Theorem 1. *Finite $L \subseteq Act^*$ is realizable (by a weak CFM) if and only if L is closed under $\models$.*

$\Longrightarrow$. Assume L is realizable. Thus, there exists a weak CFM A (a CFM without synchronization messages) such that $L = Lin(A)$. As $Lin(A)$ only contains linearizations, and every linearization is well-formed, each word in L is well formed. Let $w \in \texttt{Act}^*$, be well-formed, and assume $L \models w$. By definition of $\models$, this means that for every process p there exists a word $v^p \in L$ such that $v^p \restriction p = w \restriction p$. We show that $w \in L$. (Then it follows that L is closed under $\models$.) This goes as follows.

Let π be an accepting run of CFM A on v^p. (Such run does exist, otherwise, v^p would not belong to L.) Let $\pi \restriction p$ be the projection of run π of A by only considering the transitions along π that take place at process p. Thus, transitions along $\pi_p = \pi \restriction p$ correspond to the "local" transitions of process p. It follows from $v^p \in L$ that the local run π_p is an accepting run of local automaton (NFA) A_p (of process p) on $v^p \restriction p$ (which equals $w \restriction p$). Here, accepting means that the local run π_p ends in a local accept state of A_p. This applies to all processes $P = \{p_1, \ldots, p_n\}$ of the CFM. The local accepting runs $\pi_{p_1} = \pi \restriction p_1, \ldots, \pi_{p_n} = \pi \restriction p_n$ can be combined to obtain a run, π^w say, of CFM A on w in a straightforward manner. The run π^w is accepting, as all processes end in a local accepting state, and all channels are empty, as π was accepting and π^w is constituted from "bits" spanning π. Thus there cannot be "open" receipts. Thus $w \in L$.

$\Longleftarrow$. Assume L is closed under $\models$. As $\models$ is only defined for well-formed words, each word in L is well formed. Moreover, by definition of closure under $\models$, $L \models w$ implies $w \in L$ for each well-formed $w \in Act^*$. Proof obligation: L is realizable. This goes as follows:

Let A_p be an NFA over the alphabet $\texttt{Act}_p$ accepting $L_p = \{w \restriction p \mid w \in L\}$. As L is finite, L_p is finite and regular, thus A_p is indeed an NFA. A_p thus accepts all projections to process p of words in L. Let weak CFM $A = ((A_p)_{p \in P}, s_{init}, F)$ with $F = \prod_{p \in P} F_p$. We now claim that A is a realization of L, i.e., $Lin(A) = L$. This claim can be proven as follows:

- $\supseteq$ Let $w \in L$. By construction of CFM A, $Lin(A_p) = L_p$. But then $w \in Lin(A)$.
- $\subseteq$ Let $w \in Lin(A)$. Then $w \restriction p \in Lin(A_p)$ for each $p \in P$. By definition of $\models$, it follows $L \models w$. Since L is closed under $\models$, it follows $w \in L$.

Theorem 2. *L is safely realizable iff L is weakly closed under $\models$ and closed under $\models^{df}$.*

$\Longrightarrow$. Assume L is safely realizable. Then:

1. L is realizable, and by the theorem of Lecture 13, L is closed under $\models$. As closed under $\models$ implies weakly closed under $\models$, L is weakly closed under $\models$.
2. As L is safely realizable, there exists a deadlock-free weak CFM A with $Lin(A) = L$. As A is weak and deadlock-free, it follows that $Lin(A){=}L$ is closed under $\models^{df}$.

$\Longleftarrow$. Assume L is weakly closed under $\models$ and closed under $\models^{df}$. Let $L_p = \{w \upharpoonright p \mid w \in L\}$ for any process $p \in P$. Since L is finite, L_p is regular. Let DFA A_p (with state set Q_p, initial state s^p_{init} and set F_p of accepting states) be such that $L(A_p) = L_p$. W.l.o.g. assume that all states in A_p are productive, i.e., for any state q in A_p it is possible to reach a state in F_p. Let the weak CFM: $A = ((A_p)_{p\in P}, s_{init}, F)$ with: $s_{init} = \prod_{p\in P} s^p_{init}$, thus $s_{init} = (s^{p_1}_{init}, \ldots, s^{p_n}_{init})$, $F = \prod_{p\in P} F_p$ with $F_p \subseteq Q_p$.

Claim: $Lin(A) = L$ and CFM A is deadlock-free. These claims are proven as follows:

1. $Lin(A) = L$. This is proven by:
   - $\supseteq$. Let $w \in L$. Then, for every process p, $w \upharpoonright p \in L_p$. Thus, DFA A_p has an accepting run for $w \upharpoonright p$ and as $F = \prod_{p\in P} F_p$, CFM A has an accepting run for w. So, $w \in Lin(A)$.
   - $\subseteq$. Let $w \in Lin(A)$. As every word in $Lin(A)$ is well-formed, w is well-formed. Since $F = \prod_{p\in P} F_p$, $w \upharpoonright p \in L_p$ for each process p. Thus $L \models w$. Since L is weakly closed under $\models$, it holds $w \in L$.
2. A is deadlock-free. This is proven as follows. Assume A has successfully read the input word $w \in \texttt{Act}^*$. The word w may be either accepted or not. If it is accepted, there is nothing to prove. Assume w is not accepted. As A has successfully read w, for every process p, $w \upharpoonright p$ is a prefix of a word in L_p. Since L is closed under $\models^{df}$, it follows that $w \in pref(L)$. Let $w.u \in L$ for $u \neq \epsilon$. As A_p is deterministic, it has a unique (local) accepting run for $w.u \upharpoonright p$. This applies to every process p. As $F = \prod_{p\in P} F_p$, it follows that CFM A has a unique accepting run for $w.u$. As this applies to every input word w, A is deadlock-free.

Thus, A is deadlock-free and $Lin(A) = L$. Obviously, then L is safely realizable.