Exercises (Context-Free Languages)

C1: Construction of Context-Free Grammars

Task: Give context-free grammars that generate the following languages.

(a) $L_1 := \{a^k b^l c^{k+l} \mid k, l \in \mathbb{N}\}$

(b) $L_2 := \{a^k b^k c^l d^l \mid k, l \in \mathbb{N}\}$

(c) $L_3 := \{a^k b^l \mid k \geq 1, l > k\}$

(d) $L_4 := \{w \in \{a, b\}^* \mid |w| \text{ odd}, a \text{ in middle position}\}$ $(= \{uav \in \{a, b\}^* \mid |u| = |v|\})$

(e) $L_5 = \{a^k b^l c^m \mid k, l, m \in \mathbb{N}, k = l \text{ or } k = m\}$

$L_0 = \{a^n b^n \mid n \geq 0\}$, $S \to aSb \mid \varepsilon$

(a) $S \to aSc \mid B$

$B \to bBc \mid \varepsilon$

not! $S \to aSc \mid bSc \mid \varepsilon$

$S \Rightarrow aSc \Rightarrow abScc \Rightarrow abaSccc \Rightarrow abaccc$ ↯

generates $L = \{wc^n \mid w \in \{a,b\}^*, |w| = n\}$

(b) $L_2 = L_0 \cdot L_0'$

$S \to AB$

$A \to aAb \mid \varepsilon$ $B \to cBd \mid \varepsilon$

(c) CFG for $L_3 = \{a^k b^\ell \mid k \geq 1, \ell > k\}$

$S \to AB$ $\quad = \{a^k b^{k+m} \mid k \geq 1, m \geq 1\}$

$A \to aAb \mid ab \qquad B \to bB \mid b$

(d) $L_4 = \{w \in \{a,b\}^* \mid |w| \text{ odd},$ a in middle position$\}$

$= \{uav \in \{a,b\}^* \mid |u| = |v|\}$

$S \to a \mid aSa \mid aSb \mid bSa \mid bSb$

Ex: abaab:

$S \Rightarrow aSb \Rightarrow abSab \Rightarrow abaab$

shorter: $S \to a \mid ASA \quad A \to a \mid b$

(e) $L_5 = \{a^k b^\ell c^m \mid k = \ell$ or $k = m\}$

Idea: represent L_5 as union

$S \to A \mid B$

$A \to A' C$

$(a^k b^k c^m)$

$A' \to aA'b \mid \varepsilon \qquad C \to cC \mid \varepsilon$

$B \to aBc \mid D \qquad (a^k b^l c^k)$

$D \to bD \mid \varepsilon$

C2: From Regular to Context-Free Languages

Task: Show that every regular expression can directly be translated into an equivalent context-free grammar.

L regular $\Rightarrow$ L context-free

Lesson: $L = L(\mathcal{O})$ for DFA $\mathcal{O}$

$\Rightarrow$ CFG $G_{\mathcal{O}}$ with $L(G_{\mathcal{O}}) = L$

Basic: $q \xrightarrow{a} q'$ in $\mathcal{O}$

$\Rightarrow q \rightarrow aq'$ in P

$q \in F \Rightarrow q \rightarrow \varepsilon \in P$

Alternative proof: reg. expr. $\rightsquigarrow$ CFG

Case: $\alpha = \emptyset$: $P = \emptyset$ Example: see (a)

$\alpha = \varepsilon$: $P = \{S \rightarrow \varepsilon\}$

$\alpha = a$: $P = \{S \rightarrow a\}$

$\alpha = \alpha_1 \mid \alpha_2$: $P = \{S \rightarrow S_1 \mid S_2\} \cup P_1 \cup P_2$

(S_1 for α_1, S_2 for α_2)

$\alpha = \alpha_1 \cdot \alpha_2$: $P = \{S \rightarrow S_1 S_2\} \cup P_1 \cup P_2$

$\alpha = \alpha_1^{*}$: $P = \{S \rightarrow S_1 S \mid \varepsilon\} \cup P_1$

(a) Example reg. expr. $\leadsto$ CFG:

$$\alpha = (a \mid bb)^*$$

$A \to a$

$B \to b$

$C \to BB$ (simplified: $C \to bb$)

$D \to A \mid C$

$S \to DS \mid \varepsilon$

For $w = abb \in L(\alpha)$:

$$S \Rightarrow DS \Rightarrow DDS \Rightarrow DD$$
$$\Rightarrow AD \Rightarrow AC \Rightarrow aA$$
$$\Rightarrow aBB \Rightarrow abB \Rightarrow abb$$

C4: The Word Problem for Context-Free Languages

baa

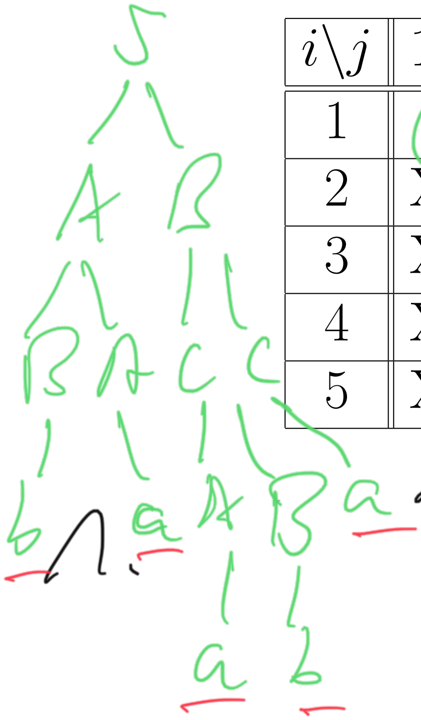
Task: Let G be the following context-free grammar:

$$
\begin{aligned}
S &\to AB \mid BC \\
A &\to BA \mid a \\
B &\to CC \mid b \\
C &\to AB \mid a
\end{aligned}
$$

and let $w := baaba$. Employ the CYK-Algorithm to show that $w \in L(G)$. Use the following table to compute the sets

$$N_{i,j} := \{A \in N \mid A \Rightarrow^* w[i,j]\} \qquad (1 \le i \le j \le 5)$$

where $w[i,j] := a_i \dots a_j$ for $w = a_1 a_2 a_3 a_4 a_5$.



$i \backslash j$	1	2	3	4	5
1	B	A, S	–	–	A, S, C
2	X	A, C	B	B	S, A, C
3	X	X	A, C	S, C	B
4	X	X	X	B	A, S
5	X	X	X	X	A, C

1. for $1 \le i \le 5$, collect all $A \in N$ such that $A \to a$

$N_{1,2} = \{D \in N \mid D \Rightarrow^* ba$

2. if $A \Rightarrow^* w[i, j']$

$B \Rightarrow^* w[j'+1, k]$

$C \Rightarrow AB$

$\Rightarrow C \Rightarrow^* w[i,k]$

$S \in N_{1,5} \Rightarrow S \Rightarrow^* baaba$

$\Rightarrow baaba \in L(G)$