

Theorem : the decision problem "does CMSG G have a safe, accepting path" is undecidable.

Proof: by a reduction from the Post's correspondence problem (PCP). Proof idea:

PCP instance $(U, W) \mapsto$ CMSG $G_{U, W}$

such that (U, W) has a solution if and only if

$G_{U, W}$ has a safe, accepting path.

let $U = \{u_1, u_2, \dots, u_n\}$ and $W = \{w_1, \dots, w_n\}$ where $u_i \in \Sigma^*$, $w_i \in \Sigma^*$ for some alphabet Σ .

Components of the CMSG $G_{U, W}$:

$P = \{P_1, P_2, P_3, P_4\}$, set of processes

$C = \Sigma \cup \{end\} \cup \{1, \dots, n\}$, set of message contents

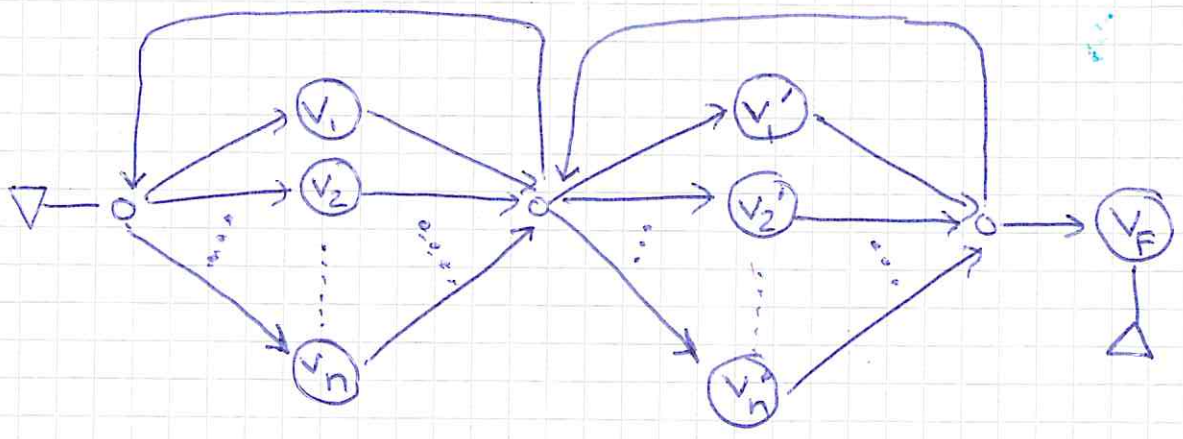
$V = \{v_1, \dots, v_n\} \cup \{v'_1, \dots, v'_n\} \cup \{v_F\}$

$F = \{v_F\}$

$\lambda(v_i) =$ CMSG that "corresponds" to the word u_i

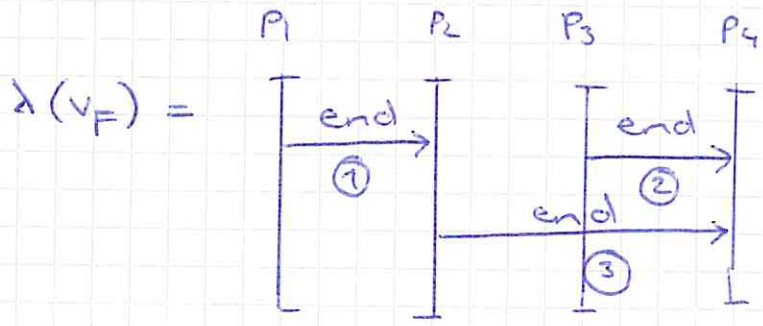
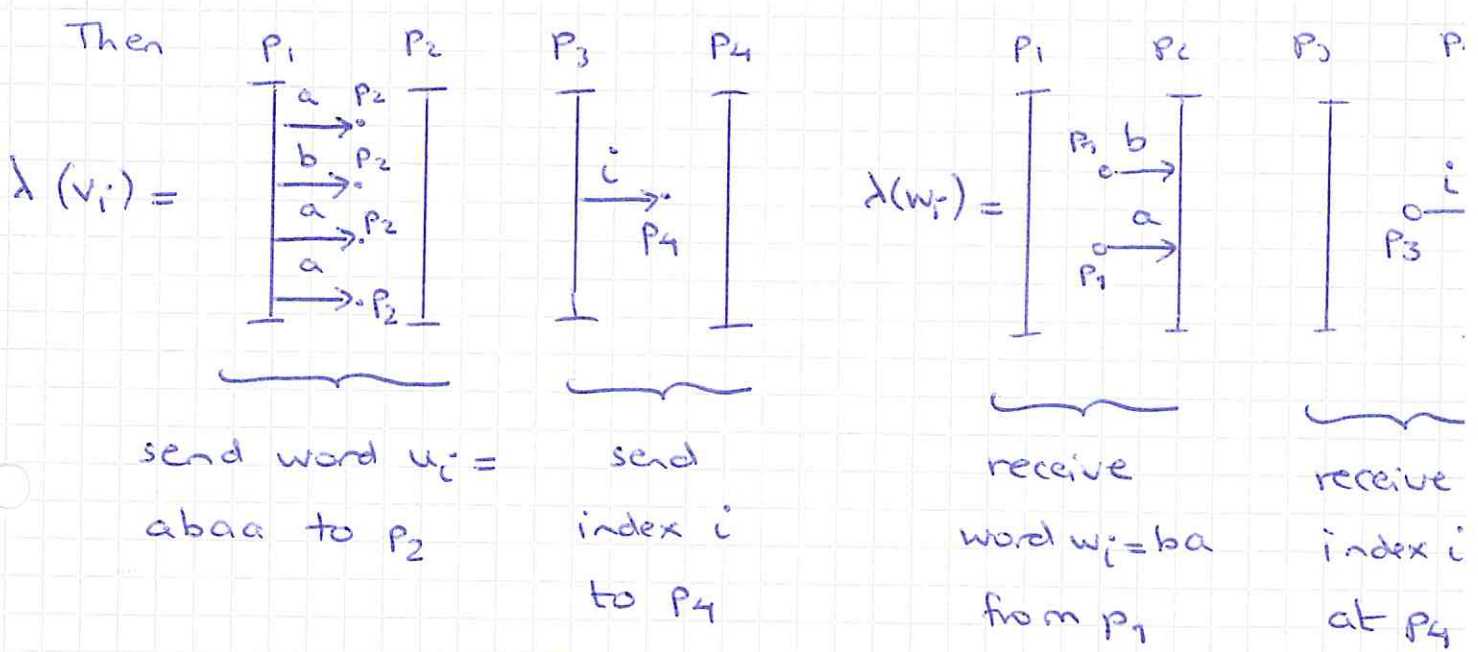
$\lambda(v'_i) =$ " " " " " " " w_i

$G_{U, W}$:



How do the vertices v_i^o , v_i^i and v_F look like?

By example. let $\Sigma = \{a, b\}$, $u_i = abaa$, $w_i = ba$



- Interpretation:
- ① indicates that P_1 has sent all its messages to P_2 and if ① is received by P_2 , all receive events — possibly unmatched — have taken place at process P_2
 - ② similar as ①, but now for the "index" messages (belonging to $\{1, \dots, n\}$)
 - ③ indicates that all possible message transfers of symbols in Σ from P_1 to P_2 and all "index" messages between P_3 and P_4 have been transferred.

It remains to prove that the reduction:

$$\text{PCP instance } (U, W) \longmapsto \text{CMSSG } G_{U, W}$$

is correct. That is, we need to prove that:

$$(U, W) \text{ has a solution} \iff G_{U, W} \text{ has a safe accepting path}$$

Proof: " \Rightarrow ": Let index sequence i_1, i_2, \dots, i_k be a solution of PCP (U, W) . Then there is an accepting

path $\pi = \underbrace{V_{i_1} V_{i_2} \dots V_{i_k}}_{\text{traverse the vertices corresponding to } u_{i_1}, u_{i_2}, \dots, u_{i_k}} \underbrace{V'_{i_1} V'_{i_2} \dots V'_{i_k}}_{\text{traverse the vertices corresponding to } w_{i_1}, w_{i_2}, \dots, w_{i_k}} V_F$ in $G_{U, W}$

As i_1, i_2, \dots, i_k is a solution of PCP instance (U, W) , and by construction of vertices V_{i_j} and V'_{i_j} it follows that

$$\lambda(V_{i_1}) \circ \lambda(V_{i_2}) \circ \dots \circ \lambda(V_{i_k}) \circ \lambda(V'_{i_1}) \circ \lambda(V'_{i_2}) \circ \dots \circ \lambda(V'_{i_k})$$

(all left-bracketed; remember that \circ is not associative on CMSSGs) is an MSC. Intuitively, this follows from the

fact that all messages sent in $\lambda(V_{i_1}) \circ \dots \circ \lambda(V_{i_k})$ are all received in $\lambda(V'_{i_1}) \circ \dots \circ \lambda(V'_{i_k})$, as otherwise

i_1, i_2, \dots, i_k would not be solution to PCP instance (U, W)

Thus: $\boxed{\pi \text{ is safe and accepting}}$

" \Leftarrow "

Let π be a safe, accepting path in $G_{u,w}$.

Assume

$$\pi = \underbrace{v_{i_1} v_{i_2} \dots v_{i_m}}_{\text{traverse first part of } G_{u,w} \text{ } m \text{ times}} \underbrace{v'_{j_1} v'_{j_2} \dots v'_{j_k}}_{\text{traverse second part of } G_{u,w} \text{ } k \text{ times}} v_F$$

with $i_1, \dots, i_m \in \{1, \dots, n\}$ and $j_1, \dots, j_k \in \{1, \dots, n\}$.

Since π is safe and ends in vertex v_F it follows:

1) as (p_4, p_3, end) occurs in v_F , all unmatched sends by process p_3 in sub-path $v_{i_1} \dots v_{i_m}$ are matched by corresponding ~~sends~~ receipts by p_4 in the sub-path $v'_{j_1} \dots v'_{j_k}$. As in each vertex v_{i_ℓ}

one message is sent from p_3 , and in each vertex v'_{j_ℓ} one message is received at p_4 (from p_3), it

follows that

$$\underbrace{|v_{i_1} \dots v_{i_m}|}_{\text{length of } v_{i_1} \dots v_{i_m}} = \underbrace{|v'_{j_1} \dots v'_{j_k}|}_{\text{length of } v'_{j_1} \dots v'_{j_k}}$$

and thus $m = k$.

5

2) As π is safe it follows that

$V_{i_1} \dots V_{i_m} \quad V'_{j_1} \dots V'_{j_m}$ is safe and FIF

Thus all "index" messages i_1, i_2, \dots, i_m sent by

P_3 in $V_{i_1} \dots V_{i_m}$ are received by process P_4 in

$V'_{j_1} \dots V'_{j_m}$ in the same order as they were sent.

Thus $i_1 = j_1, i_2 = j_2, \dots, i_m = j_m$.

So: $\pi = V_{i_1} V_{i_2} \dots V_{i_m} V'_{i_1} V'_{i_2} \dots V'_{i_m} V_F$ is a safe, accepting path in $G_{U,W}$

As π is completed by $? (P_4, P_2, \text{end})$ and $! (P_2, P_4, \text{end})$

can only occur after $? (P_2, P_1, \text{end})$ it follows that

once P_4 has received all "index" messages, P_2 has

received all messages sent by process P_1 in $V_{i_1} V_{i_2} \dots V_{i_m}$.

Process P_1 has sent the words $u_{i_1}, u_{i_2} \dots u_{i_m}$ (to P_2).

Process P_2 received the words $w_{i_1}, w_{i_2} \dots w_{i_m}$.

Since π is safe, it follows $u_{i_1}, u_{i_2} \dots u_{i_m} = w_{i_1}, w_{i_2} \dots w_{i_m}$

Thus: i_1, i_2, \dots, i_m is a solution to the PCP instance (U, W)

⊠