Foundations of Informatics: a Bridging Course

Part C: Context-Free Languages
Lesson 6: Closure Properties of Context-Free Languages

Thomas Noll
Software Modeling and Verification Group
RWTH Aachen University

https://moves.rwth-aachen.de/people/noll/

Software Modeling and Verification Chair
RWTH AACHEN UNIVERSITY

Positive Results

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The set of CFLs is closed under concatenation, union, and iteration.

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- L_1^* is generated by $G := \langle N, \Sigma, P, S \rangle$ with $N := \{S\} \cup N_1$ and
$$P := \{S \to \varepsilon \mid S_1 S\} \cup P_1$$

$\square$

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$$L_1 := \{a^k b^k c^l \mid k, l \in \mathbb{N}\} \qquad \text{(generated by } S \to AC, A \to aAb \mid \varepsilon, C \to Cc \mid \varepsilon\text{)}$$

  and

$$L_2 := \{a^k b^l c^l \mid k, l \in \mathbb{N}\} \qquad \text{(generated by } S \to AB, A \to aA \mid \varepsilon, B \to bBc \mid \varepsilon\text{)}$$

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- Complement: if CFLs were closed under complement, then also under intersection (as $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$).

□

Overview of Decidability and Closure Results

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Class	$w \in L$	$L = \emptyset$	$L_1 = L_2$
Reg	+	+	+
CFL	+	+	−

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Closure Results					
Class	$L_1 \cdot L_2$	$L_1 \cup L_2$	$L_1 \cap L_2$	$\overline{L}$	L^*
Reg	+	+	+	+	+
CFL	+	+	−	−	+

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Next:

- Automata model for CFLs