Foundations of Informatics: a Bridging Course

Part C: Context-Free Languages
Lesson 8: Pushdown Automata and Context-Free Languages

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PDA and Context-Free Languages I

Theorem

A language is context-free iff it is PDA-recognisable.

PDA and Context-Free Languages I

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Proof.

"$\Leftarrow$": omitted

"$\Rightarrow$": let $G = \langle N, \Sigma, P, S \rangle$ be a CFG. Construction of PDA $\mathfrak{A}_G$ recognising $L(G)$:

- $\mathfrak{A}_G$ simulates a derivation of G where always the leftmost nonterminal of a sentence is replaced ("leftmost derivation")
- begin with S on pushdown
- if nonterminal on top: apply a corresponding production rule
- if terminal on top: match with next input symbol

(cf. formal construction on following slide)

$\square$

PDA and Context-Free Languages II

Proof of Theorem C.1 (continued).

"$\Rightarrow$": Formally, $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

- $Q := \{q_0\}$
- $\Gamma := N \cup \Sigma$
- $Z_0 := S$
- for each $A \to \alpha \in P$: $((q_0, A, \varepsilon), (q_0, \alpha)) \in \Delta$ ("expansion")
- for each $a \in \Sigma$: $((q_0, a, a), (q_0, \varepsilon)) \in \Delta$ ("matching")
- $F := Q$

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PDA and Context-Free Languages II

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Example ("Bracket language" given by $G : S \to \langle\rangle \mid \langle S \rangle \mid SS$)

$\mathfrak{A}_G = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ with

- $Q = F = \{q_0\}$
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- $Z_0 = S$
- Δ: $((q_0, S, \varepsilon),(q_0, \langle\rangle))$ $((q_0, \langle, \langle),(q_0, \varepsilon))$
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PDA and Context-Free Languages II

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PDA and Context-Free Languages II

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PDA and Context-Free Languages II

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PDA and Context-Free Languages II

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PDA and Context-Free Languages II

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Seen:

- Construction of PDA for given CFG ($\Rightarrow$ parser generation!)
- Reverse direction also possible
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Outlook:

- Equivalence problem for CFG and PDA ("$L(X_1) = L(X_2)$?"): generally undecidable, but decidable for DPDA
- Pumping Lemma for CFL (e.g., to prove that $\{a^n b^n c^n \mid n \geq 1\}$ not context-free)
- Greibach Normal Form for CFG
- Systematic construction of deterministic and efficient parsers for compilers (LL/LR grammars)
- Non-context-free grammars and languages (e.g., context-sensitive languages such as $\{a^n b^n c^n \mid n \geq 1\}$)