Theorem 9.46. Number of Regions

The number of clock regions is bounded from below and above as follows:

$$|C|! * \prod_{x \in C} c_x \quad \leqslant \quad | \; Eval(C)/\cong \; | \quad \leqslant \quad |C|! * 2^{|C|-1} * \prod_{x \in C} (2c_x + 2)$$

where for the upper bound it is assumed that $c_x \geqslant 1$ for all $x \in C$.

Proof: The lower and upper bounds are determined by considering a representation of clock regions such that there is a one-to-one relationship between the representation of a clock region and the clock region itself. This representation allows derivation of the bounds.

Let C be a set of clocks and $\eta \in Eval(C)$. Every clock region r can be represented by a tuple $\langle J, \wp, D \rangle$ where J is a family of intervals, $\wp$ is a permutation of a subset of clocks in C, and $D \subseteq C$ is a set of clocks such that

- $J = (J_x)_{x \in C}$ is a family of intervals with

$$J_x \;\in\; \Big\{ \; [0,0], \;\;]0,1[, \;\; [1,1], \;\;]1,2[, \;\; \ldots, \;\;]c_x{-}1, c_x[, \;\; [c_x, c_x], \;\;]c_x, \infty[\; \Big\},$$

such that $\eta(x) \in J_x$ for all clocks $x \in C$ and clock evaluations $\eta \in r$.

- Let C_{open} be the set of clocks $x \in C$ such that J_x is an open interval, i,e,

$$C_{open} = \Big\{ x \in C \mid J_x \in \{]0,1[,\ ]1,2[,\ \ldots,\ ]c_x{-}1, c_x[,\ ]c_x,\infty[\} \Big\}.$$

  $\wp = \{ x_{i_1}, \ldots, x_{i_k} \}$ is a permutation of $C_{open} = \{ x_1, \ldots, x_k \}$ such that for any $\eta \in r$ the clocks are ordered according to their fractional parts, i.e.,

$$i_h < i_j \quad \text{imples} \quad \mathit{frac}(\eta(x_{i_h})) \leqslant \mathit{frac}(\eta(x_{i_j})).$$

- $D \subseteq C_{open}$ contains all clocks in C_{open} such that for all clock evaluations $\eta' \in [\eta]$ the fractional part for clock x_{i_j} corresponds to the fractional part for its predecessor $x_{i_{j-1}}$ in the permutation $\wp$:

$$x_{i_j} \in D \text{ implies } \mathit{frac}(\eta(x_{i_{j-1}})) = \mathit{frac}(\eta(x_{i_j})).$$

There is a one-to-one relation between the clock regions and triples $\langle J, \wp, D \rangle$.

The indicated upper bound for the number of clock regions is obtained by the following combinatorial observation that there are

- exactly $\prod_{x \in C} (2c_x + 2)$ different interval families J,
- maximally $|C_{open}|! \leqslant |C|!$ different permutations over C_{open}, and
- maximally $2^{|C_{open}|-1} \leqslant 2^{|C|-1}$ different choices for $D \subseteq C \setminus \{ x_1 \}$.

The indicated lower bound is obtained when all clocks have a value in an open interval (though not the unbounded interval $]c_x, \infty[$), and all have different fractional parts. In this case $D = \varnothing$, and

$$J_x \in \Big\{]0,1[,\ ]1,2[,\ \ldots,\ ]c_x{-}1, c_x[\Big\}.$$

As there are exactly $\prod_{x \in C} c_x$ possibilities for J and maximally $|C|!$ different permutations, the lower bound follows. ■