Theorem 9.46. Number of Regions

The number of clock regions is bounded from below and above as follows:

$$|C|! \prod_{x \in C} c_x \leq |\text{Eval}(C)| \leq |C|! \cdot 2^{|C|-1} \prod_{x \in C} (2c_x + 2)$$

where for the upper bound it is assumed that $c_x \geq 1$ for all $x \in C$.

Proof: The lower and upper bounds are determined by considering a representation of clock regions such that there is a one-to-one relationship between the representation of a clock region and the clock region itself. This representation allows derivation of the bounds.

Let $C$ be a set of clocks and $\eta \in \text{Eval}(C)$. Every clock region $r$ can be represented by a tuple $(J, \varphi, D)$ where $J$ is a family of intervals, $\varphi$ is a permutation of a subset of clocks in $C$, and $D \subseteq C$ is a set of clocks such that

- $J = (J_x)_{x \in C}$ is a family of intervals with
  $$J_x \in \{ [0,0], [0,1[, [1,1[, [1,2[, \ldots, [c_x-1,c_x[, [c_x,c_x[, [c_x,\infty[ \}.$$
such that $\eta(x) \in J_x$ for all clocks $x \in C$ and clock evaluations $\eta \in r$.

- Let $C_{\text{open}}$ be the set of clocks $x \in C$ such that $J_x$ is an open interval, i.e.,
  \[ C_{\text{open}} = \left\{ x \in C \mid J_x \in \left\{ \left[0, \; 1[, \; 1, \; 2[, \; \ldots, \; \lfloor c_x - 1, \; c_x[, \; \lfloor c_x, \; \infty[ \right\} \right\}. \]

$\varphi = \{ x_{i_1}, \ldots, x_{i_k} \}$ is a permutation of $C_{\text{open}} = \{ x_1, \ldots, x_k \}$ such that for any $\eta \in r$ the clocks are ordered according to their fractional parts, i.e.,

\[ i_h < i_j \implies \text{frac}(\eta(x_{i_h})) \leq \text{frac}(\eta(x_{i_j})). \]

- $D \subseteq C_{\text{open}}$ contains all clocks in $C_{\text{open}}$ such that for all clock evaluations $\eta' \in [\eta]$ the fractional part for clock $x_{i_{j-1}}$ corresponds to the fractional part for its predecessor $x_{i_j}$ in the permutation $\varphi$:
  \[ x_{i_j} \in D \implies \text{frac}(\eta(x_{i_{j-1}})) = \text{frac}(\eta(x_{i_j})). \]

There is a one-to-one relation between the clock regions and triples $(J, \varphi, D)$.

The indicated upper bound for the number of clock regions is obtained by the following combinatorial observation that there are

- exactly $\prod_{x \in C} (2c_x + 2)$ different interval families $J$,
- maximally $|C_{\text{open}}|! \leq |C|!$ different permutations over $C_{\text{open}}$, and
- maximally $2^{|C_{\text{open}}| - 1} \leq 2^{|C| - 1}$ different choices for $D \subseteq C \setminus \{ x_1 \}$.

The indicated lower bound is obtained when all clocks have a value in an open interval (though not the unbounded interval $\lfloor c_x, \infty[\right\}$, and all have different fractional parts. In this case $D = \varnothing$, and

\[ J_x \in \left\{ \left[0, \; 1[, \; 1, \; 2[, \; \ldots, \; \lfloor c_x - 1, \; c_x[, \right\}. \]

As there are exactly $\prod_{x \in C} c_x$ possibilities for $J$ and maximally $|C|!$ different permutations, the lower bound follows. \hfill \blacksquare