C5: The Emptiness Problem for Context-Free Languages

Task: Using the marking algorithm, check whether the following grammars generate empty languages or not:

(a) $S \to XU \mid UW$
$U \to SaW \mid VbU \mid SbX$
$V \to U \mid ab \mid XW$
$W \to bXaUb \mid VbX \mid SX$
$X \to bV \mid WX$

(b) $S \to AC \mid DA$
$A \to SbC \mid BaB \mid AaD$
$B \to A \mid ba \mid DC$
$C \to aDbAa \mid BaD \mid SD$
$D \to aB \mid CD$

1st iteration
2nd iteration
3rd — " —
4rd — " —

Idea: $L(G) \neq \emptyset$ $(G = (N, \Sigma, P, S))$
$\iff S$ productive
where $A \in N$ productive
$\iff \exists w \in \Sigma^* : A \Rightarrow^* w$

1. mark all $a \in \Sigma$
2. iterate: whenever α fully marked and $A \to \alpha \Rightarrow$ mark A

(a) S not productive $\Rightarrow L(G) = \emptyset$
(b) S productive $\Rightarrow L(G) \neq \emptyset$
$S \Rightarrow DA \Rightarrow aBA \Rightarrow abaA$
$\Rightarrow abaBaB \Rightarrow^2 abababaaba$

C6: Closure Properties of Context-Free Languages

Task: Show that context-free languages are closed under the reversal operation.

$w = a_1 \dots a_n \Rightarrow w^R := a_n \dots a_1$

$L^R := \{ w^R \mid w \in L \}$

To show: $L \in CFL \Rightarrow L^R \in CFL$

Proof: Let $L = L(G)$ for

$G = (N, \Sigma, P, S)$

Let $G^R = (N, \Sigma, P', S)$

where $P' = \{ A \to \alpha^R \mid A \to \alpha \in P \}$

Example:

$G: S \to aSb \mid \varepsilon \quad (L = \{a^n b^n \mid n \in \mathbb{N}\})$

$\Rightarrow G^R: S \to bSa \mid \varepsilon$

$(\{b^n a^n \mid n \in \mathbb{N}\})$

$= L^R$

Show: L regular $\Rightarrow L^R$ regular

Proof: (a) Using DFA:

$L = L(\mathcal{A})$ where $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$

Construct ε-NFA $\mathcal{A}^R = (Q', \Sigma, \Delta', q_0', F')$

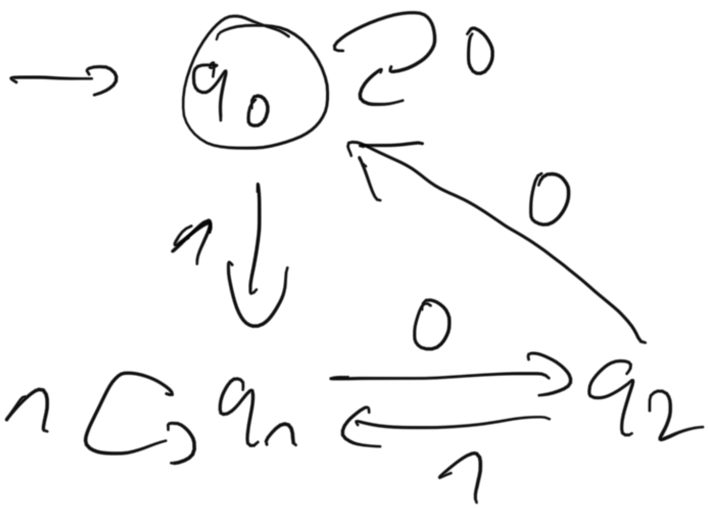
with $L(\mathcal{A}^R) = L^R$.

- $Q' = Q \cup \{q_0'\}$
- for all $q \in F$, we let $q_0' \xrightarrow{\varepsilon} q$ in Δ'
- whenever $q \xrightarrow{a} q'$ in δ, $q' \xrightarrow{a} q$ in Δ'
- $F' = \{q_0\}$

Observation: $\mathcal{A}^R$ can be nondeterministic

Example: $\{w \in \{0,1\}^* \mid$ decimal value of w is divisible by 4$\}$

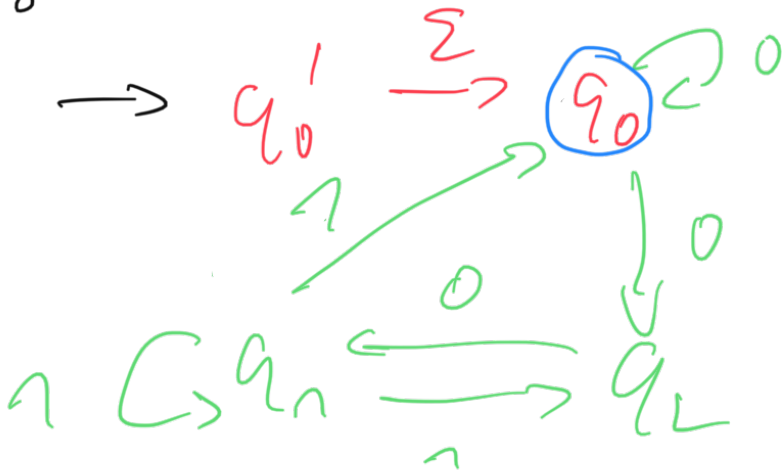
$\mathfrak{A}$:



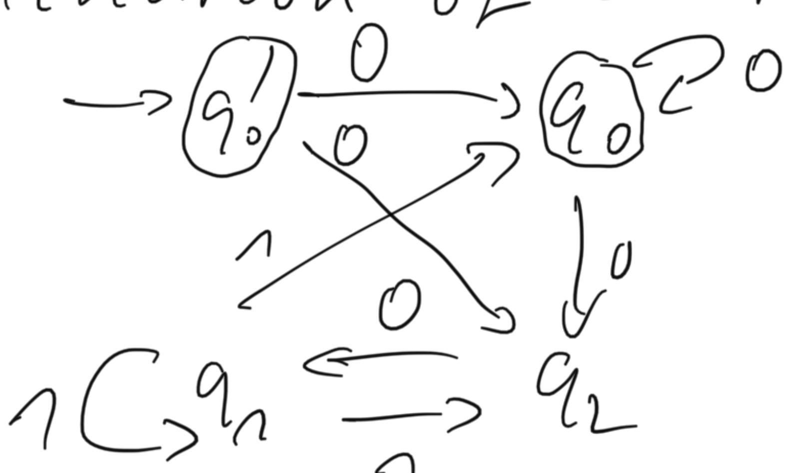
accepts ε, 0, $w \cdot 00$

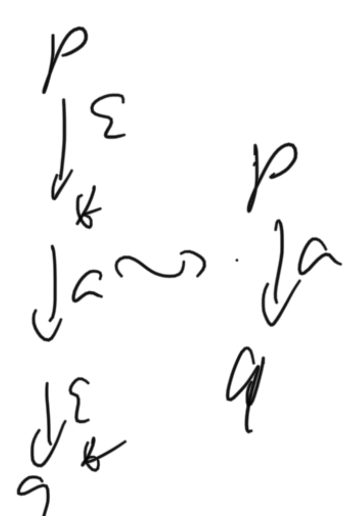
Apply reversal construction:

$\mathfrak{A}^R$:

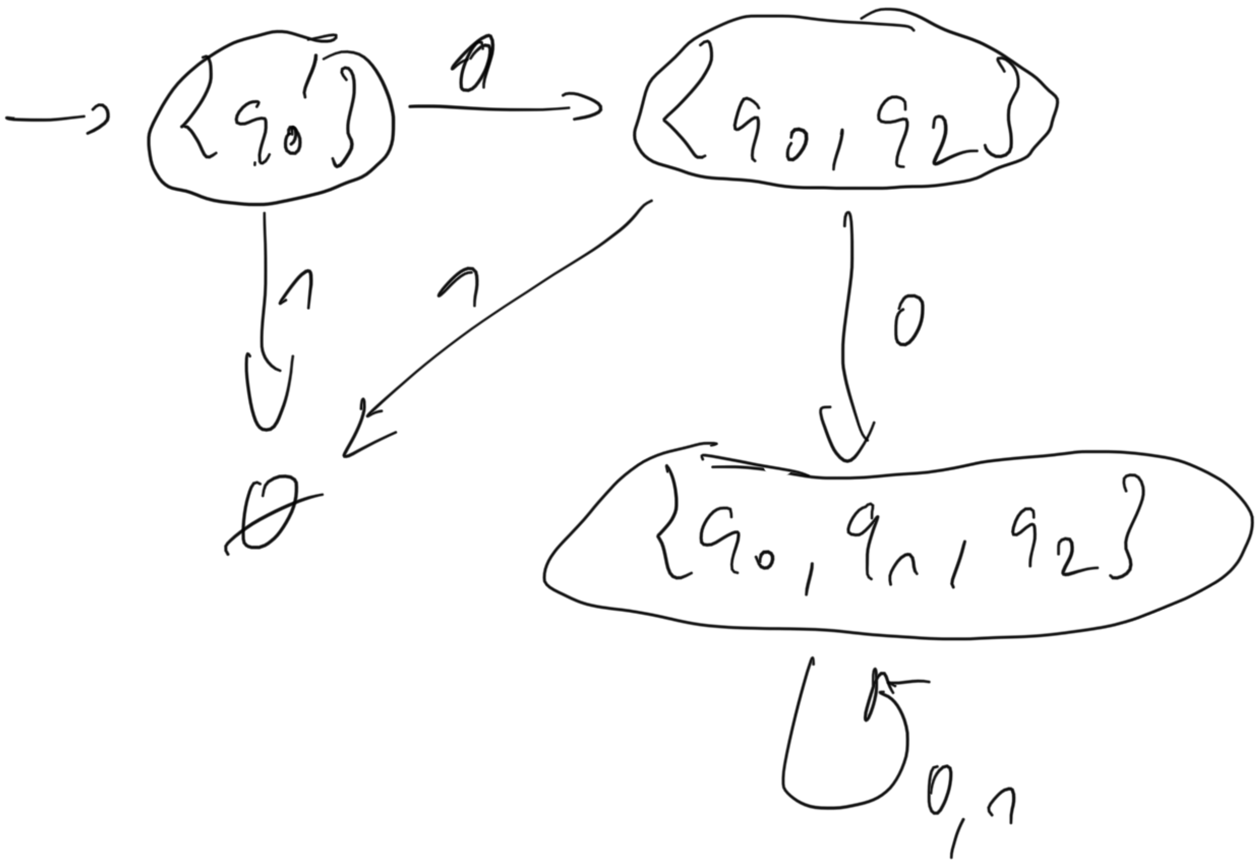


Elimination of ε-transitions:





Powerset construction:



(b) Using reg. expr:

Given α, define α^R such that $L(\alpha^R) = (L(\alpha))^R$

- $\emptyset^R = \emptyset$
- $\varepsilon^R = \varepsilon$
- $a^R = a$ (for $a \in \Sigma$)
- $(\alpha_1 | \alpha_2)^R = \alpha_1^R | \alpha_2^R$
- $(\alpha_1 \cdot \alpha_2)^R = \alpha_2^R \cdot \alpha_1^R$
- $(\alpha_1^*)^R = (\alpha_1^R)^*$

C8: From Context-Free Grammars to Pushdown Automata

Task: Construct a PDA that accepts the language generated by the grammar

$$S \to aSa \mid bSb \mid \varepsilon$$

(which is $L = \{ww^R \mid w \in \{a,b\}^*\}$, i.e., palindromes of even length).

$G = (N, \Sigma, P, S)$

Corresponding PDA $\mathfrak{A} = (Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F)$

- $Q = \{q_0\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{a, b, S\}$
- Δ:

$$\left.\begin{array}{l} ((q_0, S, \varepsilon), (q_0, aSa)) \\ (\text{— '' —}, (q_0, bSb)) \\ (\text{— '' —}, (q_0, \varepsilon)) \end{array}\right\} \text{"expansion"}$$

$$\left.\begin{array}{l} ((q_0, a, a), (q_0, \varepsilon)) \\ ((q_0, b, b), (q_0, \varepsilon)) \end{array}\right\} \text{"matching"}$$

Example computation:

$(q_0, S, abba) \vdash (q_0, aSa, abba)$
$\vdash (q_0, Sa, bba) \vdash (q_0, bSba, bba)$
$\vdash (q_0, Sba, ba) \vdash (q_0, ba, ba) \vdash^2 (q_0, \varepsilon, \varepsilon)$