let $\wp$ be an ordering on Var.

(a) For each sw. func. f on Var. $\exists$. a $\wp$-ROBDD $\mathfrak{B}$ with $f_{\mathfrak{B}} = f$

 (b) For any $\wp$-ROBDD $\mathfrak{C}$ with $f_{\mathfrak{C}} = f$,

$\mathfrak{B}$ and $\mathfrak{C}$ are isomorphic.

Proof: ad (a). Clearly, the constant functions 0 and 1 can be represented by a $\wp$-ROBDD consisting of a single drain. Given a nonconstant switching function f for *Var* and a variable ordering $\wp$ for *Var*, we construct a reduced $\wp$-OBDD $\mathfrak{B}$ for f as follows. Let V be the set of $\wp$-consistent cofactors of f. The root of $\mathfrak{B}$ is f. The constant cofactors are the drains with the obvious values. For $f' \in V$, $f' \notin \{0,1\}$, let

$$var(f') = \min_{<_\wp}\{ z \in Var \mid z \text{ is essential for } f' \}$$

i.e. $f'|_{z=0} \neq f'|_{z=1}$

~~be the first essential variable where the minimum is taken according to the total order $<_\wp$ induced by $\wp$~~. (We use here the trivial fact that any nonconstant switching function has at least one essential variable.) The successor functions are given by

$$succ_0(f') = f'|_{z=0}, \quad succ_1(f') = f'|_{z=1}$$

where $z = var(f')$. The definition of $var(\cdot)$ yields that $\mathfrak{B}$ is a $\wp$-OBDD. By the Shannon expansion we get that the semantics of $f' \in V$ (i.e., the switching function of f' as a node of $\mathfrak{B}$) is f'. In particular, this yields that $f_{\mathfrak{B}} = f$ (the function for the root f) and the reducedness of $\mathfrak{B}$ (as any two nodes represent different cofactors of f).

To prove the statement in (b), it suffices to show that any reduced $\wp$-OBDD $\mathfrak{C}$ with $f_{\mathfrak{C}} = f$ is isomorphic to the $\wp$-ROBDD $\mathfrak{B}$ constructed above. Let $V^{\mathfrak{C}}$ be the node set of $\mathfrak{C}$, $v_0^{\mathfrak{C}}$ the root of $\mathfrak{C}$, $var^{\mathfrak{C}}$ the variable labeling function, and $succ_0^{\mathfrak{C}}$, $succ_1^{\mathfrak{C}}$ the successor functions in $\mathfrak{C}$. Let function $\imath : V^{\mathfrak{C}} \to V$ be given by $\imath(v) = f_v$. (Recall that the functions f_v are $\wp$-consistent cofactors of $f_{\mathfrak{C}} = f$. This ensures that $f_v \in V$.) Since $\mathfrak{C}$ is reduced, $\imath$ is a bijection. It remains to show that $\imath$ preserves the variable labeling of inner nodes and maps the successors of an inner node v of $\mathfrak{C}$ to the successors of $f_v = \imath(v)$ in $\mathfrak{B}$.

minimal

Let v be an inner node of $\mathfrak{C}$, say a z-node, and let w_0 and w_1 be the 0- and 1-successors of v in $\mathfrak{C}$. Then, the cofactor $f_v|_{z=0}$ agrees with f_{w_0}, and similarly, $f_{w_1} = f_v|_{z=1}$. (This holds in any OBDD.) As $\mathfrak{C}$ is reduced, f_v is nonconstant (since otherwise $f_v = f_{w_0} = f_{w_1}$). Variable z must be the first essential variable of f_v according to $<_\wp$, i.e., $z = var(f_v)$. Let us see why. Let $y = var(f_v)$. The assumption $z <_\wp y$ yields that z is not essential for f_v, and therefore $f_{w_0} = f_v|_{z=0} = f = f_v|_{z=1} = f_{w_1}$. But then w_0, w_1 and v represent the same function. Since $w_0 \neq v$ and $w_1 \neq v$, this contradicts the assumption that $\mathfrak{C}$ is reduced. The assumption $y <_\wp z$ is also impossible since then no y-node would appear in the sub-OBDD $\mathfrak{C}_v$, which is impossible as $y = var(f_v)$ is essential for f_v by definition.

But then $var(\imath(v)) = z = var^{\mathfrak{C}}(v)$ and, for $b \in \{0,1\}$:

$$succ_b(\imath(v)) = f_v|_{z=b} = f_{succ_b^{\mathfrak{C}}(v)} = \imath(succ_b^{\mathfrak{C}}(v))$$

Hence, $\imath$ is an isomorphism. ■