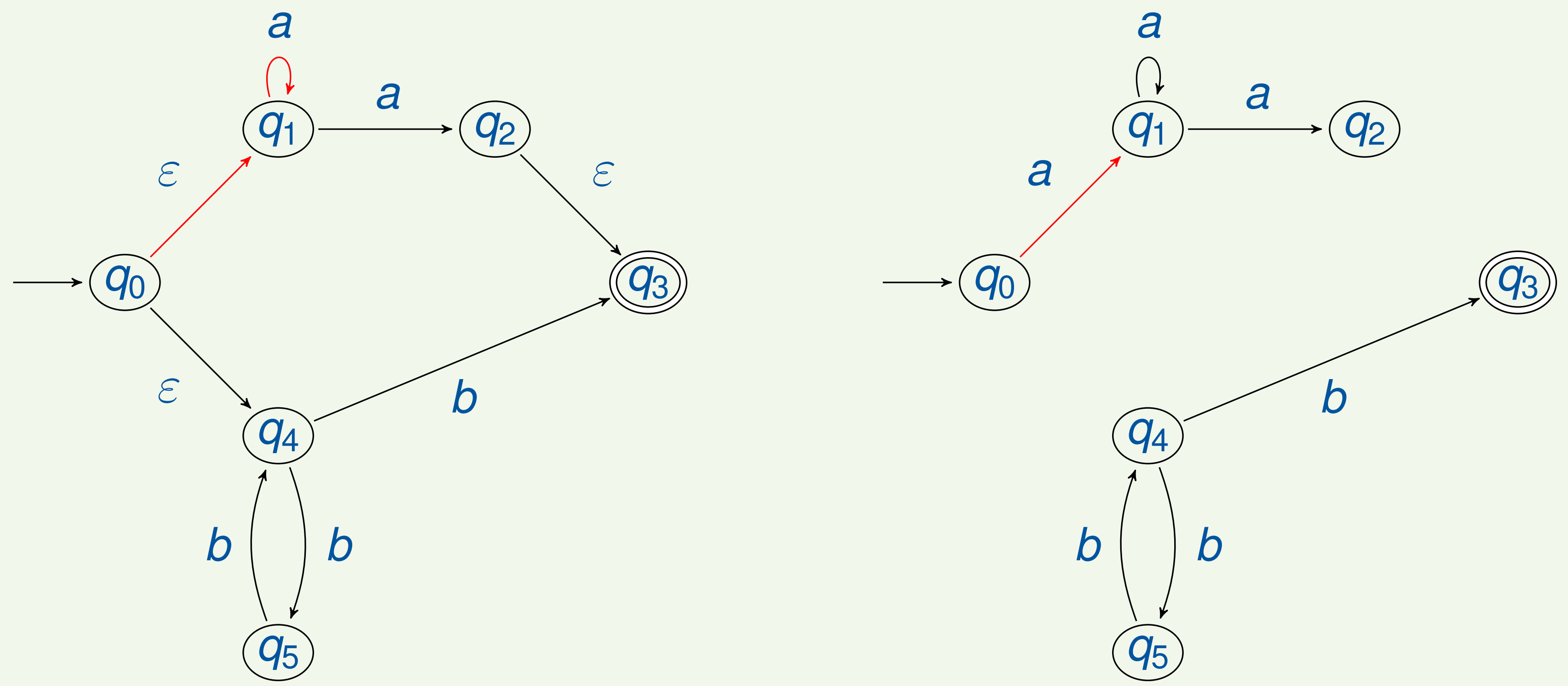

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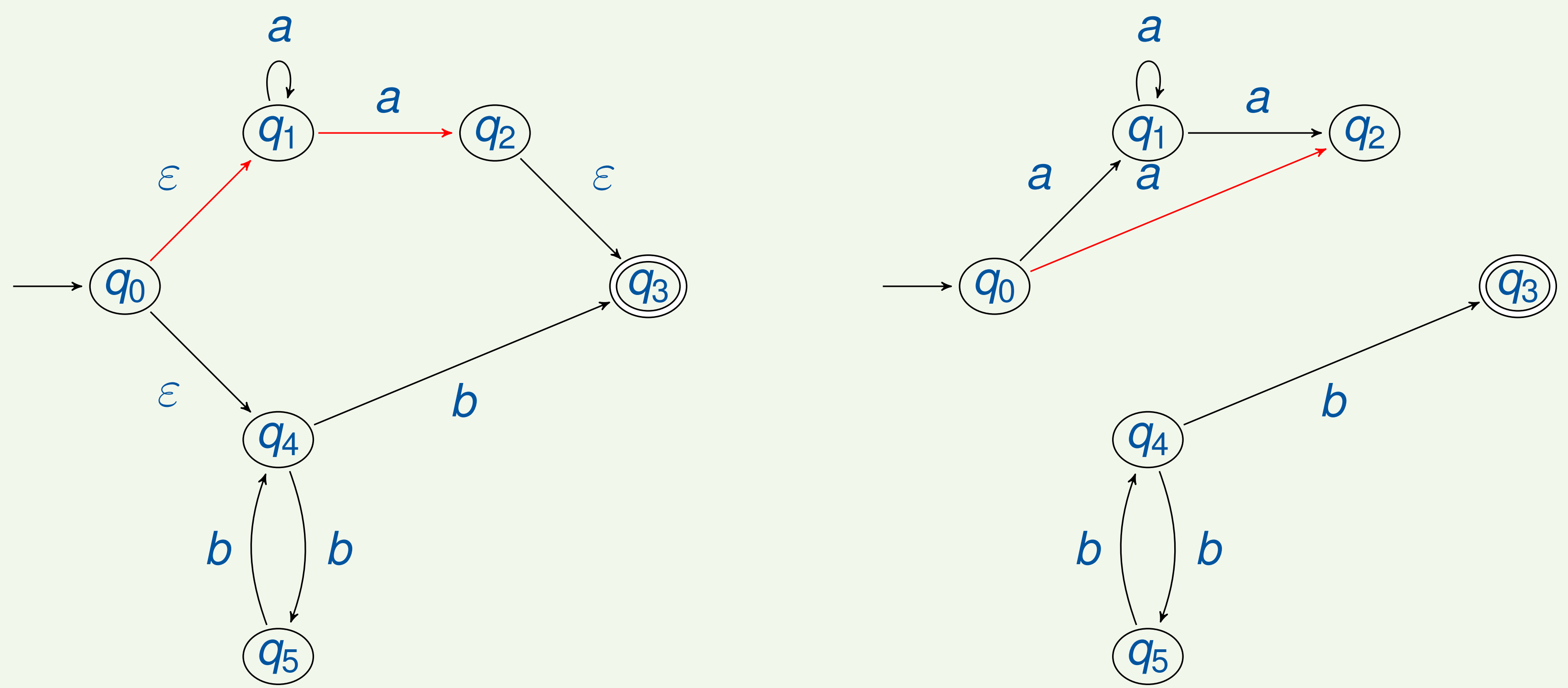

From ε-NFA to NFA II

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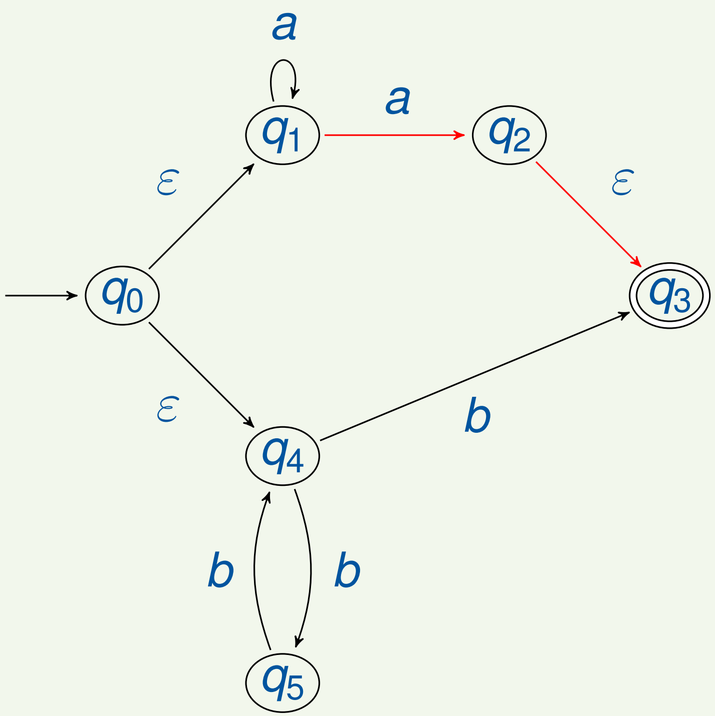

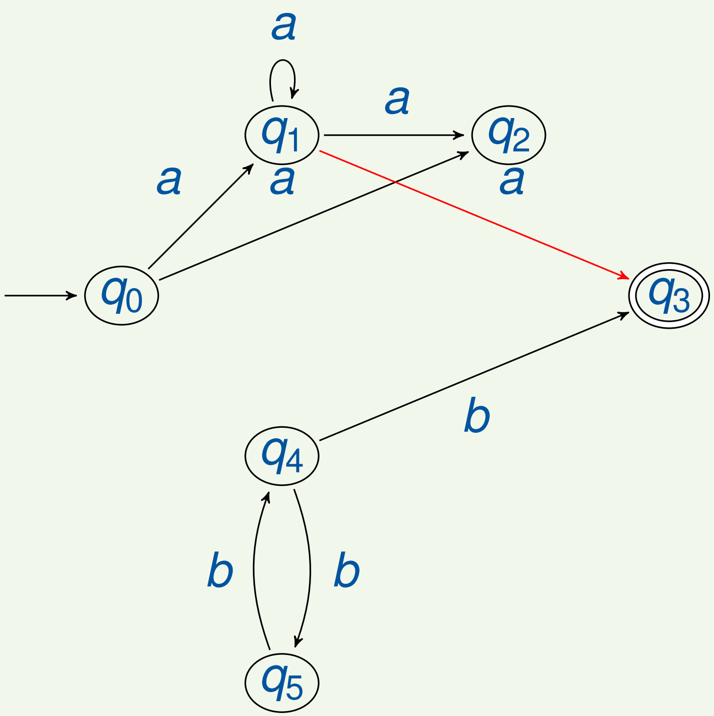

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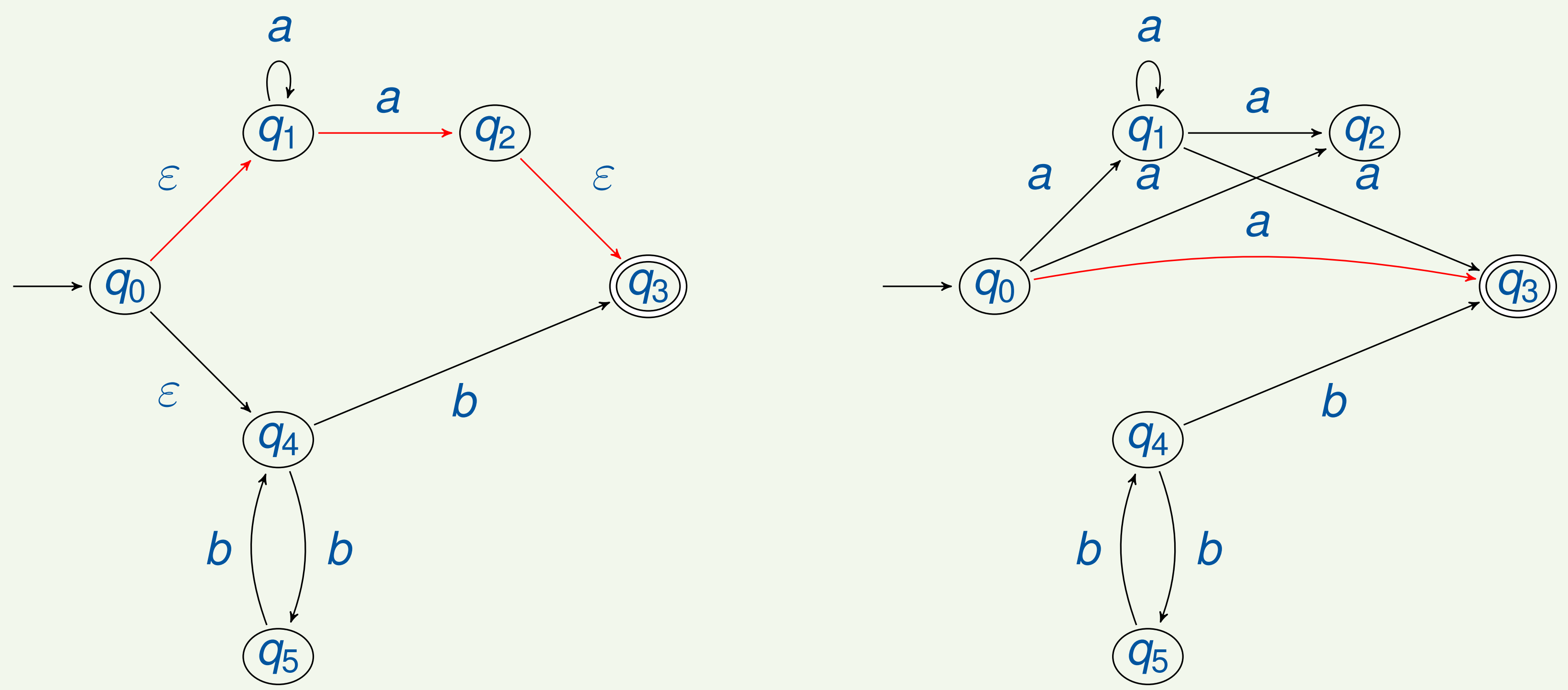

From ε-NFA to NFA II

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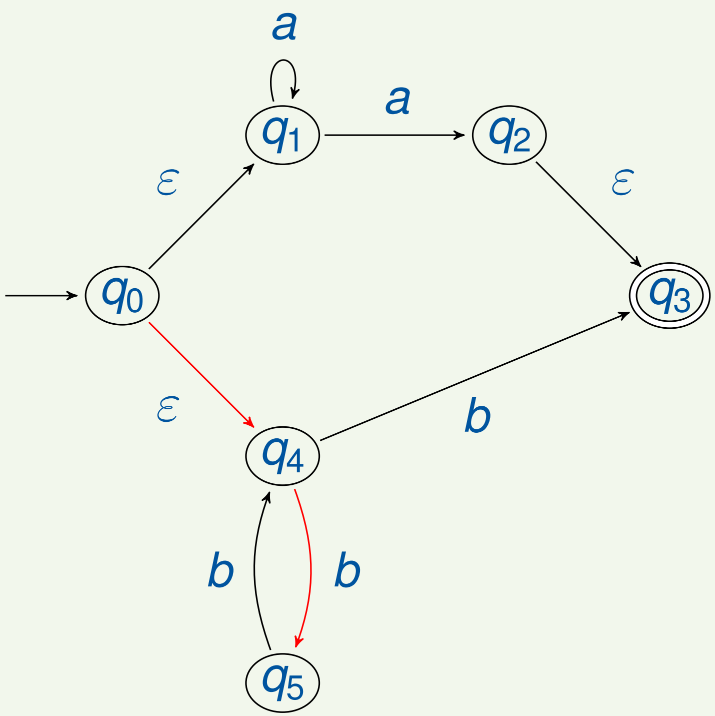

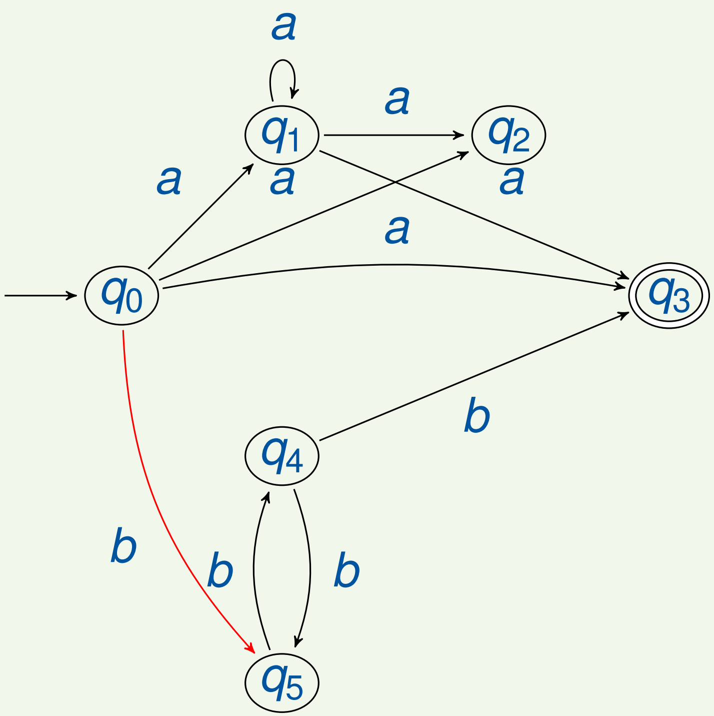

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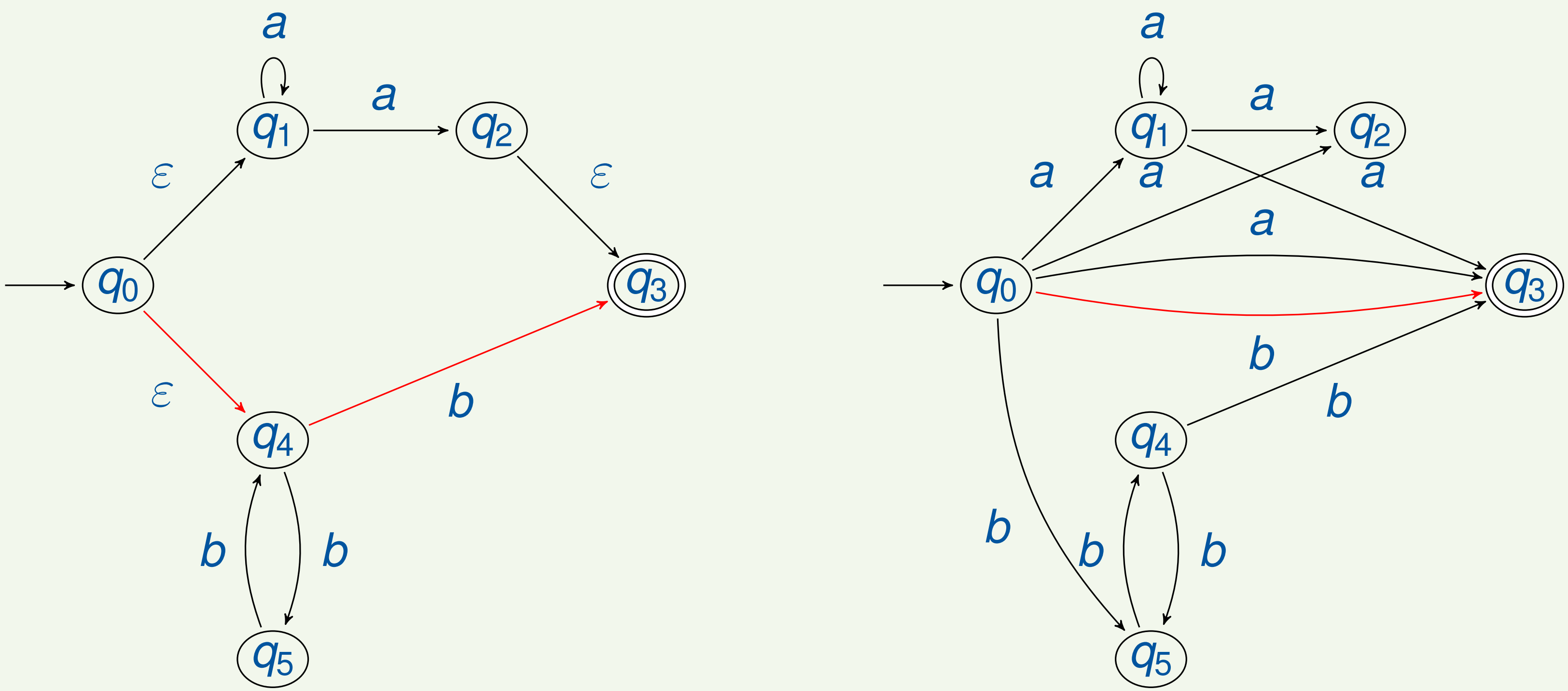

Nondeterministic Finite Automata

Seen:

- Definition of ε-NFA
- Determinisation of (ε-)NFA

Nondeterministic Finite Automata

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- Definition of ε-NFA
- Determinisation of (ε-)NFA

Next:

- More decidability results

Outline of Part A

The Word Problem Revisited

Definition A.36

The word problem for DFA is specified as follows:

Given a DFA $\mathfrak{A}$ and a word $w \in \Sigma^*$, decide whether

$$w \in L(\mathfrak{A}).$$

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As we have seen (Def. A.10, Alg. A.24, Thm. A.33):

Theorem A.37

The word problem for DFA (NFA, ε-NFA) is decidable.

The Emptiness Problem

Definition A.38

The emptiness problem for DFA is specified as follows:
Given a DFA $\mathfrak{A}$, decide whether $L(\mathfrak{A}) = \emptyset$.

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Remark: important result for formal verification (unreachability of bad [= final] states)

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Theorem A.39

The emptiness problem for DFA (NFA, ε-NFA) is decidable.

Proof.

It holds that $L(\mathfrak{A}) \neq \emptyset$ iff in $\mathfrak{A}$ some final state is reachable from the initial state (simple graph-theoretic problem). □

The Equivalence Problem

Definition A.40

The equivalence problem for DFA is specified as follows:
Given two DFA $\mathfrak{A}_1, \mathfrak{A}_2$, decide whether $L(\mathfrak{A}_1) = L(\mathfrak{A}_2)$.

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$$L(\mathfrak{A}_1) = L(\mathfrak{A}_2)$$

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Proof.

$$\begin{aligned} & L(\mathfrak{A}_1) = L(\mathfrak{A}_2) \\ \iff \quad & L(\mathfrak{A}_1) \subseteq L(\mathfrak{A}_2) \text{ and } L(\mathfrak{A}_2) \subseteq L(\mathfrak{A}_1) \end{aligned}$$

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Proof.

$$\begin{aligned} & L(\mathfrak{A}_1) = L(\mathfrak{A}_2) \\ \iff & L(\mathfrak{A}_1) \subseteq L(\mathfrak{A}_2) \text{ and } L(\mathfrak{A}_2) \subseteq L(\mathfrak{A}_1) \\ \iff & (L(\mathfrak{A}_1) \setminus L(\mathfrak{A}_2)) = \emptyset \text{ and } (L(\mathfrak{A}_2) \setminus L(\mathfrak{A}_1)) = \emptyset \end{aligned}$$

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\iff\ & (L(\mathfrak{A}_1) \setminus L(\mathfrak{A}_2)) = \emptyset \text{ and } (L(\mathfrak{A}_2) \setminus L(\mathfrak{A}_1)) = \emptyset \\
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\iff\ & \underbrace{\underbrace{\underbrace{(L(\mathfrak{A}_1) \cap \underbrace{\overline{L(\mathfrak{A}_2)}}_{\text{DFA-recognisable (Thm. A.14)}})}_{\text{DFA-recognisable (Thm. A.16)}} \cup \underbrace{(L(\mathfrak{A}_2) \cap \underbrace{\overline{L(\mathfrak{A}_1)}}_{\text{DFA-recognisable (Thm. A.14)}})}_{\text{DFA-recognisable (Thm. A.16)}}}_{\text{DFA-recognisable (Thm. A.18)}} = \emptyset}_{\text{decidable (Thm. A.39)}}
\end{aligned}
$$

Seen:

- Decidability of word problem
- Decidability of emptiness problem
- Decidability of equivalence problem

Finite Automata

Seen:

- Decidability of word problem
- Decidability of emptiness problem
- Decidability of equivalence problem

Next:

- Non-algorithmic description of languages

Outline of Part A

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An Example

Example A.42

Consider the set of all words over $\Sigma := \{a, b\}$ which

1. start with one or three a symbols
2. continue with a (potentially empty) sequence of blocks, each containing at least one b and exactly two a's
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Corresponding regular expression:

$$\underbrace{(a \mid aaa)}_{(1)} \underbrace{(\underbrace{bb^*ab^*ab^*}_{b \text{ before } a\text{'s}} \mid \underbrace{b^*abb^*ab^*}_{b \text{ between } a\text{'s}} \mid \underbrace{b^*ab^*abb^*}_{b \text{ after } a\text{'s}})^*}_{(2)} \underbrace{b^*}_{(3)}$$

Syntax of Regular Expressions

Definition A.43

The set of regular expressions over Σ is inductively defined by:

- $\emptyset$ and ε are regular expressions
- every $a \in \Sigma$ is a regular expression
- if α and β are regular expressions, then so are
  - $\alpha \mid \beta$
  - $\alpha \cdot \beta$
  - α^*

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  - $\alpha \mid \beta$
  - $\alpha \cdot \beta$
  - α^*

Notation:

- $\cdot$ can be omitted
- * binds stronger than $\cdot$, $\cdot$ binds stronger than $\mid$
  - thus: $a \mid bc^* := a \mid (b \cdot (c^*))$
- α^+ abbreviates $\alpha \cdot \alpha^*$

Semantics of Regular Expressions

Definition A.44

Every regular expression α defines a language $L(\alpha)$:

$$
\begin{aligned}
L(\emptyset) &:= \emptyset \\
L(\varepsilon) &:= \{\varepsilon\} \\
L(a) &:= \{a\} \\
L(\alpha \mid \beta) &:= L(\alpha) \cup L(\beta) \\
L(\alpha \cdot \beta) &:= L(\alpha) \cdot L(\beta) \\
L(\alpha^*) &:= (L(\alpha))^*
\end{aligned}
$$

Semantics of Regular Expressions

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L(a) &:= \{a\} \\
L(\alpha \mid \beta) &:= L(\alpha) \cup L(\beta) \\
L(\alpha \cdot \beta) &:= L(\alpha) \cdot L(\beta) \\
L(\alpha^*) &:= (L(\alpha))^*
\end{aligned}
$$

A language L is called regular if it is definable by a regular expression, i.e., if $L = L(\alpha)$ for some regular expression α.

Example A.45

1. $\{aa\}$ is regular since

$$L(a \cdot a) = L(a) \cdot L(a) = \{a\} \cdot \{a\} = \{aa\}$$

Regular Languages

Example A.45

1. $\{aa\}$ is regular since

$$L(a \cdot a) = L(a) \cdot L(a) = \{a\} \cdot \{a\} = \{aa\}$$

2. $\{a, b\}^*$ is regular since

$$L((a \mid b)^*) = (L(a \mid b))^* = (L(a) \cup L(b))^* = (\{a\} \cup \{b\})^* = \{a, b\}^*$$

Example A.45

1. $\{aa\}$ is regular since

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$$L((a \mid b)^*) = (L(a \mid b))^* = (L(a) \cup L(b))^* = (\{a\} \cup \{b\})^* = \{a, b\}^*$$

3. The set of all words over $\{a, b\}$ containing *abb* is regular since

$$L((a \mid b)^* \cdot a \cdot b \cdot b \cdot (a \mid b)^*) = \{a, b\}^* \cdot \{abb\} \cdot \{a, b\}^*$$

Outline of Part A

Theorem A.46 (Kleene's Theorem)

To each regular expression there corresponds an ε-NFA, and vice versa.

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Proof.

$\Rightarrow$: by induction over the given regular expression α, we construct an ε-NFA $\mathfrak{A}_\alpha$ with exactly one final state q_f and without transitions into the initial/leaving the final state:

Regular Languages and Finite Automata I

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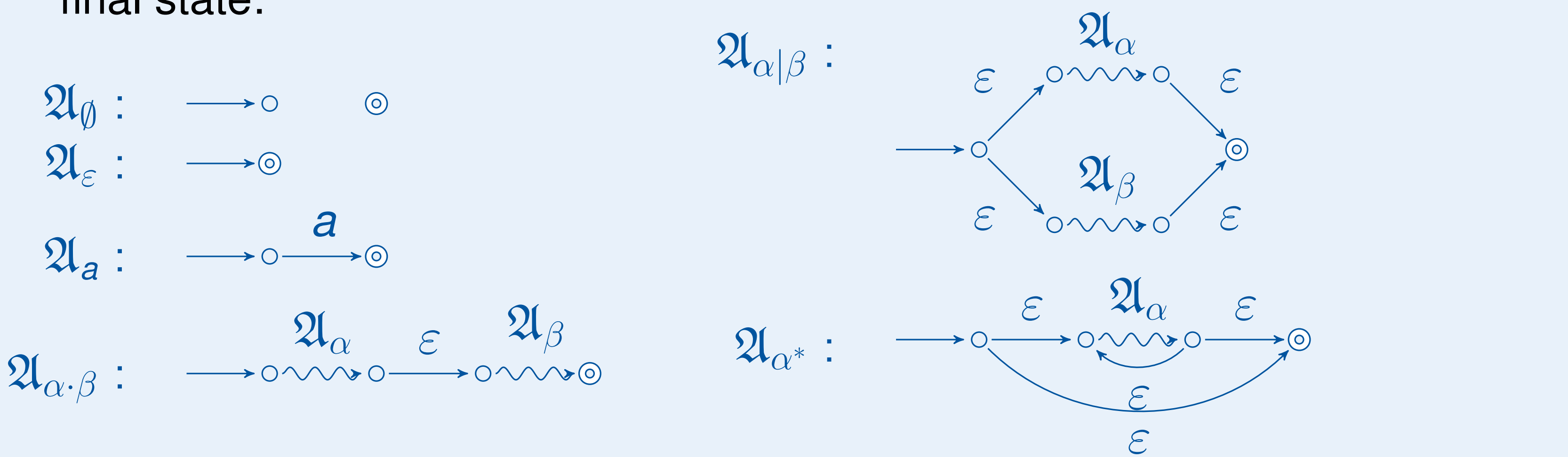

Regular Languages and Finite Automata I

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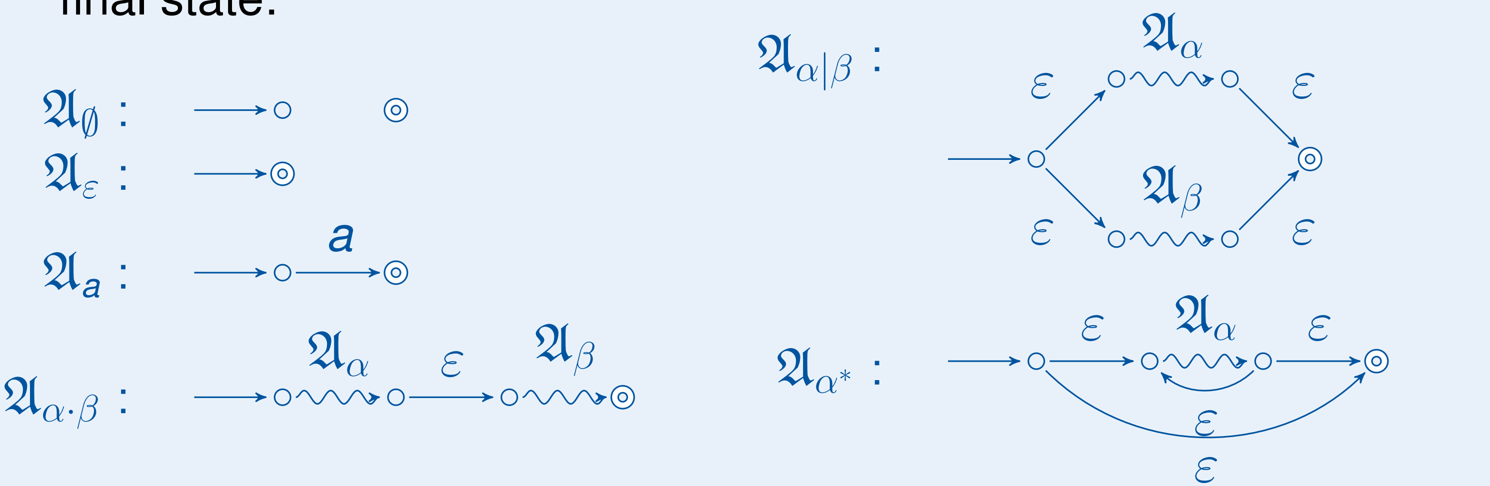
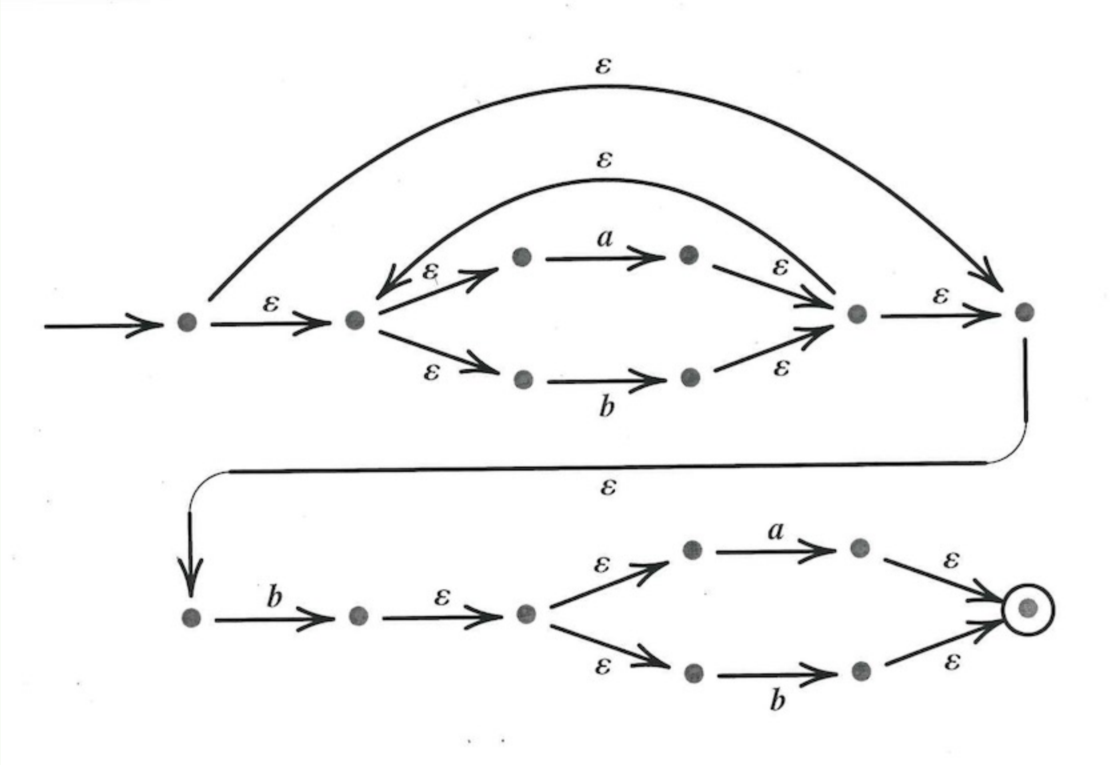


- $\Leftarrow$: by solving a regular equation system (details omitted) □

Example A.47

For the regular expression $(a \mid b)^* \cdot b \cdot (a \mid b)$, we obtain the following ε-NFA:



Corollary A.48

The following properties are equivalent:

- L *is regular*
- L *is DFA-recognisable*
- L *is NFA-recognisable*
- L *is* ε*-NFA-recognisable*

Implementation of Pattern Matching

Algorithm A.49 (Pattern Matching)

Input: *regular expression* α *and* $w \in \Sigma^*$

Question: *does* w *contain some* $v \in L(\alpha)$*?*

Procedure:

1. *let* $\beta := (a_1 \mid \ldots \mid a_n)^* \cdot \alpha$ *(for* $\Sigma = \{a_1, \ldots, a_n\}$*)*
2. *determine* ε*-NFA* $\mathfrak{A}_\beta$ *for* β
3. *eliminate* ε*-transitions*
4. *apply powerset construction to obtain DFA* $\mathfrak{A}$
5. *let* $\mathfrak{A}$ *run on* w

Output: *"yes" if* $\mathfrak{A}$ *passes through some final state, otherwise "no"*

Remark: in UNIX/LINUX implemented by `grep` and `lex`

Regular Expressions in UNIX (`grep`, `flex`, ...)

Syntax	Meaning
printable character	this character
`\n`, `\t`, `\123`, etc.	newline, tab, octal representation, etc.
`.`	any character except `\n`
`[`*Chars*`]`	one of *Chars*; ranges possible (“`0-9`”)
`[^`*Chars*`]`	none of *Chars*
`\\`, `\.`, `\[`, etc.	`\`, `.`, `[`, etc.
`"`*Text*`"`	*Text* without interpretation of `.`, `[`, `\`, etc.
`^`α	α at beginning of line
α`$`	α at end of line
α`?`	zero or one α
α`*`	zero or more α
α`+`	one or more α
α`{`*n*, *m*`}`	between *n* and *m* times α (“, *m*” optional)
`(`α`)`	α
$\alpha_1\alpha_2$	concatenation
$\alpha_1 \mid \alpha_2$	alternative

Regular Expressions

Seen:

- Definition of regular expressions
- Equivalence of regular and DFA-recognisable languages

Regular Expressions

Seen:

- Definition of regular expressions
- Equivalence of regular and DFA-recognisable languages

Next:

- “Optimisation” of finite automata

Outline of Part A

Motivation

Goal: space-efficient implementation of regular languages

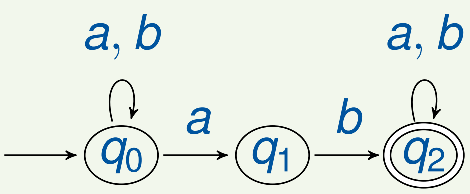
Given: DFA $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$

Wanted: DFA $\mathfrak{A}_{min} = \langle Q', \Sigma, \delta', q'_0, F' \rangle$ such that $L(\mathfrak{A}_{min}) = L(\mathfrak{A})$ and $|Q'|$ minimal

State Equivalence

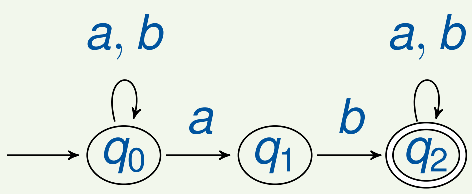
Example A.50

NFA for accepting $(a \mid b)^* ab(a \mid b)^*$:

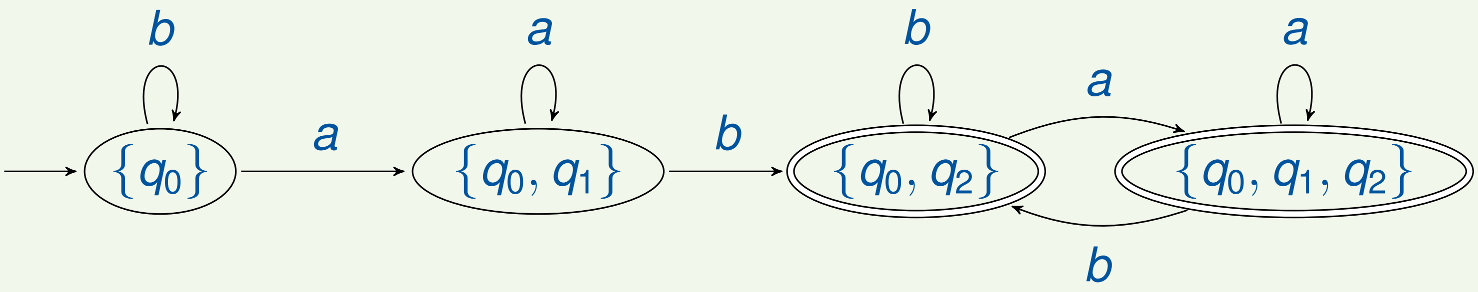


Example A.50

NFA for accepting $(a \mid b)^* ab(a \mid b)^*$:



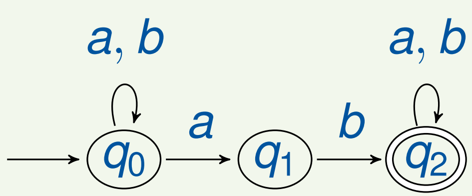
Powerset construction yields DFA $\mathfrak{A}$:



State Equivalence

Example A.50

NFA for accepting $(a \mid b)^* ab(a \mid b)^*$:



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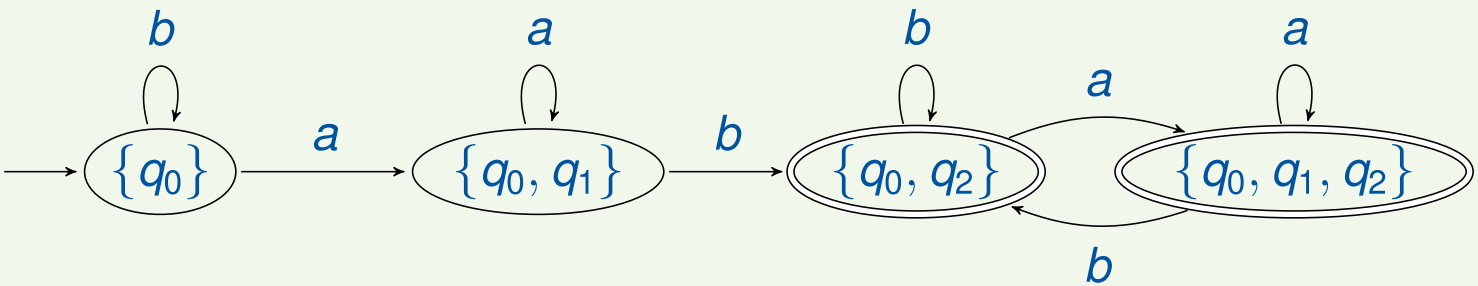

Observation: $\{q_0, q_2\}$ and $\{q_0, q_1, q_2\}$ are equivalent (every suffix accepted)

State Equivalence

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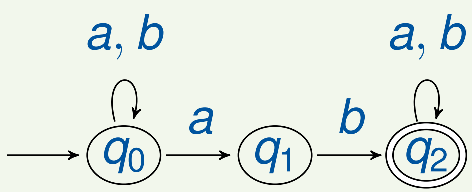
NFA for accepting $(a \mid b)^* ab(a \mid b)^*$:



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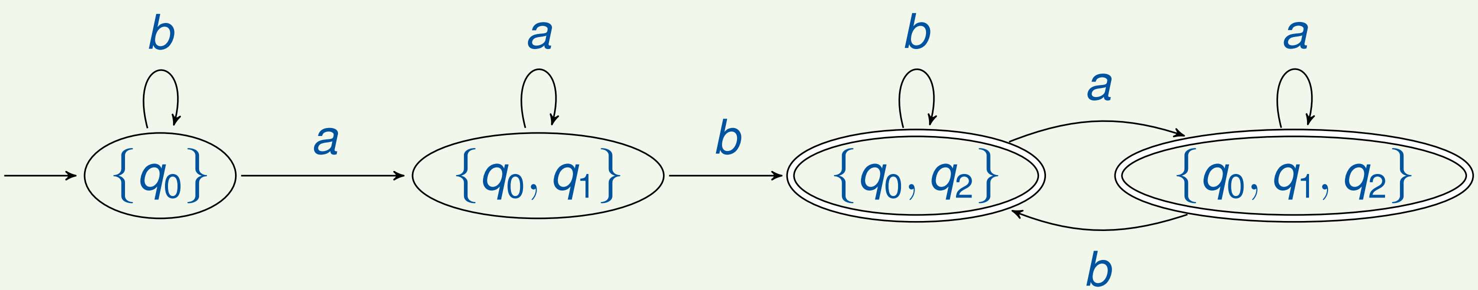


Observation: $\{q_0, q_2\}$ and $\{q_0, q_1, q_2\}$ are equivalent (every suffix accepted)

Definition A.51

Given DFA $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F\rangle$, states $p, q \in Q$ are equivalent if

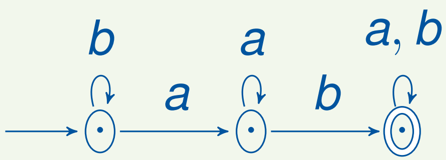
$$\forall w \in \Sigma^* : \delta^*(p, w) \in F \iff \delta^*(q, w) \in F.$$

State Merging

Minimisation: merging of equivalent states

Example A.52 (cf. Example A.50)

DFA after merging of $\{q_0, q_2\}$ and $\{q_0, q_1, q_2\}$:

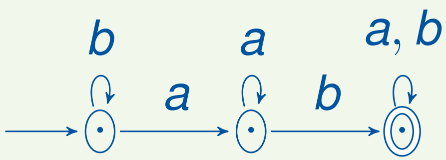


State Merging

Minimisation: merging of equivalent states

Example A.52 (cf. Example A.50)

DFA after merging of $\{q_0, q_2\}$ and $\{q_0, q_1, q_2\}$:



Problem: identification of equivalent states

Approach: iterative computation of inequivalent states by refinement

Corollary A.53

$p, q \in Q$ are inequivalent if there exists $w \in \Sigma^*$ such that

$$\delta^*(p, w) \in F \text{ and } \delta^*(q, w) \notin F$$

(or vice versa, i.e., p and q can be distinguished by w)

Computing State (In-)Equivalence

Lemma A.54

Inductive characterisation of state inequivalence:

- $w = \varepsilon$: $p \in F, q \notin F \implies p, q$ *inequivalent (by* ε*)*
- $w = av$: p', q' *inequivalent (by* v*)*, $p \xrightarrow{a} p', q \xrightarrow{a} q'$
  $\implies p, q$ *inequivalent (by* w*)*

Software Modeling and Verification Chair
RWTH AACHEN UNIVERSITY

Computing State (In-)Equivalence

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- $w = av$: p', q' *inequivalent (by* v*),* $p \xrightarrow{a} p', q \xrightarrow{a} q'$
  $\implies p, q$ *inequivalent (by* w*)*

Algorithm A.55 (State Equivalence for DFA)

Input: DFA $\mathfrak{A} = \langle Q, \Sigma, \Delta, q_0, F \rangle$

Procedure: Computation of "equivalence matrix" over $Q \times Q$

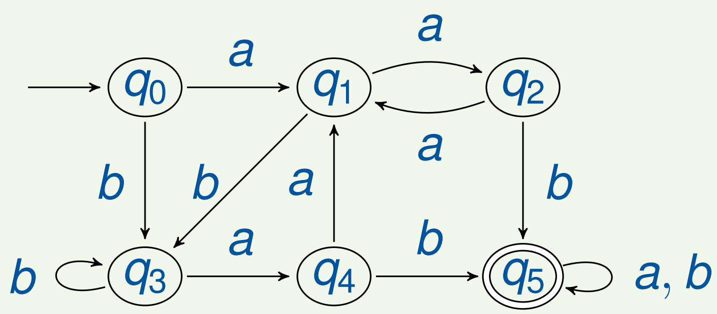
1. *mark every pair* (p, q) *with* $p \in F, q \notin F$ *by* ε
2. *for every unmarked pair* (p, q) *and every* $a \in \Sigma$*:*
   if $(\delta(p, a), \delta(q, a))$ *marked by* v*, then mark* (p, q) *by* av
3. *repeat until no change*

Output: all equivalent (= unmarked) pairs of states

Minimisation Example

Example A.56

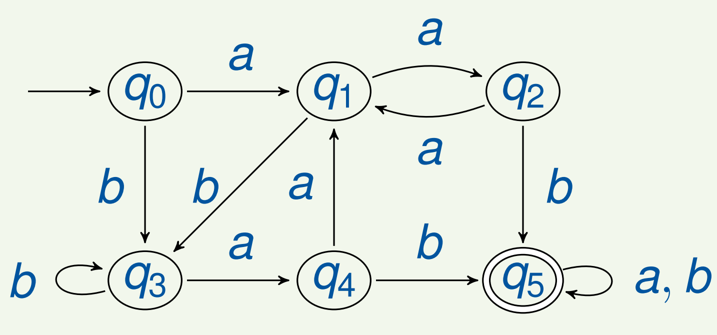
Given DFA:



Minimisation Example

Example A.56

Given DFA:



Equivalence matrix:

	q_0	q_1	q_2	q_3	q_4	q_5
q_0	X					
q_1	X	X				
q_2	X	X	X			
q_3	X	X	X	X		
q_4	X	X	X	X	X	
q_5	X	X	X	X	X	X

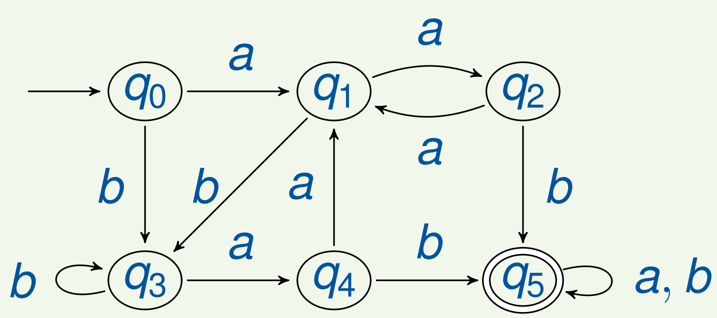
Remarks:

- entries (q_i, q_i) not needed as always equivalent
- entries (q_i, q_j) with $i > j$ not needed due to symmetry

Minimisation Example

Example A.56

Given DFA:



Equivalence matrix:

	q_0	q_1	q_2	q_3	q_4	q_5
q_0	X					ε
q_1	X	X				ε
q_2	X	X	X			ε
q_3	X	X	X	X		ε
q_4	X	X	X	X	X	ε
q_5	X	X	X	X	X	X

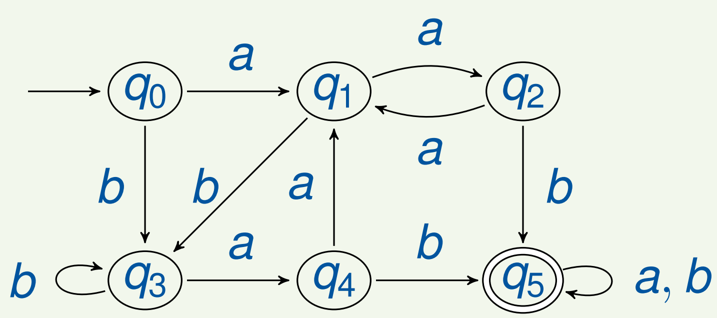
Algorithm A.55:

1. Mark every pair (p, q) with $p \in F, q \notin F$ by ε

Minimisation Example

Example A.56

Given DFA:



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q_1	X	X				ε
q_2	X	X	X			ε
q_3	X	X	X	X		ε
q_4	X	X	X	X	X	ε
q_5	X	X	X	X	X	X

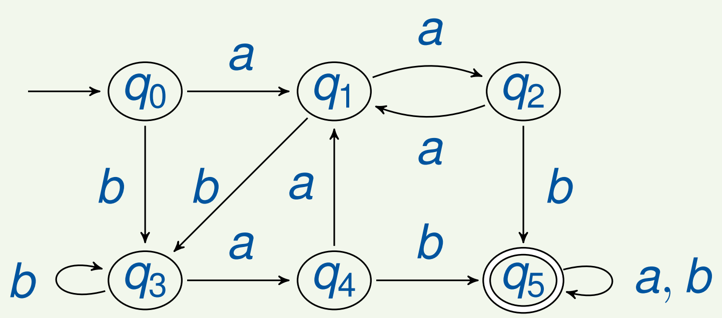
Algorithm A.55:

2. If $(\delta(p, a), \delta(q, a))$ marked by ε, then mark (p, q) by a (not applicable)

Minimisation Example

Example A.56

Given DFA:



Equivalence matrix:

	q_0	q_1	q_2	q_3	q_4	q_5
q_0	X		b		b	ε
q_1	X	X	b		b	ε
q_2	X	X	X	b		ε
q_3	X	X	X	X	b	ε
q_4	X	X	X	X	X	ε
q_5	X	X	X	X	X	X

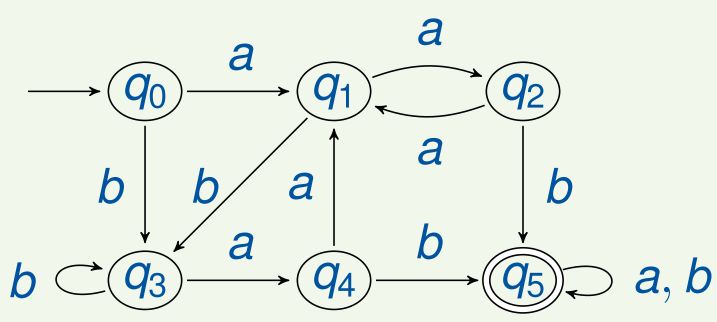
Algorithm A.55:

2. If $(\delta(p, b), \delta(q, b))$ marked by ε, then mark (p, q) by b

Minimisation Example

Example A.56

Given DFA:



Equivalence matrix:

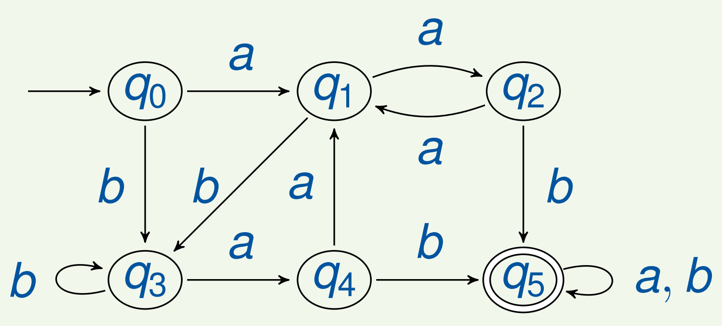
	q_0	q_1	q_2	q_3	q_4	q_5
q_0	X	ab	b	ab	b	ε
q_1	X	X	b		b	ε
q_2	X	X	X	b		ε
q_3	X	X	X	X	b	ε
q_4	X	X	X	X	X	ε
q_5	X	X	X	X	X	X

Algorithm A.55:

2. If $(\delta(p,a),\delta(q,a))$ marked by $c \in \{a,b\}$, then mark (p,q) by ac

Example A.56

Given DFA:



Equivalence matrix:

	q_0	q_1	q_2	q_3	q_4	q_5
q_0	X	ab	b	ab	b	ε
q_1	X	X	b		b	ε
q_2	X	X	X	b		ε
q_3	X	X	X	X	b	ε
q_4	X	X	X	X	X	ε
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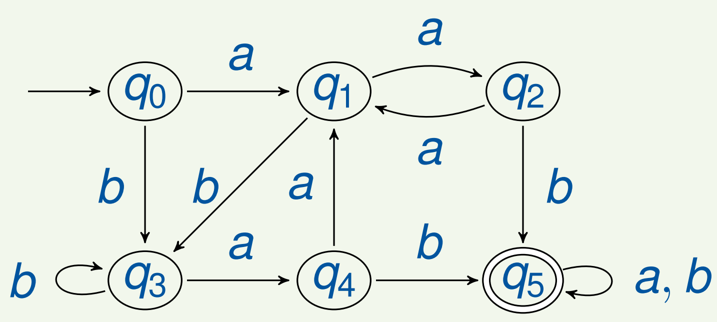
Algorithm A.55:

2. If $(\delta(p, b), \delta(q, b))$ marked by $c \in \{a, b\}$, then mark (p, q) by bc (not applicable)

Minimisation Example

Example A.56

Given DFA:



Equivalence matrix:

	q_0	q_1	q_2	q_3	q_4	q_5
q_0	X	ab	b	ab	b	ε
q_1	X	X	b	✓	b	ε
q_2	X	X	X	b	✓	ε
q_3	X	X	X	X	b	ε
q_4	X	X	X	X	X	ε
q_5	X	X	X	X	X	X

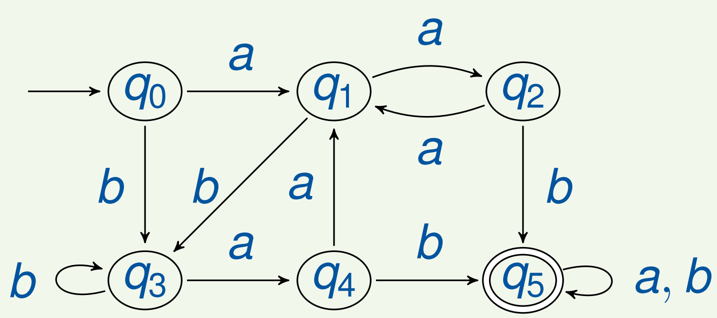
Algorithm A.55:

3. No further changes $\Longrightarrow$ $(q_1, q_3), (q_2, q_4)$ equivalent

Minimisation Example

Example A.56

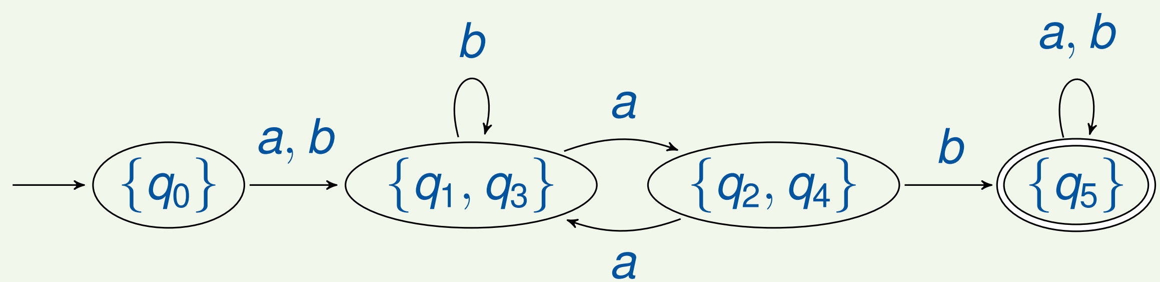
Given DFA:



Equivalence matrix:

	q_0	q_1	q_2	q_3	q_4	q_5
q_0	X	ab	b	ab	b	ε
q_1	X	X	b	✓	b	ε
q_2	X	X	X	b	✓	ε
q_3	X	X	X	X	b	ε
q_4	X	X	X	X	X	ε
q_5	X	X	X	X	X	X

Resulting minimal DFA:



Correctness of Minimisation

Theorem A.57

For every DFA $\mathfrak{A}$,

$$L(\mathfrak{A}) = L(\mathfrak{A}_{min})$$

Correctness of Minimisation

Theorem A.57

For every DFA $\mathfrak{A}$,

$$L(\mathfrak{A}) = L(\mathfrak{A}_{min})$$

Remark: the minimal DFA is unique, in the following sense:

$$\forall \text{DFA } \mathfrak{A}, \mathfrak{B} : L(\mathfrak{A}) = L(\mathfrak{B}) \implies \mathfrak{A}_{min} \approx \mathfrak{B}_{min}$$

where $\approx$ refers to automata isomorphism (= identity up to naming of states)

Outline of Part A

Outlook

- Pumping Lemma (to prove non-regularity of languages)
  - can be used to show that $\{a^n b^n \mid n \geq 1\}$ is not regular
- More language operations (homomorphisms, ...)
- Construction of scanners for compilers