Proof: ad (a). Clearly, the constant functions 0 and 1 can be represented by a $\varphi$-ROBDD consisting of a single drain. Given a nonconstant switching function $f$ for Var and a variable ordering $\varphi$ for Var, we construct a reduced $\varphi$-OBDD $\mathcal{B}$ for $f$ as follows. Let $V$ be the set of $\varphi$-consistent cofactors of $f$. The root of $\mathcal{B}$ is $f$. The constant cofactors are the drains with the obvious values. For $f' \in V$, $f' \neq \{0, 1\}$, let
\[
\text{var}(f') = \min \{ z \in \text{Var} \mid z \text{ is essential for } f' \} \quad \text{i.e. } f'_{z=0} \neq f'_{z=1}
\]
be the first essential variable where the minimum is taken according to the total order $\varphi$. (We use here the trivial fact that any nonconstant switching function has at least one essential variable.) The successor functions are given by
\[
succ_0(f') = f'_{z=0}, \quad succ_1(f') = f'_{z=1}
\]
where $z = \text{var}(f')$. The definition of $\text{var}(\cdot)$ yields that $\mathcal{B}$ is a $\varphi$-OBDD. By the Shannon expansion we get that the semantics of $f' \in V$ (i.e., the switching function of $f'$ as a node of $\mathcal{B}$) is $f'$. In particular, this yields that $f_\mathcal{B} = f$ (the function for the root $f$) and the reducedness of $\mathcal{B}$ (as any two nodes represent different cofactors of $f$).

To prove the statement in (b), it suffices to show that any reduced $\varphi$-OBDD $\mathcal{C}$ with $f_\mathcal{C} = f$ is isomorphic to the $\varphi$-ROBDD $\mathcal{B}$ constructed above. Let $V^\mathcal{C}$ be the node set of $\mathcal{C}$, $v_0^\mathcal{C}$ the root of $\mathcal{C}$, $\text{var}^\mathcal{C}$ the variable labeling function, and $\text{succ}_0^\mathcal{C}, \text{succ}_1^\mathcal{C}$ the successor functions in $\mathcal{C}$. Let function $\iota: V^\mathcal{C} \to V$ be given by $\iota(v) = v_0$. (Recall that the functions $f_\mathcal{C}$ are $\varphi$-consistent cofactors of $f_\mathcal{C} = f$. This ensures that $f_\mathcal{C} \in V$.) Since $\mathcal{C}$ is reduced, $\iota$ is a bijection. It remains to show that $\iota$ preserves the variable labeling of inner nodes and maps the successors of an inner node $v$ of $\mathcal{C}$ to the successors of $f_\mathcal{C} = \iota(v)$ in $\mathcal{B}$.

Let $v$ be an inner node of $\mathcal{C}$, say a $z$-node, and let $w_0$ and $w_1$ be the 0- and 1-successors of $v$ in $\mathcal{C}$. Then, the cofactor $f_\mathcal{C} = f$ agrees with $f_{w_0}$, and similarly, $f_{w_1} = f_{v_{z=1}}$. (This holds in any OBDD.) As $\mathcal{C}$ is reduced, $f_\mathcal{C}$ is nonconstant (since otherwise $f_\mathcal{C} = f_{w_0} = f_{w_1}$). Variable $z$ must be the first essential variable of $f_\mathcal{C}$ according to $\varphi$, i.e., $z = \text{var}(f_\mathcal{C})$. Let us see why. Let $y = \text{var}(f_\mathcal{C})$. The assumption $z \varphi y$ yields that $z$ is not essential for $f_\mathcal{C}$, and therefore $f_{w_0} = f_{v_{z=0}} = f = f_{v_{z=1}} = f_{w_1}$. But then $w_0, w_1$ and $v$ represent the same function. Since $w_0 \neq v$ and $w_1 \neq v$, this contradicts the assumption that $\mathcal{C}$ is reduced. The assumption $y \varphi z$ is also impossible since then no $y$-node would appear in the sub-OBDD $\mathcal{C}_z$, which is impossible as $y = \text{var}(f_\mathcal{C})$ is essential for $f_\mathcal{C}$ by definition.

But then $\text{var}(\iota(v)) = z = \text{var}^\mathcal{C}(v)$ and, for $b \in \{0, 1\}$:
\[
succ_b^\mathcal{C}(\iota(v)) = f_{\iota(v)_{z=b}} = f_{\text{succ}_b^\mathcal{C}(v)} = \iota(\text{succ}_b^\mathcal{C}(v))
\]
Hence, $\iota$ is an isomorphism. ■