# On the Complexity of Reachability in Parametric MDPs

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## Overview

Parametric Markov models

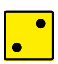
Main contributions

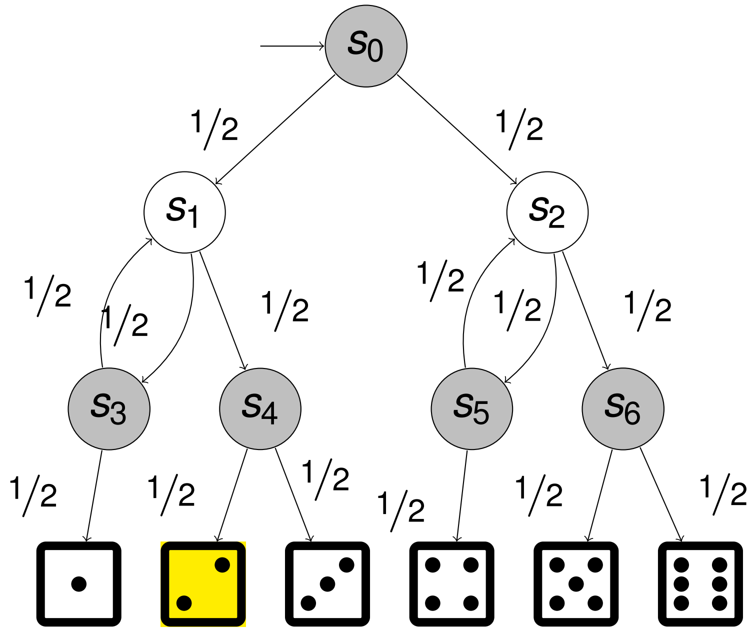
Open problems

Conclusion

## Knuth-Yao Die

Simulate 6-sided die by repeatedly throwing a **fair** coin

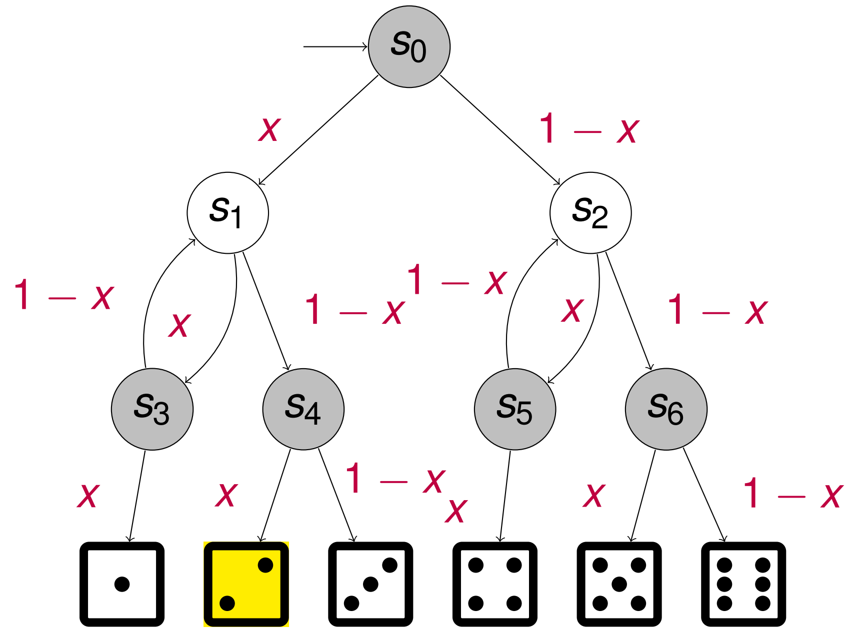
$Pr(reach$  $) = 1/6$ ✓



## Knuth-Yao Die with parametric coin

What if the coin is a little bit **unfair**?

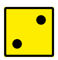
$Pr(reach\ \square) = ?$

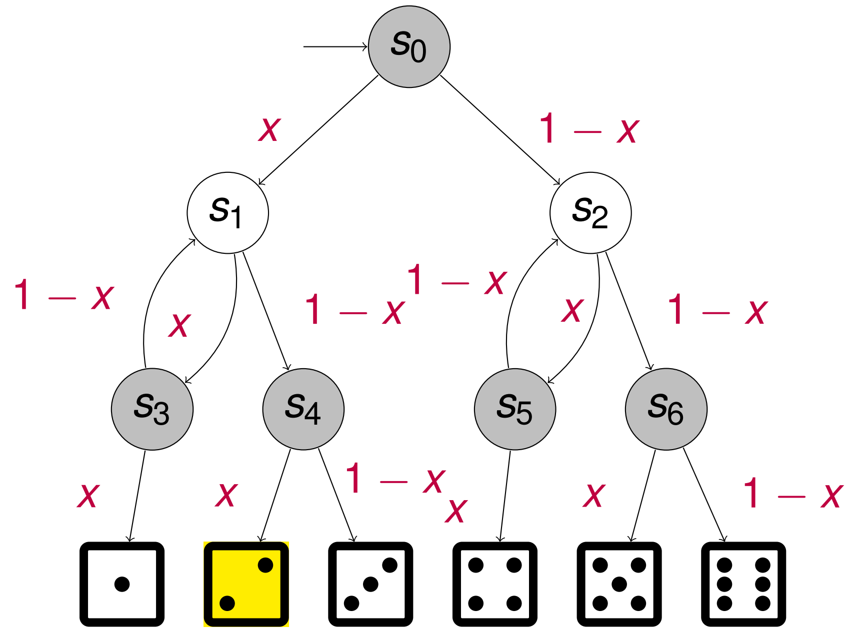



$$x \in \left[\frac{49}{100}, \frac{51}{100}\right] \overset{?}{\Longrightarrow} Pr(reach\ \square) \in \left[\frac{9}{60}, \frac{11}{60}\right]$$

## Knuth-Yao Die with parametric coin

What if the coin is a little bit **unfair**?

$Pr(reach$  $) = \frac{x^2-x^3}{x^2-x+1}$

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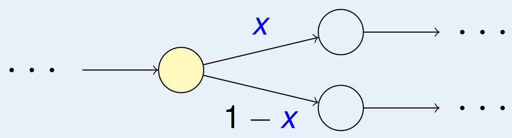
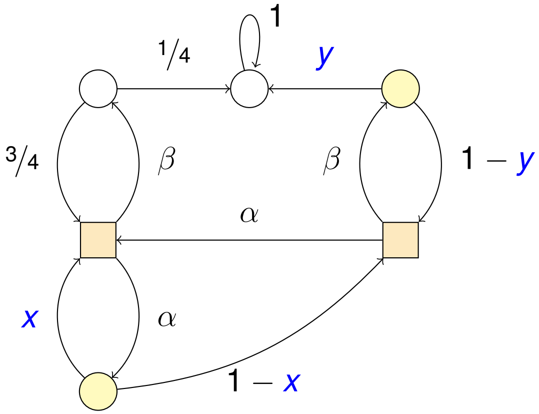
# Parametric Markov models

## Definition (Daws '05, Lanotte et al. '07)

- A **parametric MDP** is an MDP that contains parametric probabilistic branchings of the form




where $x \in Var$, a set of variables.




Scheduler:
$\sigma : S \rightarrow Act$

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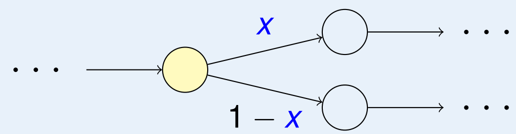
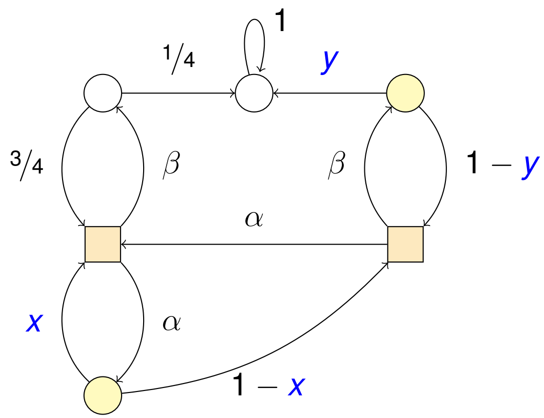



  where $x \in Var$, a set of variables.
- A **parametric Markov chain** is the special case without non-determinism.

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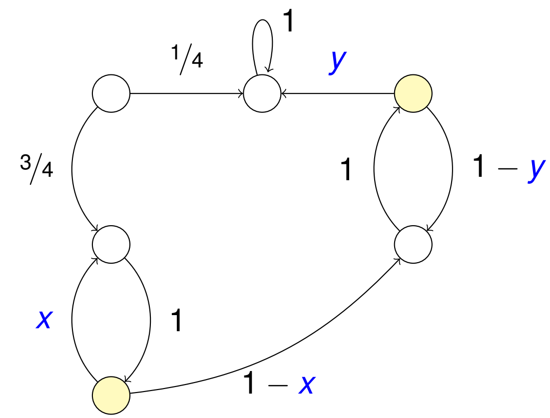
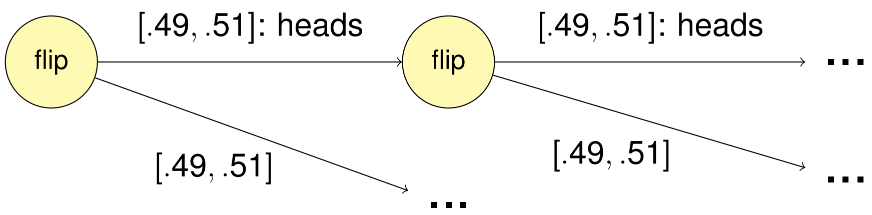


# Parametric Markov models

## Why parametric models matter

- Exact probabilities often not available
- Interval models too pessimistic




- Extensive tool support
  - dedicated tools: *PARAM* [Hahn et al. '10], *PROPhESY* [Dehnert et al. '15]
  - general purpose probabilistic model checkers: *PRISM*, *STORM*, *ePMC*

Many open complexity questions

# Main contributions

## 2 basic formal decision problems for reachability

- $\exists$Reach $\overset{def}{\iff}$ $\exists \vec{x}$: $Pr(reach ☺) \geq 1/2$? (for Markov chains)
- $\exists\forall$Reach $\overset{def}{\iff}$ $\exists \vec{x}$ $\forall \sigma$ : $Pr(reach ☺) \geq 1/2$? (for MDPs)

### Theorem

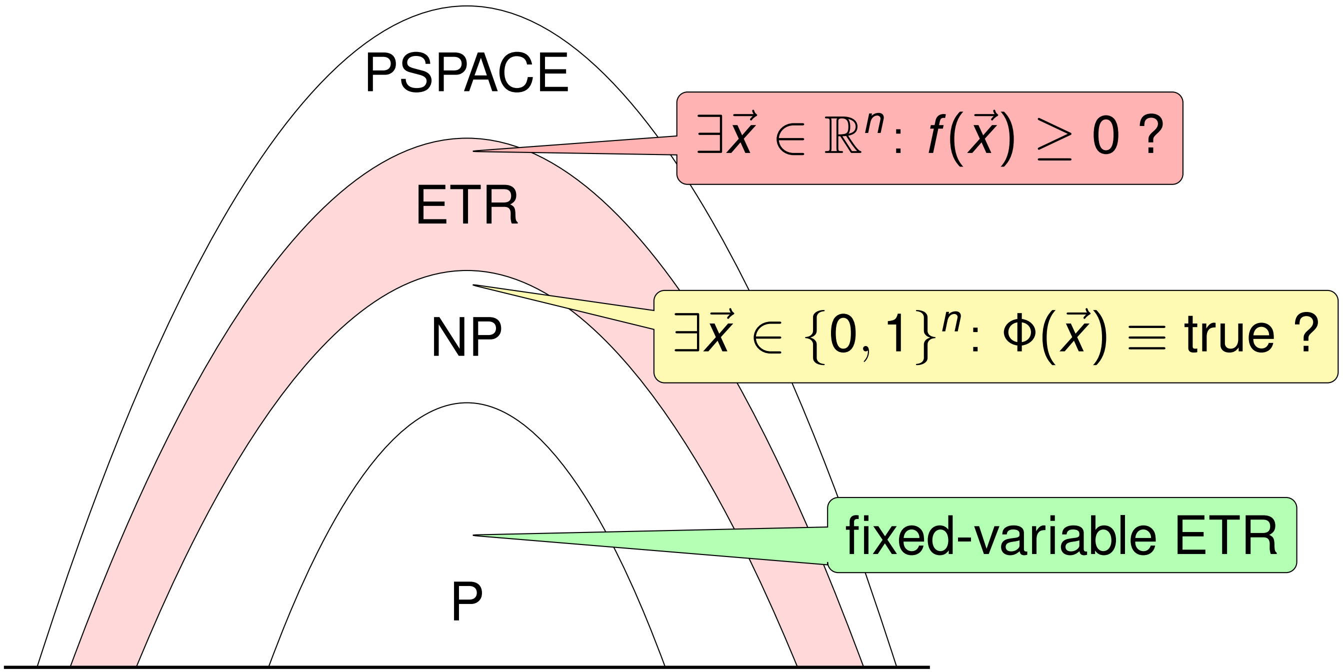
| | # params fixed | # params arbitrary | |
|---|---|---|---|
| $\exists$Reach | in P [Hutschenreiter et al. '17] | ETR-complete | $\leftarrow$ Only for $\geq$, $\leq$ |
| $\exists\forall$Reach | in NP | ETR-complete | $\leftarrow$ For $\geq$, $\leq$, $>$, $<$ |

- Further variants in paper

# Main contributions

## ETR as a complexity class

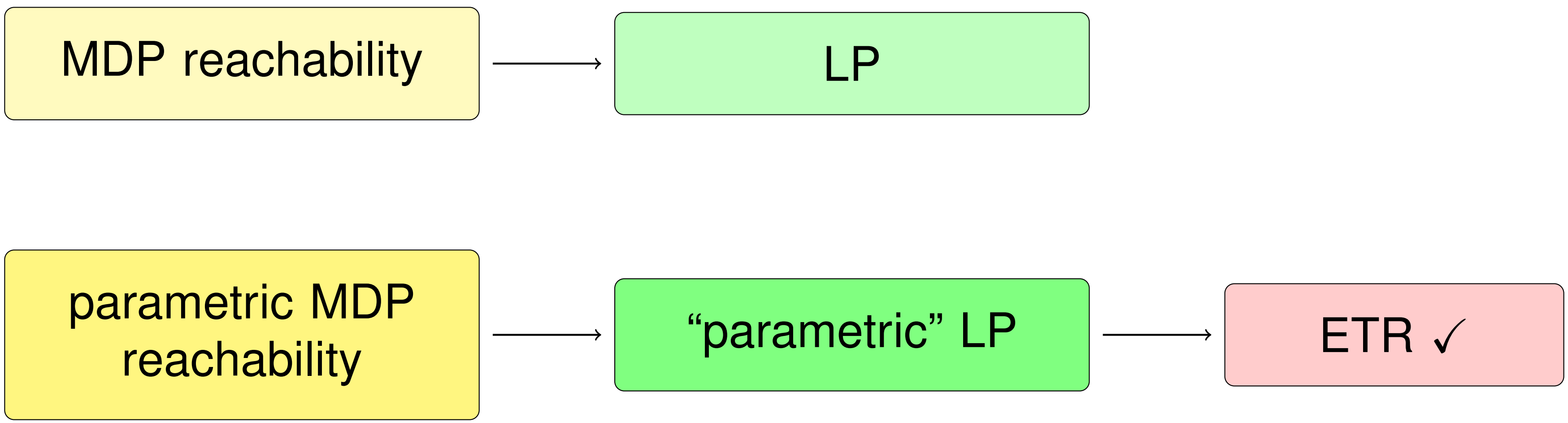
ETR = **E**xistential **T**heory of the **R**eals = $\exists$-fragment of the FO theory $(\mathbb{R}, +, \cdot)$




Also ETR-complete: Several problems about Nash equilibria in 3-player games, problems related to planar graph drawing, and many other problems from topology and geometry

# Main contributions

$\exists\forall\text{Reach} \iff \exists\vec{x}\forall\sigma : Pr(reach \text{☺}) \overset{?}{\geq} \frac{1}{2}$ **is in ETR**



# Main contributions

$\exists$Reach $\iff$ $\exists \vec{x}\colon Pr(reach ☺) \geq \frac{1}{2}$ **is ETR-hard**

ETR $\xrightarrow{\text{red.}}$ $\exists \vec{x} \in \mathbb{R}^n\colon f(\vec{x}) \geq 0$ ? $\xrightarrow{\text{red.}}$ $\exists$Reach

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$\exists$Reach $\iff$ $\exists\vec{x}\colon Pr(reach ☺) \geq \frac{1}{2}$ **is ETR-hard**

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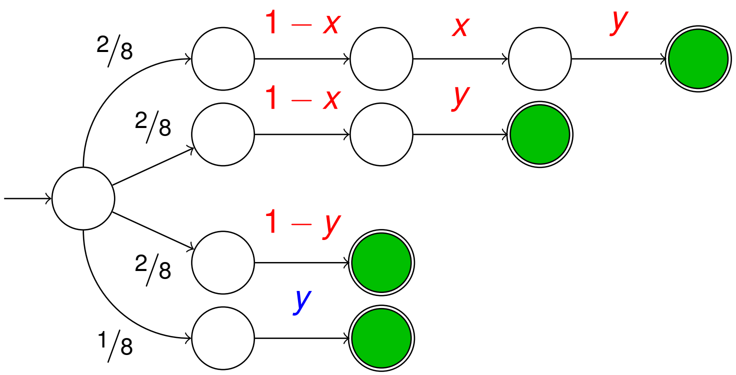
$$-2x^2y + y - 5 \geq 0$$



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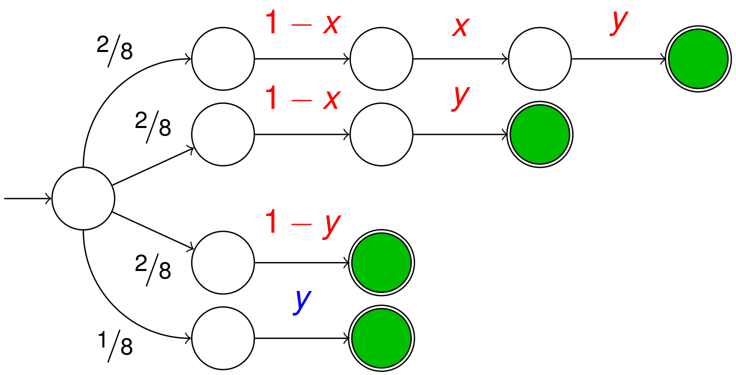
(rewrite) $\Updownarrow$

$$2((1-x)xy + (1-x)y + (1-y) - 1) + y \geq 5$$



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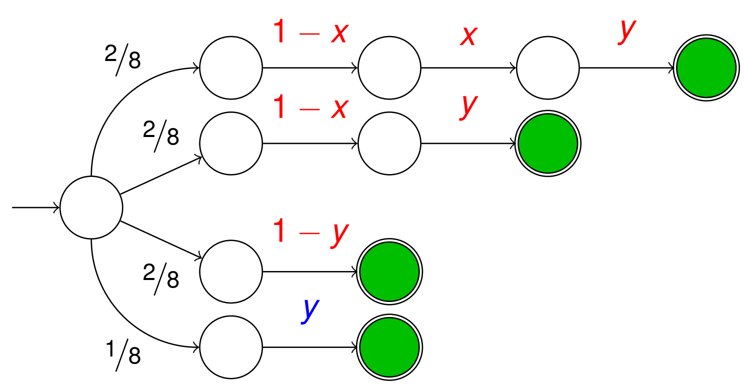
$$2((1-x)xy + (1-x)y + (1-y) - 1) + y \geq 5$$

(scale) $\Updownarrow$

$$\underbrace{\frac{2}{8}(1-x)xy + \frac{2}{8}(1-x)y + \frac{2}{8}(1-y) + \frac{1}{8}y}_{\text{sum of coefficients } \leq 1} \geq \frac{2 \cdot 1 + 5}{8}$$




This “trick” was first observed in [Chonev arXiv ’17]

## Main contributions

### Practice: Often just a few parameters

Recall: fixed-variable ETR in P

| | *# params fixed* | *# params arbitrary* |
|---|---|---|
| ∃Reach | in P [Hutschenreiter et al. '17] | ETR-complete |
| ∃∀Reach | **in NP** | ETR-complete |

Lower complexity for fixed number of parameters ✓

# Main contributions

$\exists\forall$Reach $\iff \exists\vec{x}\forall\sigma: Pr(reach ☺) \geq \frac{1}{2}$ **is in NP (for fixed # of params)**

- Use good parameters as polynomial certificate?
- Use a scheduler instead – which?

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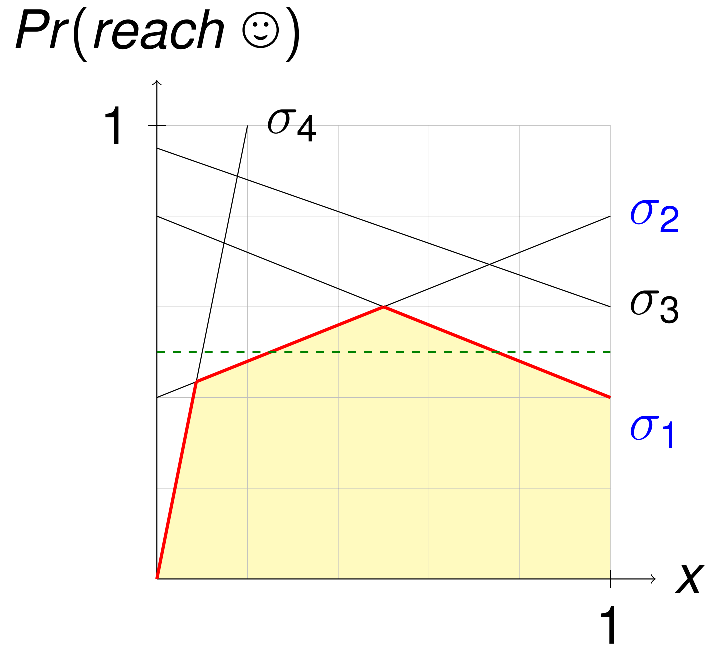

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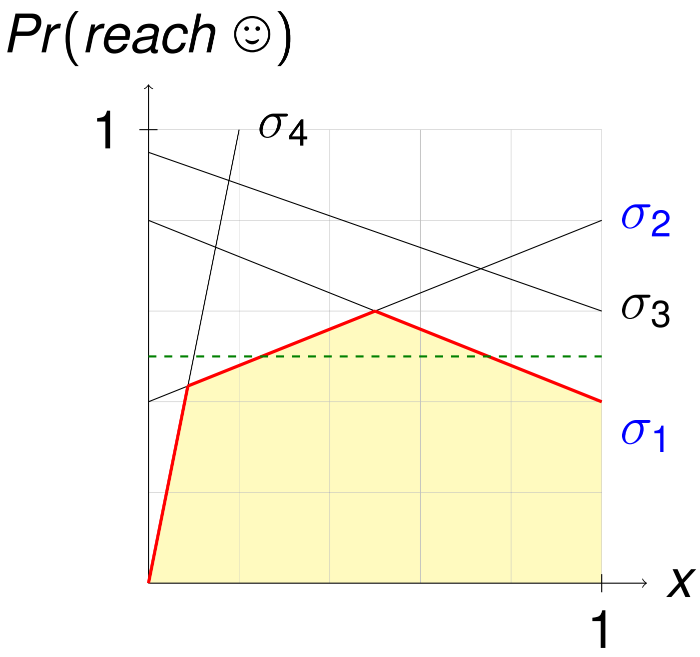
Check $\sigma$ via fixed-param ETR query

$$\bigwedge_{s,a} Pr^{\sigma}(\text{reach ☺ from s}) \leq \sum_{s'} P(s,a,s') \underbrace{Pr^{\sigma}(\text{reach ☺ from } s')}_{\text{fixed-param rational funct.}}$$



# Main contributions

## More refined results in paper

| | | *# params fixed* | *# params arbitrary* | |
|---|---|---|---|---|
| | | | well-defined, [0, 1] | graph-preserving, (0, 1) |
| pMC | $\exists$Reach$^{\geq/\leq}$ | in P | — ETR-complete — | |
| | $\exists$Reach$^{>}$ | ” | NP-hard | $\exists$Reach$^{>}_{\mathrm{wd}}$-complete |
| | $\exists$Reach$^{<}$ | ” | ” | $\exists$Reach$^{>}_{\mathrm{wd}}$-complete |
| pMDP | $\exists\exists$Reach$^{\geq/\leq}$ | in NP | — ETR-complete — | |
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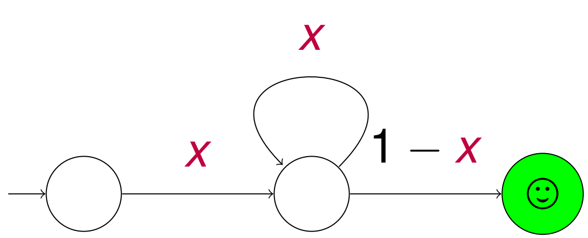
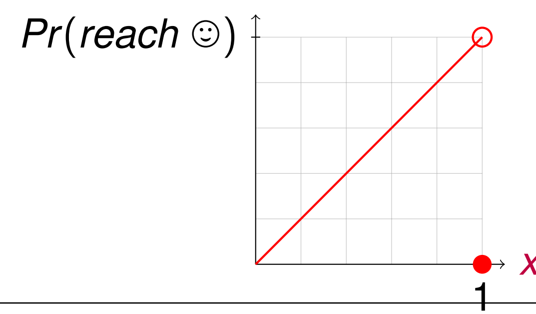
| | $\exists\exists$Reach$^{<}$ | ” | $\exists$Reach$^{<}_{\mathrm{wd}}$-complete | $\exists$Reach$^{>}_{\mathrm{wd}}$-hard |
| | $\exists\forall$Reach$^{\bowtie}$ | in NP | — ETR-complete — | |



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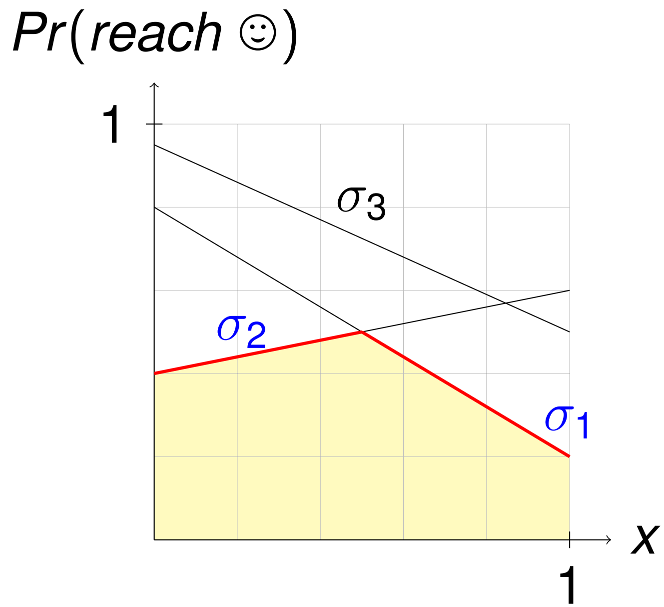
Additionally: Robust strategies, i.e. $\exists\sigma\forall\vec{x}\colon Pr(reach☺) \geq \frac{1}{2}$ under *deterministic memoryless* schedulers

# Open problems

## 1. Better complexity bounds

| | *# params fixed* | ... |
|---|---|---|
| ∃Reach | in P [Hutschenreiter et al. '17] | ... |
| ∃∀Reach | in NP ← tight? | ... |

Can we show a *coNP* upper bound on fixed-param-∃∀Reach ?



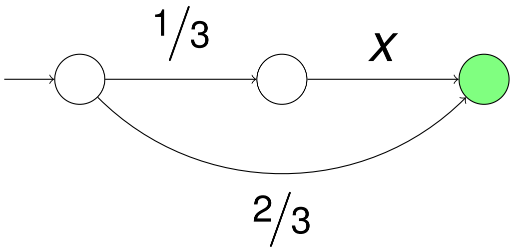

$\{\sigma_1, \sigma_2\}$ = minimal optimal scheduler set

∃ polynomially sized optimal scheduler set $\implies$ ∃∀Reach ∈ *coNP*

# Open problems

## 2. Connection pMC $\longleftrightarrow$ polynomials

- $Pr(reach ☺)$ is a polynomial for acyclic pMCs
- For which polynomials $f$ exists a pMC with $Pr(reach ☺) = f$?




$\Rightarrow \frac{1}{3}x + \frac{2}{3}$

No pMC for $-2x^2y + y - 5$

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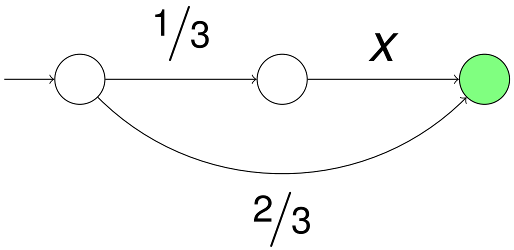


$\Rightarrow \frac{1}{3}x + \frac{2}{3}$

No pMC for $-2x^2y + y - 5$

We showed for univariate $f$:

> If $f(x) \in (0,1)$ for $x \in (0,1)$, then there is a pMC with $Pr(reach ☺) = f$.

Questions:

- How big is the resulting pMC?
- What about multivariate polynomials?

# Conclusion

Acyclic Markov chains with parametric $x/1\text{-}x$ transitions are already **hard**, even for *graph-preserving* parameter valuations.

Any Boolean combination of *polynomial* constraints can be encoded in a pMC reachability problem.

A *fixed number of parameters* implies lower complexity for both pMC & pMDP.

**Thank you for your attention!**