Foundations of Informatics: a Bridging Course

Part C: Context-Free Languages
Lesson 2: Context-Free vs. Regular Languages

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Context-Free vs. Regular Languages

Theorem

1. *Every regular language is context-free.*
2. *There exist CFLs which are not regular.*

(Thus: regular languages are a proper subset of CFLs.)

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Proof.

1. Let L be a regular language, and let $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F\rangle$ be a DFA which recognises L. $G_\mathfrak{A} := \langle N, \Sigma, P, S\rangle$ is defined as follows:
   - $N := Q$, $S := q_0$
   - if $\delta(q, a) = q'$, then $q \to aq' \in P$
   - if $q \in F$, then $q \to \varepsilon \in P$

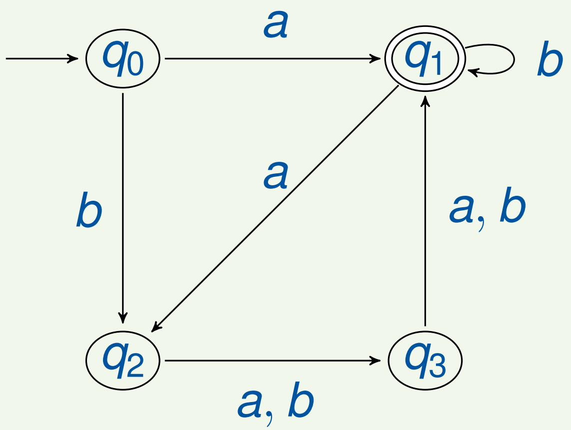
   Obviously a w-labelled run in $\mathfrak{A}$ from q_0 to F corresponds to a derivation of w in $G_\mathfrak{A}$, and vice versa. Thus $L(\mathfrak{A}) = L(G_\mathfrak{A})$ (example on the following slide).
2. An example is $\{a^n b^n \mid n \in \mathbb{N}\}$ (see Lesson 1).
   Intuitive reason for non-regularity: recognising this language requires "unbounded counting" capability.

$\square$

From Regular to Context-Free Languages

Example

DFA $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$:



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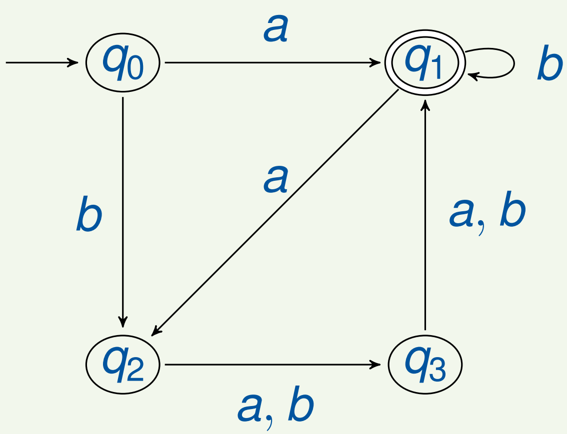

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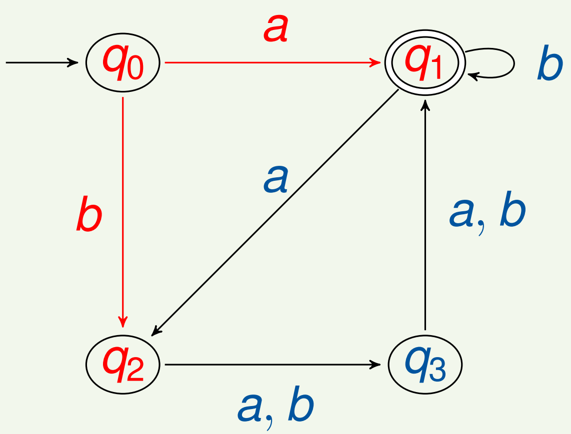

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$$q_0 \rightarrow a\, q_1 \mid b\, q_2$$

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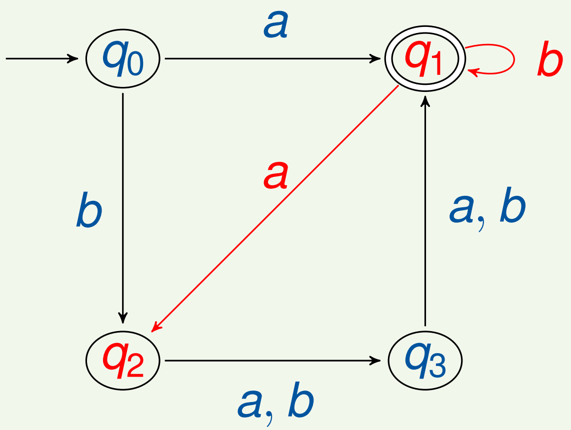

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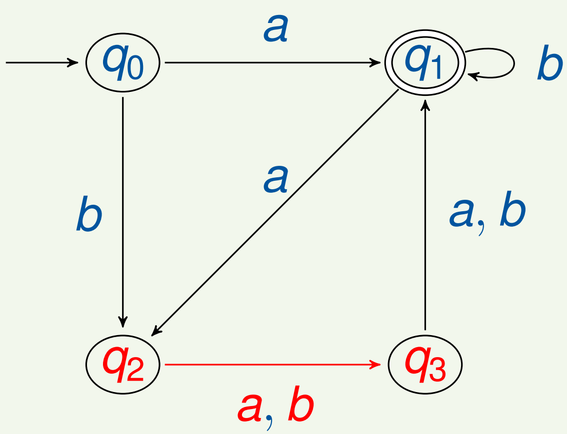

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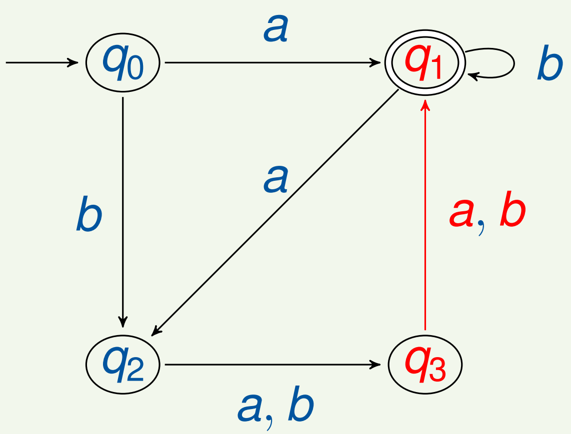

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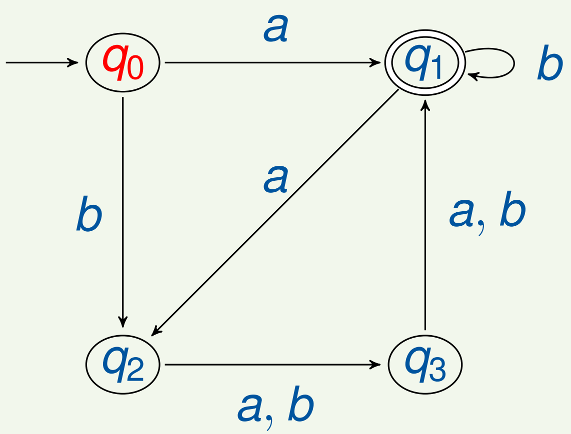

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E.g., $\mathfrak{A}$'s run on input $baab \in L(\mathfrak{A})$ is simulated by the following derivation in $G_{\mathfrak{A}}$:

$$q_0$$

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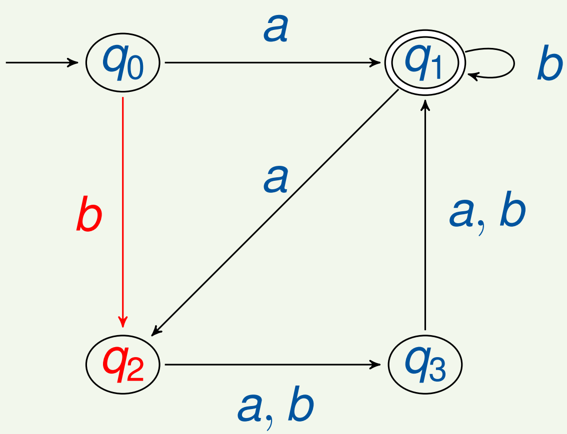

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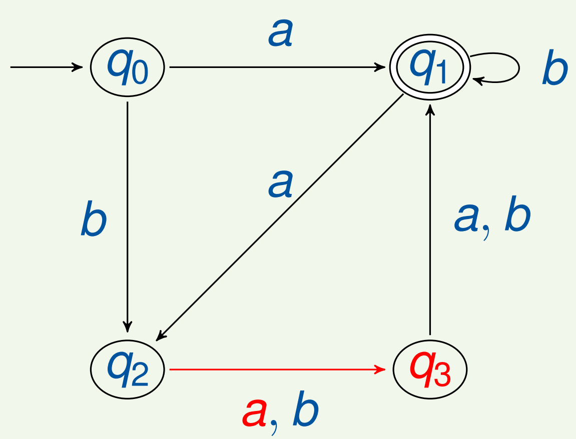

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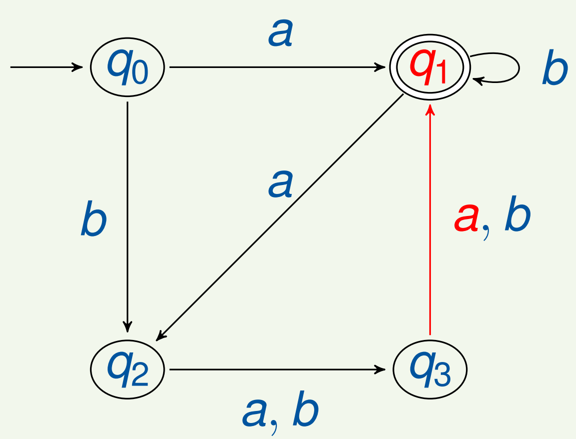

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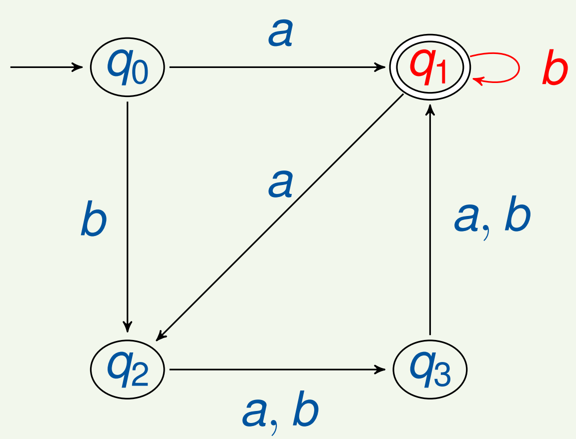

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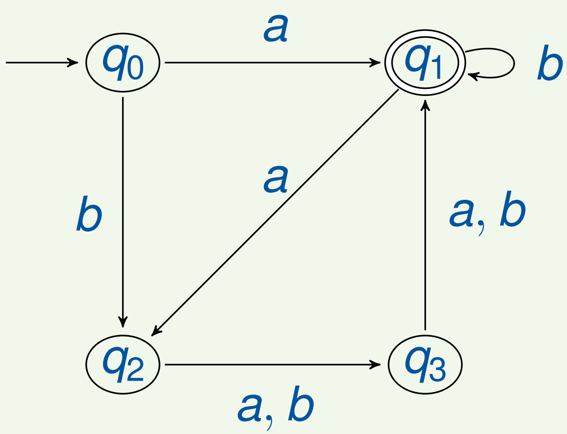

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Next:

- Decidability of word problem