Foundations of Informatics: a Bridging Course

Part C: Context-Free Languages
Lesson 7: Pushdown Automata

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Pushdown Automata I

- **Goal:** introduce an automata model which exactly accepts CFLs
- **Clear:** DFA not sufficient
  (missing "counting capability", e.g. for $\{a^n b^n \mid n \geq 1\}$)

Pushdown Automata I

- **Goal:** introduce an automata model which exactly accepts CFLs
- **Clear:** DFA not sufficient
  (missing "counting capability", e.g. for $\{a^n b^n \mid n \geq 1\}$)
- DFA will be extended to pushdown automata by
  - adding a pushdown store which stores symbols from a pushdown alphabet and uses a special bottom symbol
  - adding push and pop operations to transitions

Pushdown Automata II

Definition

A pushdown automaton (PDA) is of the form $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ where

- Q is a finite set of states
- Σ is the (finite) input alphabet
- Γ is the (finite) pushdown alphabet
- $\Delta \subseteq (Q \times \Gamma \times \Sigma_\varepsilon) \times (Q \times \Gamma^*)$ is a finite set of transitions
- $q_0 \in Q$ is the initial state
- Z_0 is the (pushdown) bottom symbol
- $F \subseteq Q$ is a set of final states

Interpretation of $((q, Z, x), (q', \delta)) \in \Delta$: if the PDA $\mathfrak{A}$ is in state q where Z is on top of the stack and x is the next input symbol (or empty), then $\mathfrak{A}$ reads x, replaces Z by δ, and changes into the state q'.

Configurations, Runs, Acceptance

Definition

Let $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ be a PDA.

- An element of $Q \times \Gamma^* \times \Sigma^*$ is called a configuration of $\mathfrak{A}$.
- The initial configuration for input $w \in \Sigma^*$ is given by (q_0, Z_0, w).
- The set of final configurations is given by $F \times \{\varepsilon\} \times \{\varepsilon\}$.
- If $((q, Z, x), (q', \delta)) \in \Delta$, then $(q, Z\gamma, xw) \vdash (q', \delta\gamma, w)$ for every $\gamma \in \Gamma^*$, $w \in \Sigma^*$.

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- If $((q, Z, x), (q', \delta)) \in \Delta$, then $(q, Z\gamma, xw) \vdash (q', \delta\gamma, w)$ for every $\gamma \in \Gamma^*$, $w \in \Sigma^*$.
- $\mathfrak{A}$ accepts $w \in \Sigma^*$ if $(q_0, Z_0, w) \vdash^* (q, \varepsilon, \varepsilon)$ for some $q \in F$.
- The language accepted by $\mathfrak{A}$ is $L(\mathfrak{A}) := \{w \in \Sigma^* \mid \mathfrak{A} \text{ accepts } w\}$.
- A language L is called PDA-recognisable if $L = L(\mathfrak{A})$ for some PDA $\mathfrak{A}$.
- Two PDA $\mathfrak{A}_1, \mathfrak{A}_2$ are called equivalent if $L(\mathfrak{A}_1) = L(\mathfrak{A}_2)$.

Examples of PDA I

Example (PDA for $L = \{a^n b^n \mid n \geq 1\}$)

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- Observation: no transitions for
  - (q_0, Z_0, b): input must start with a
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Accepting run of PDA for input $w = aabb$:

(remember: if $((q, Z, x),(q', \delta)) \in \Delta$, then $(q, Z\gamma, xw) \vdash (q', \delta\gamma, w)$)

$$(q_0, Z_0, aabb)$$

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Examples of PDA II

Example (PDA for $L = \{ww^R \mid w \in \{a, b\}^*\}$ (palindromes of even length))

Idea:
1. $\mathfrak{A}$ pushes input w
2. switches nondeterministically to the w^R recognition phase
3. compares w and w^R symbol-wise by matching steps
4. accepts with empty pushdown

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Accepting run of PDA for input $w = abba$:

$(q_0, Z_0, abba)$

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Examples of PDA II

Example (PDA for $L = \{ww^R \mid w \in \{a,b\}^*\}$ (palindromes of even length))

Idea:
1. $\mathfrak{A}$ pushes input w
2. switches nondeterministically to the w^R recognition phase
3. compares w and w^R symbol-wise by matching steps
4. accepts with empty pushdown

Formally: $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$

- $Q = \{q_0, q_1, q_2\}$
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- Δ:

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Accepting run of PDA for input $w = abba$:

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Observation: $\mathfrak{A}$ is nondeterministic – in a configuration of the form (q_0, cv, cw) ($c \in \Sigma$, $v, w \in \Sigma^*$), both (1) and (2) are applicable.

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$$(q_0, Z_0, abba) \vdash (q_0, aZ_0, bba) \vdash (q_0, baZ_0, ba) \vdash (q_0, bbaZ_0, a) \vdash (q_0, abbaZ_0, \varepsilon) \not\vdash$$

Deterministic PDA

Definition

A PDA $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is called deterministic (DPDA) if for every $q \in Q, Z \in \Gamma$,

1. for every $x \in \Sigma_\varepsilon$, there is at most one (q, Z, x)-transition in Δ and
2. if there is a (q, Z, a)-transition in Δ for some $a \in \Sigma$, then there is no (q, Z, ε)-transition in Δ.

Remark: this excludes two types of nondeterminism:

1. if $((q, Z, x), (q'_1, \delta_1)), ((q, Z, x), (q'_2, \delta_2)) \in \Delta$:
$$(q'_1, \delta_1\gamma, w) \dashv (q, Z\gamma, xw) \vdash (q'_2, \delta_2\gamma, w)$$
2. if $((q, Z, a), (q'_1, \delta_1)), ((q, Z, \varepsilon), (q'_2, \delta_2)) \in \Delta$:
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Corollary

In a DPDA, every configuration has at most one $\vdash$-successor.

Expressiveness of DPDA

One can show: determinism restricts the set of acceptable languages (DPDA-recognisable languages are closed under complement, which is generally not true for PDA-recognisable languages)

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Example

The set of palindromes of even length is PDA-recognisable, but not DPDA-recognisable (without proof).

Summary: Pushdown Automata

Seen:

- Extension of finite automata by pushdown store
- Enables "counting" (e.g., $\{a^n b^n \mid n \geq 1\}$)
- Determinism restricts expressivity (in contrast to finite automata)

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Next:

- Relation between PDA and context-free languages