Exercises (Context-Free Languages)

C1: Construction of Context-Free Grammars

Task: Give context-free grammars that generate the following languages.

(a) $L_1 := \{a^k b^l c^{k+l} \mid k, l \in \mathbb{N}\}$

(b) $L_2 := \{a^k b^k c^l d^l \mid k, l \in \mathbb{N}\}$

(c) $L_3 := \{a^k b^l \mid k \geq 1, l > k\}$

(d) $L_4 := \{w \in \{a,b\}^* \mid |w| \text{ odd}, a \text{ in middle position}\}$ $(= \{uav \in \{a,b\}^* \mid |u| = |v|\})$

(e) $L_5 = \{a^k b^l c^m \mid k, l, m \in \mathbb{N}, k = l \text{ or } k = m\}$

$G = (\Sigma, N, P, S)$ $\quad A \to \alpha \; (A \in \Sigma, \; \alpha \in (\Sigma \cup N)^*,$

$S \in N$

implies $\beta A \gamma \Rightarrow \beta \alpha \gamma$

$L(G) = \{w \in \Sigma^* \mid S \Rightarrow^* w\}$

e.g. $S \to aSb \mid \varepsilon$

$\Rightarrow L(G) = \{a^n b^n \mid n \in \mathbb{N}\}$

(a) $S \xrightarrow{(1,2)} aSc \mid B$

$B \xrightarrow{(3,4)} bBc \mid \varepsilon$

$S \xRightarrow{(1)}{}^{k} a^k S c^k \xRightarrow{(2)} a^k B c^k \xRightarrow{(3)}{}^{l} a^k b^l B c^l c^k \xrightarrow{(4)} a^k b^k c^{k+l}$

(b) $S \to AB$

$A \to aAb \mid \varepsilon \quad B \to cBd \mid \varepsilon$

(c) $S \xrightarrow{1} AB$

$A \xrightarrow{(2,3)} aAb \mid ab \quad B \xrightarrow{(4,5)} b \mid Bb$

$S \overset{(1)}{\Longrightarrow} AB \overset{(2)}{\Longrightarrow}{}^{k-1} a^{k-1}Ab^{k-1}B \overset{(3)}{\Longrightarrow} a^k b^k B$

$\overset{(4)}{\Longrightarrow}{}^{l-k-1} a^k b^k B b^{l-k-1}$

$\overset{(5)}{\Longrightarrow} a^k b^k b^{l-k} = a^k b^l$

(d) $S \to ASA \mid a$

$A \to a \mid b$

Alternative: $S \to aSa \mid aSb \mid bSa \mid$
$bSb \mid a$

(e) [Remark: $\{a^n b^n c^n \mid n \in \mathbb{N}\} \notin CFL$]

$S \to AB \mid C$

$A \to aAb \mid \varepsilon$

$B \to cB \mid \varepsilon$

$C \to aCc \mid D$

$D \to bD \mid \varepsilon$

$L_5 = L_5' \cup L_5''$

$L_5' = \{a^k b^k c^m \mid k, m \in \mathbb{N}\}$

$L_5'' = \{a^k b^l c^k \mid k, l \in \mathbb{N}\}$

Implementation of L_5':

$$S \Rightarrow AB \Rightarrow^k a^k A b^k B$$
$$\Rightarrow a^k b^k B \Rightarrow^m a^k b^k c^m B$$
$$\Rightarrow a^k b^k c^m \quad \checkmark$$

(and L_5'' similarly using $S \Rightarrow C$ in first step)

C2: From Regular to Context-Free Languages

Task: Show that every regular expression can directly be translated into an equivalent context-free grammar.

Given reg. expr. α over Σ,
we inductively construct $G_\alpha = (N, \Sigma, P, S)$
such that $L(G) = L(\alpha)$

- $\alpha = \emptyset$: $P = \{S \to S\}$
  $(L(\alpha) = \emptyset)$

- $\alpha = \varepsilon$: $P = \{S \to \varepsilon\}$
  $(L(\alpha) = \{\varepsilon\})$

- $\alpha = a$ $(a \in \Sigma)$: $P = \{S \to a\}$
  $(L(\alpha) = \{a\})$

- $\alpha = \alpha_1 \mid \alpha_2$: Given $G_{\alpha_i} = (N_i, \Sigma, P_i, S_i)$ $(i = 1, 2)$ with $L(G_{\alpha_i}) = L(\alpha_i)$,
  $(L(\alpha) = L(\alpha_1) \cup L(\alpha_2))$ $P = \{S \to S_1 \mid S_2\} \cup P_1 \cup P_2$

- $\alpha = \alpha_1 \cdot \alpha_2$:
  $(L(\alpha) = L(\alpha_1) \cdot L(\alpha_2))$ $P = \{S \to S_1 S_2\} \cup P_1 \cup P_2$

- $\alpha = \alpha_1^*$
  $(L(\alpha) = (L(\alpha_1))^*)$ $P = \{S \to S_1 S \mid \varepsilon\} \cup P_1$

C4: The Word Problem for Context-Free Languages

Task: Let G be the following context-free grammar:

$$\begin{aligned} S &\to AB \mid BC \\ A &\to BA \mid a \\ B &\to CC \mid b \\ C &\to AB \mid a \end{aligned}$$

and let $w :=$ *baaba*. Employ the CYK-Algorithm to show that $w \in L(G)$. Use the following table to compute the sets

$$N_{i,j} := \{A \in N \mid A \Rightarrow^* w[i,j]\} \qquad (1 \leq i \leq j \leq 5)$$

where $w[i,j] := a_i \dots a_j$ for $w = a_1 a_2 a_3 a_4 a_5$.

$i \backslash j$	1	2	3	4	5
1	B	A, S	$\emptyset$	$\emptyset$	S, A, C
2	X	A, C	B	B	S, A, C
3	X	X	A, C	S, C	B
4	X	X	X	B	A, S
5	X	X	X	X	A, C

$w[1,2] = b \cdot a = w[1,1] \cdot w[2,2]$

$w[1,3] = w[1,2] \cdot w[3,3]$

$\quad = w[1,1] \cdot w[2,3]$

$\vdots$

$S \in N_{1,5} \implies S \Rightarrow^* w[1,5]$

$\quad = baaba$