Exercises (Context-Free Languages)

C1: Construction of Context-Free Grammars

Task: Give context-free grammars that generate the following languages.

(a) $L_1 := \{a^k b^l c^{k+l} \mid k, l \in \mathbb{N}\}$

(b) $L_2 := \{a^k b^k c^l d^l \mid k, l \in \mathbb{N}\}$

(c) $L_3 := \{a^k b^l \mid k \geq 1, l > k\}$

(d) $L_4 := \{w \in \{a,b\}^* \mid |w| \text{ odd}, a \text{ in middle position}\}$ $(= \cancel{\{uav \in \{a,b\}^*} \mid |u| = |v|\})$

(e) $L_5 = \{a^k b^l c^m \mid k, l, m \in \mathbb{N}, k = l \text{ or } k = m\}$

(a) $a^* b^* c^*$ $\boxed{G': S \to aS;\ L(G') = \emptyset}$

$S \to aSc \mid B \quad B \to bBc \mid \varepsilon$

Derivations: $S \Rightarrow^k a^k S c^k$

$\Rightarrow a^k B c^k$

$\Rightarrow^l a^k b^l B c^k c^l$

$\Rightarrow a^k b^l c^{k+l}$

(b) $a^* b^* c^* d^*$

$L_2 = L_2' \cdot L_2''$ $L_2' = \{a^k b^k \mid k \in \mathbb{N}\}$

$L_2'' = \{c^l d^l \mid l \in \mathbb{N}\}$

$S \to AC \quad A \to aAb \mid \varepsilon$

$C \to cCd \mid \varepsilon$

(c) $L_3 = L_3' \cdot L_3''$ $L_3' = \{a^k b^k \mid k \in \mathbb{N}\}$

$L_3'' = b^+$

$k \geq 0$: $S \to AB \quad A \to aAb \mid \varepsilon \quad B \to b \mid Bb$

$k \geq 1$: $S \to aAbB$ $A \to \ldots$ $B \to \ldots$

OR: $S \to AB$ $A \to aAb \mid ab$

$B \to b \mid Bb$

(d) $S \to a \mid aSa \mid aSb \mid bSa \mid bSb$

or shorter: $A \to a \mid b$

$S \to AaA \mid a$

(e) $L_5 = \{a^k b^l c^m \mid k, l, m \in \mathbb{N}$

$k = l$ or $k = m\}$

$L_5 = L_5' \cup L_5''$

$L_5' = \{a^k b^k c^m \mid k, m \in \mathbb{N}\}$

$L_5'' = \{a^k b^l c^k \mid k, l \in \mathbb{N}\}$

L_5: $S \to A \mid B$

L_5': $A \to DC$ $D \to aDb \mid \varepsilon$ $C \to Cc \mid \varepsilon$

L_5'': $B \to aBc \mid F$ $F \to \varepsilon \mid bF$

C4: The Word Problem for Context-Free Languages

Task: Let G be the following context-free grammar:

$$\begin{aligned} S &\to AB \mid BC \\ A &\to BA \mid a \\ B &\to CC \mid b \\ C &\to AB \mid a \end{aligned}$$

and let $w := baaba$. Employ the CYK-Algorithm to show that $w \in L(G)$. Use the following table to compute the sets

$$N_{i,j} := \{A \in N \mid A \Rightarrow^* w[i,j]\} \qquad (1 \le i \le j \le 5)$$

where $w[i,j] := a_i \dots a_j$ for $w = a_1a_2a_3a_4a_5$.

$i \backslash j$	1	2	3	4	5
1	B	S, A	∅	∅	S, A, C ← S ∈ ?
2	X	A, C	B	B	S, C, A
3	X	X	A, C	S, C	B
4	X	X	X	B	S, A
5	X	X	X	X	A, C

$w \in L(G) \iff S \Rightarrow^* w$

For $|w| = n$ and $1 \le i \le j \le n$,

$N_{ij} = \{A \in N \mid A \Rightarrow^* w[i,j]\}$

where $w[i,j] = a_i \dots a_j$

for $w = a_1 \dots a_n$

Then: $S \Rightarrow^* w \iff S \in N_{1,n}$

Here: $S \in N_{1,5} \Rightarrow baaba \in L(G)$

C5: The Emptiness Problem for Context-Free Languages

Task: Using the marking algorithm, check whether the following grammars generate empty languages or not:

(a)
$$\begin{aligned} S &\to XU \mid UW \\ U &\to SaW \mid VbU \mid SbX \\ V &\to U \mid ab \mid XW \\ W &\to bXaUb \mid VbX \mid SX \\ X &\to bV \mid WX \end{aligned}$$

0. initialisation
1. iteration
2. iteration
3. iteration

S unmarked $\Rightarrow L(G) = \emptyset$

(b)
$$\begin{aligned} S &\to AC \mid DA \\ A &\to SbC \mid BaB \mid AaD \\ B &\to A \mid ba \mid DC \\ C &\to aDbAa \mid BaD \mid SD \\ D &\to aB \mid CD \end{aligned}$$

0. initialisation
1. iteration
2. iteration
3. iteration

S marked $\Rightarrow$ S productive $\Rightarrow L(G) \neq \emptyset$

$S \Rightarrow DA \Rightarrow aBA \Rightarrow abaA$

$\Rightarrow abaBaB \Rightarrow^2 abababa$

C8: From Context-Free Grammars to Pushdown Automata

Task: Construct a PDA that accepts the language generated by the grammar

$$S \to aSa \mid bSb \mid \varepsilon$$

(which is $L = \{ww^R \mid w \in \{a,b\}^*\}$, i.e., palindromes of even length).

In general: $\mathcal{A}_G = (Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F)$

- $Q = \{q_0\}$
- $\Gamma = N \cup \Sigma$
- $Z_0 = S$
- for each $A \to \alpha \in P$: ("expansion")
  $((q_0, A, \varepsilon), (q_0, \alpha)) \in \Delta$
- for each $a \in \Sigma$: ("matching")
  $((q_0, a, a), (q_0, \varepsilon))$
- $F = Q$

Here:
- $Q = \{q_0\}$
- $\Gamma = \{S, a, b\}$
- $Z_0 = S$
- Δ: $((q_0, S, \varepsilon), (q_0, aSa))$,
  $((q_0, S, \varepsilon), (q_0, bSb))$,
  $((q_0, S, \varepsilon), (q_0, \varepsilon))$,

$((q_0, a, a), (q_0, \varepsilon))$

$((q_0, b, b), (q_0, \varepsilon))$

For example:

$$
\begin{aligned}
&(q_0, S, abba) \\
\vdash\ &(q_0, aSa, abba) \\
\vdash\ &(q_0, Sa, bba) \\
\vdash\ &(q_0, bSba, bba) \\
\vdash\ &(q_0, Sba, ba) \\
\vdash\ &(q_0, ba, ba) \\
\vdash\ &(q_0, a, a) \\
\vdash\ &(q_0, \varepsilon, \varepsilon) \quad \checkmark
\end{aligned}
$$

$$S \Rightarrow aSa \Rightarrow abSba \Rightarrow abba$$

$\Rightarrow abba \in L(G)$