On the Complexity of Reachability in Parametric MDPs

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Overview

Parametric Markov models

Main contributions

Open problems

Conclusion
Knuth-Yao Die

Simulate 6-sided die by repeatedly throwing a fair coin

\[ \Pr(\text{reach } \square ) = \frac{1}{6} \]
Parametric Markov models

Knuth-Yao Die with parametric coin

What if the coin is a little bit unfair?

\[ \Pr(\text{reach } \text{∅}) = ? \]

\[
x \in \left[ \frac{49}{100}, \frac{51}{100} \right] \Rightarrow \Pr(\text{reach } \text{∅}) \in \left[ \frac{9}{60}, \frac{11}{60} \right]
\]
Parametric Markov models

Knuth-Yao Die with parametric coin

What if the coin is a little bit \textit{unfair}?

\[ Pr(\text{reach } \heartsuit) = \frac{x^2 - x^3}{x^2 - x + 1} \]

\[ x \in \left[ \frac{49}{100}, \frac{51}{100} \right] \quad \Rightarrow \quad Pr(\text{reach } \heartsuit) \in \left[ \frac{9}{60}, \frac{11}{60} \right] \]
Definition (Daws ’05, Lanotte et al. ’07)

A **parametric MDP** is an MDP that contains parametric probabilistic branchings of the form

\[
\begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\end{array}
\]

\[
\begin{array}{c}
x \\
1 - x \\
\end{array}
\]

where \( x \in \text{Var} \), a set of variables.

**Scheduler:**

\[ \sigma : S \rightarrow \text{Act} \]
Definition (Daws ’05, Lanotte et al. ’07)

- A **parametric MDP** is an MDP that contains **parametric** probabilistic branchings of the form

  ![Diagram](image)

  where $x \in \text{Var}$, a set of variables.

- A **parametric Markov chain** is the special case without non-determinism.

Scheduler: 
$\sigma : S \rightarrow \text{Act}$
Parametric Markov models

Why parametric models matter

- Exact probabilities often not available
- Interval models too pessimistic

- Extensive tool support
  - dedicated tools: PARAM [Hahn et al. ’10], PROPhESY [Dehnert et al. ’15]
  - general purpose probabilistic model checkers: PRISM, STORM, ePMCE

Many open complexity questions
Main contributions

2 basic formal decision problems for reachability

- $\exists \text{Reach} \iff \exists \vec{x} : \Pr(\text{reach} \odot) \geq \frac{1}{2}$? (for Markov chains)
- $\exists \forall \text{Reach} \iff \exists \vec{x} \forall \sigma : \Pr(\text{reach} \odot) \geq \frac{1}{2}$? (for MDPs)

Theorem

<table>
<thead>
<tr>
<th>$\exists \text{Reach}$</th>
<th>$\exists \forall \text{Reach}$</th>
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</thead>
<tbody>
<tr>
<td># params fixed</td>
<td># params arbitrary</td>
</tr>
<tr>
<td>$\text{in P}$ [Hutschenreiter et al. '17]</td>
<td>ETR-complete ← Only for $\geq, \leq$</td>
</tr>
<tr>
<td>in NP</td>
<td>ETR-complete ← For $\geq, \leq, &gt;, &lt;$</td>
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Further variants in paper
Main contributions

ETR as a complexity class

\[ \text{ETR} = \text{E}x\text{istential } \text{T}heory \text{ of the } \text{R}eals = \exists\text{-fragment of the FO theory } (\mathbb{R}, +, \cdot) \]

Also ETR-complete: Several problems about Nash equilibria in 3-player games, problems related to planar graph drawing, and many other problems from topology and geometry.
Main contributions

\[ \exists \forall \text{Reach} \iff \exists \forall \sigma: Pr(\text{reach } \text{Smiley}) \geq \frac{1}{2} \text{ is in ETR} \]

Diagram:

- MDP reachability \rightarrow LP
- parametric MDP reachability \rightarrow “parametric” LP \rightarrow ETR ✓
Main contributions

\[ \exists \text{Reach} \iff \exists \vec{x} : Pr(\text{reach } \odot) \geq \frac{1}{2} \text{ is ETR-hard} \]

ETR \quad \text{red.} \quad \exists \vec{x} \in \mathbb{R}^n: f(\vec{x}) \geq 0 \quad \text{red.} \quad \exists \text{Reach}
Main contributions

\[ \exists \text{Reach} \iff \exists \vec{x}: \Pr(\text{reach } \ominus) \geq \frac{1}{2} \text{ is ETR-hard} \]

\[ \text{ETR} \xrightarrow{\text{red.}} \exists \vec{x} \in [0, 1]^n: f(\vec{x}) \geq 0? \xrightarrow{\text{red.}} \exists \text{Reach} \]

This "trick" was first observed in [Chonevar ArXiv '17].
Main contributions

∃Reach $\iff$ $\exists \vec{x} : Pr(\text{reach } \otimes) \geq \frac{1}{2}$ is ETR-hard

ETR↓ red. $\exists \vec{x} \in [0, 1]^n : f(\vec{x}) \geq 0$ ? red. $\exists \text{Reach}$

$-2x^2y + y - 5 \geq 0$

This "trick" was first observed in [Chonevar ArXiv '17]
Main contributions

\[ \exists \text{Reach} \iff \exists \vec{x} : \Pr(\text{reach } \circledast) \geq \frac{1}{2} \text{ is ETR-hard} \]

**ETR** \(\xrightarrow{\text{red.}}\) \(\exists \vec{x} \in [0, 1]^n : f(\vec{x}) \geq 0 ?\) \(\xrightarrow{\text{red.}}\) \(\exists \text{Reach}\)

\[-2x^2y + y - 5 \geq 0\]

(rewrite) \(\Updownarrow\)

\[2((1 - x)xy + (1 - x)y + (1 - y) - 1) + y \geq 5\]

This "trick" was first observed in [Chonevar arXiv '17](#)
Main contributions

$\exists \text{Reach} \iff \exists \vec{x}: Pr(\text{reach} \odot) \geq \frac{1}{2}$ is ETR-hard

This “trick” was first observed in [Chonev arXiv ’17]
Main contributions

Practice: Often just a few parameters

Recall: fixed-variable ETR in P

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<td>in P [Hutschenreiter et al. ‘17]</td>
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<tr>
<td>( \forall \text{Reach} )</td>
<td>in NP</td>
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Lower complexity for fixed number of parameters ✓
Main contributions

\( \exists \forall \text{Reach} \iff \exists \check{x} \forall \sigma: \Pr(\text{reach} \; \check{x}) \geq \frac{1}{2} \) is in NP (for fixed # of params)

- Use good parameters as polynomial certificate?
- Use a scheduler instead – which?
Main contributions

∃∀Reach \iff \exists \vec{x} \forall \sigma: Pr(reach \, \sigma) \geq \frac{1}{2} \text{ is in NP (for fixed # of params)}

- Use good parameters as polynomial certificate?
- Use a scheduler instead – which? → a minimal one

\[
Pr(reach \, \sigma)
\]

\[
\begin{align*}
\sigma_1 & \quad \sigma_2 & \quad \sigma_3 & \quad \sigma_4 \\
1 & \quad 1 & \quad 1 & \quad 1
\end{align*}
\]
Main contributions

\[ \exists \forall \text{Reach} \iff \exists \forall \sigma : \Pr(\text{reach } \nexists) \geq \frac{1}{2} \text{ is in NP (for fixed # of params)} \]

- Use good parameters as polynomial certificate?
- Use a scheduler instead – which? \(\rightarrow\) a minimal one

Check \(\sigma\) via fixed-param ETR query

\[ \bigwedge_{s,a} \Pr^\sigma(\text{reach } \nexists \text{ from } s) \leq \sum_{s'} P(s, a, s') \Pr^\sigma(\text{reach } \nexists \text{ from } s') \]

\(Pr(\text{reach } \nexists)\)

fixed-param rational funct.
Main contributions

More refined results in paper

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Pr(\( \text{reach } \odot \))
Main contributions

More refined results in paper

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Additionally: Robust strategies, i.e. $\exists\sigma\forall\vec{x}: Pr(\text{reach} \odot) \geq \frac{1}{2}$ under deterministic memoryless schedulers
Open problems

1. Better complexity bounds

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<td>( \forall ) Reach ( \in \mathbf{NP} \leftarrow \text{tight?} )</td>
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Can we show a \( \mathbf{coNP} \) upper bound on fixed-param-
\( \forall \) Reach?

\( \{ \sigma_1, \sigma_2 \} = \text{minimal optimal scheduler set} \)

\( \exists \) polynomially sized optimal scheduler set \( \implies \forall \) Reach \( \in \mathbf{coNP} \)
Open problems

2. Connection pMC $\leftrightarrow$ polynomials

- $Pr(\text{reach } \odot)$ is a polynomial for acyclic pMCs
- For which polynomials $f$ exists a pMC with $Pr(\text{reach } \odot) = f$?

$$\frac{1}{3} x + \frac{2}{3}$$

No pMC for $-2x^2y + y - 5$
2. Connection pMC $\leftrightarrow$ polynomials

- $Pr(\text{reach } \odot)$ is a polynomial for acyclic pMCs
- For which polynomials $f$ exists a pMC with $Pr(\text{reach } \odot) = f$?

We showed for univariate $f$:

If $f(x) \in (0, 1)$ for $x \in (0, 1)$, then there is a pMC with $Pr(\text{reach } \odot) = f$.

Questions:
- How big is the resulting pMC?
- What about multivariate polynomials?
Conclusion

Acyclic Markov chains with parametric $x/1-x$ transitions are already hard, even for graph-preserving parameter valuations.

Any Boolean combination of polynomial constraints can be encoded in a pMC reachability problem.

A fixed number of parameters implies lower complexity for both pMC & pMDP.

Thank you for your attention!