Exercises (Context-Free Languages)

C1: Construction of Context-Free Grammars

Task: Give context-free grammars that generate the following languages.

(a) $L_1 := \{a^k b^l c^{k+l} \mid k, l \in \mathbb{N}\}$

(b) $L_2 := \{a^k b^k c^l d^l \mid k, l \in \mathbb{N}\}$

(c) $L_3 := \{a^k b^l \mid k \geq 1, l > k\}$

(d) $L_4 := \{w \in \{a, b\}^* \mid |w| \text{ odd}, a \text{ in middle position}\}$ $(= \{uav \in \{a, b\}^* \mid |u| = |v|\})$

(e) $L_5 = \{a^k b^l c^m \mid k, l, m \in \mathbb{N}, k = l \text{ or } k = m\}$

Solution:

(a) $S \rightarrow aSc \mid B$
$B \rightarrow bBc \mid \varepsilon$

(b) $S \rightarrow AC$
$A \rightarrow aAb \mid \varepsilon$
$C \rightarrow cCd \mid \varepsilon$

(c) $S \rightarrow AB$
$A \rightarrow aAb \mid ab$
$B \rightarrow bB \mid b$

(d) $S \rightarrow aSa \mid aSb \mid bSa \mid bSb \mid a$ or shorter: $S \rightarrow ASA \mid a$
$A \rightarrow a \mid b$

(e) $S \rightarrow \underbrace{AC}_{k=l} \mid \underbrace{D}_{k=m}$
$A \rightarrow aAb \mid \varepsilon$ % $a^k b^k$
$C \rightarrow cC \mid \varepsilon$ % c^m
$D \rightarrow aDc \mid B$ % $a^k B c^k$
$B \rightarrow bB \mid \varepsilon$ % b^l

C2: From Regular to Context-Free Languages

Task: Show that every regular expression can directly be translated into an equivalent context-free grammar.

Solution: Given regular expression α over Σ, we inductively construct CFG $G_\alpha = \langle N, \Sigma, P, S\rangle$ with $L(G_\alpha) = L(\alpha)$ as follows:

- $\alpha = \emptyset$: $P = \emptyset$
- $\alpha = \varepsilon$: $P = \{S \to \varepsilon\}$
- $\alpha = a$: $P = \{S \to a\}$
- $\alpha = \alpha_1 + \alpha_2$: $P = \{S \to S_1 \mid S_2\} \cup P_1 \cup P_2$ (where $G_{\alpha_i} = \langle N_i, \Sigma, P_i, S_i\rangle$ for $i = 1, 2$)
- $\alpha = \alpha_1 \cdot \alpha_2$: $P = \{S \to S_1 S_2\} \cup P_1 \cup P_2$ (where G_{α_i} as above)
- $\alpha = \alpha_1^*$: $P = \{S \to S_1 S \mid \varepsilon\} \cup P_1$ (where G_{α_1} as above)

C3: Chomsky Normal Form

Task: Transform the following grammar into Chomsky Normal Form:

$$
\begin{aligned}
S &\to xAx \mid CyBA \mid BB \mid z \mid xxx \\
A &\to C \mid xy \\
B &\to A \\
C &\to yyy \mid B
\end{aligned}
$$

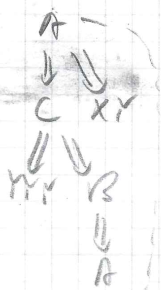
Solution:

$S \to xAx \mid CyBA \mid BB \mid z \mid xxx$
$A \to C \mid xy$
$B \to A$
$C \to yyy \mid B$

1st step: terminals

$S \to XAX \mid CYBA \mid BB \mid z \mid Xxx$
$A \to C \mid XY$
$B \to A$
$C \to YYY \mid B$
$X \to x \quad Y \to y$

2nd step: elim. of chain rules



$S \to XAX \mid CYBA \mid BB \mid z \mid XXX$
$A \to YYY \mid XY$
$B \to YYY \mid XY$
$C \to YYY \mid XY$
$X \to x \quad Y \to y$

3rd step: elim. of long rules

$S \to XD \mid CE \mid BB \mid z \mid XG$
$A \to YH \mid XY$
$B \to YH \mid XY$
$C \to YH \mid XY$
$X \to x \quad Y \to y$
$D \to AX \quad E \to YF \quad F \to BA \quad G \to XX \quad H \to YY$

C4: The Word Problem for Context-Free Languages

Task: Let G be the following context-free grammar:

$$\begin{aligned} S &\to AB \mid BC \\ A &\to BA \mid a \\ B &\to CC \mid b \\ C &\to AB \mid a \end{aligned}$$

and let $w := baaba$. Employ the CYK-Algorithm to show that $w \in L(G)$. Use the following table to compute the sets

$$N_{i,j} := \{A \in N \mid A \Rightarrow^* w[i,j]\} \qquad (1 \leq i \leq j \leq 5)$$

where $w[i,j] := a_i \dots a_j$ for $w = a_1 a_2 a_3 a_4 a_5$.

$i \backslash j$	1	2	3	4	5
1					
2	X				
3	X	X			
4	X	X	X		
5	X	X	X	X	

Solution:

$i \backslash j$	1	2	3	4	5
1	B	S, A	$\emptyset$	$\emptyset$	S, A, C
2	X	A, C	B	B	S, A, C
3	X	X	A, C	S, C	B
4	X	X	X	B	S, A
5	X	X	X	X	A, C

$S \in N_{1,5} \Rightarrow w \in L(G)$