Foundations of Informatics: a Bridging Course

Week 3: Formal Languages and Processes
Part C: Context-Free Languages
March 11–15, 2024

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https://moves.rwth-aachen.de/teaching/ws-23-24/foi/

Outline of Part C

Introductory Example I

Example C.1

Syntax definition of programming languages by “Backus-Naur” rules

Here: simple arithmetic expressions

$$\begin{aligned}\langle \mathit{Expression} \rangle \; ::= \; & 0 \\ \mid \; & 1 \\ \mid \; & \langle \mathit{Expression} \rangle + \langle \mathit{Expression} \rangle \\ \mid \; & \langle \mathit{Expression} \rangle * \langle \mathit{Expression} \rangle \\ \mid \; & (\langle \mathit{Expression} \rangle)\end{aligned}$$

Meaning:

An expression is either 0 or 1, or it is of the form $u + v$, $u * v$, *or* (u) *where* u, v *are again expressions*

Introductory Example II

Example C.1 (continued)

Here we abbreviate $\langle Expression \rangle$ as E, and use "$\longrightarrow$" instead of "$::=$". Thus:

$$E \longrightarrow 0 \mid 1 \mid E + E \mid E * E \mid (E)$$

Introductory Example II

Example C.1 (continued)

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$$E \Rightarrow E * E$$

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Context-Free Grammars I

Definition C.2

A context-free grammar (CFG) is a quadruple

$$G = \langle N, \Sigma, P, S \rangle$$

where

- N is a finite set of nonterminal symbols
- Σ is the (finite) alphabet of terminal symbols (disjoint from N)
- P is a finite set of production rules of the form $A \to \alpha$ where $A \in N$ and $\alpha \in (N \cup \Sigma)^*$
- $S \in N$ is a start symbol

Context-Free Grammars II

Example C.3

For the above example, we have:

- $N = \{E\}$
- $\Sigma = \{0, 1, +, *, (,)\}$
- $P = \{E \to 0, E \to 1, E \to E + E, E \to E * E, E \to (E)\}$
- $S = E$

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Naming conventions:

- nonterminals start with uppercase letters
- terminals start with lowercase letters
- start symbol = symbol on LHS of first production

$\Rightarrow$ grammar completely defined by productions

Context-Free Languages I

Definition C.4

Let $G = \langle N, \Sigma, P, S \rangle$ be a CFG.

- A sentence $\gamma \in (N \cup \Sigma)^*$ is directly derivable from $\beta \in (N \cup \Sigma)^*$ if there exist $\pi = A \to \alpha \in P$ and $\delta_1, \delta_2 \in (N \cup \Sigma)^*$ such that $\beta = \delta_1 A \delta_2$ and $\gamma = \delta_1 \alpha \delta_2$ (notation: $\beta \stackrel{\pi}{\Rightarrow} \gamma$ or just $\beta \Rightarrow \gamma$).

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- A derivation (of length $n \in \mathbb{N}$) of γ from β is a sequence of direct derivations of the form $\delta_0 \Rightarrow \delta_1 \Rightarrow \ldots \Rightarrow \delta_n$ where $\delta_0 = \beta$, $\delta_n = \gamma$, and $\delta_{i-1} \Rightarrow \delta_i$ for every $i \in \{1, \ldots, n\}$ (notation: $\beta \Rightarrow^* \gamma$).

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- A **derivation** (of length $n \in \mathbb{N}$) of γ from β is a sequence of direct derivations of the form $\delta_0 \Rightarrow \delta_1 \Rightarrow \ldots \Rightarrow \delta_n$ where $\delta_0 = \beta$, $\delta_n = \gamma$, and $\delta_{i-1} \Rightarrow \delta_i$ for every $i \in \{1, \ldots, n\}$ (notation: $\beta \Rightarrow^* \gamma$).
- A word $w \in \Sigma^*$ is called **derivable** in G if $S \Rightarrow^* w$.

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- A language $L \subseteq \Sigma^*$ is called context-free (CFL) if it is generated by some CFG.

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- A word $w \in \Sigma^*$ is called derivable in G if $S \Rightarrow^* w$.
- The language generated by G is $L(G) := \{w \in \Sigma^* \mid S \Rightarrow^* w\}$.
- A language $L \subseteq \Sigma^*$ is called context-free (CFL) if it is generated by some CFG.
- Two grammars G_1, G_2 are equivalent if $L(G_1) = L(G_2)$.

Context-Free Languages II

Example C.5

The language

$$\{a^n b^n \mid n \in \mathbb{N}\}$$

is context-free. It is generated by the grammar $G = \langle N, \Sigma, P, S \rangle$ with

- $N = \{S\}$
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(proof: generating $a^n b^n$ requires exactly n applications of the first and one concluding application of the second rule)

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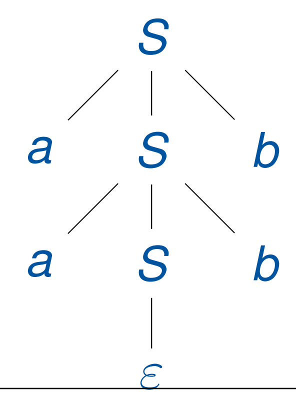
Remark: illustration of derivations by derivation trees

- root labelled by start symbol
- leaves labelled by terminal symbols
- successors of node labelled according to right-hand side of production rule
- sequence of leaf symbols = generated word



Summary: Context-Free Grammars and Languages

Seen:

- Context-free grammars
- Derivations
- Context-free languages

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Next:

- Relation between context-free and regular languages

Outline of Part C

Context-Free vs. Regular Languages

Theorem C.6

1. *Every regular language is context-free.*
2. *There exist CFLs which are not regular.*

(Thus: regular languages are a proper subset of CFLs.)

Context-Free vs. Regular Languages

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2. *There exist CFLs which are not regular.*

(Thus: regular languages are a proper subset of CFLs.)

Proof.

1. Let L be a regular language, and let $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$ be a DFA which recognises L.
   $G_\mathfrak{A} := \langle N, \Sigma, P, S \rangle$ is defined as follows:
   - $N := Q$, $S := q_0$
   - if $\delta(q, a) = q'$, then $q \to aq' \in P$
   - if $q \in F$, then $q \to \varepsilon \in P$

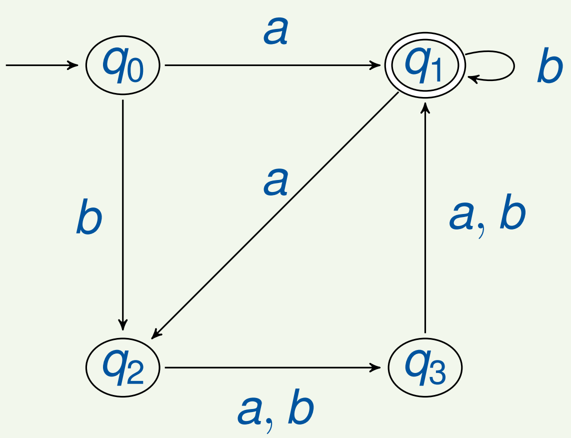
   Obviously a w-labelled run in $\mathfrak{A}$ from q_0 to F corresponds to a derivation of w in $G_\mathfrak{A}$, and vice versa. Thus $L(\mathfrak{A}) = L(G_\mathfrak{A})$ (example on the following slide).
2. An example is $\{a^n b^n \mid n \in \mathbb{N}\}$ (see Lesson 1).
   Intuitive reason for non-regularity: recognising this language requires "unbounded counting" capability.

□

From Regular to Context-Free Languages

Example C.7

DFA $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$:



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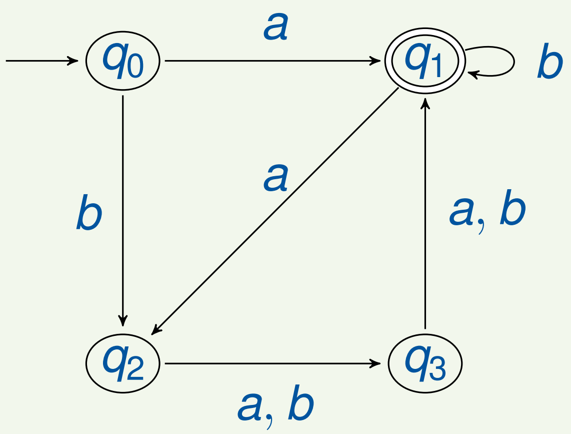

Corresponding CFG $G_{\mathfrak{A}} := \langle N, \Sigma, P, S \rangle$ with $N := Q$, $S := q_0$:

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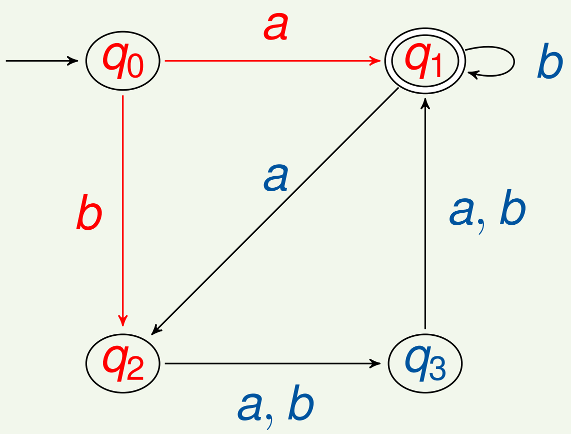

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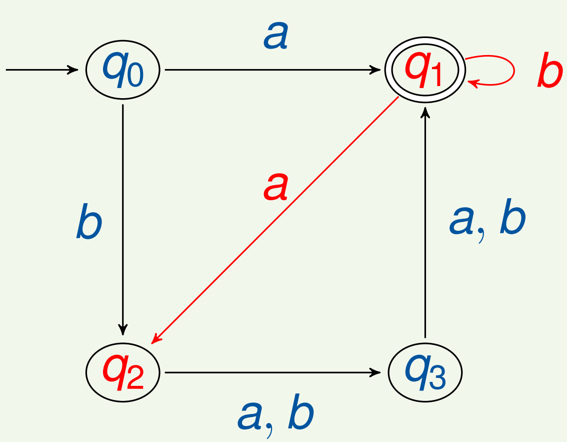

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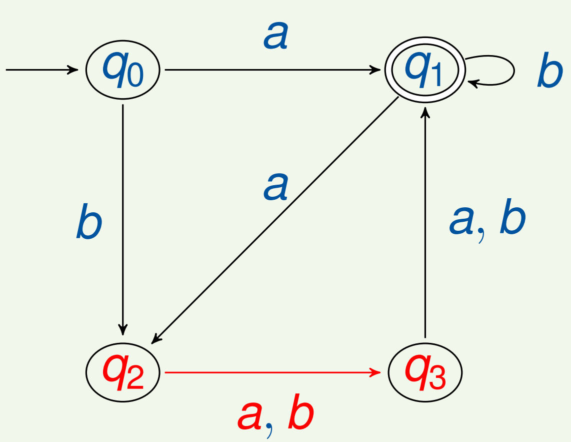

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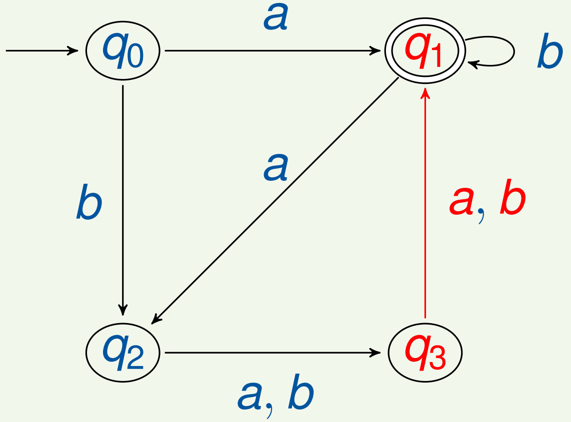

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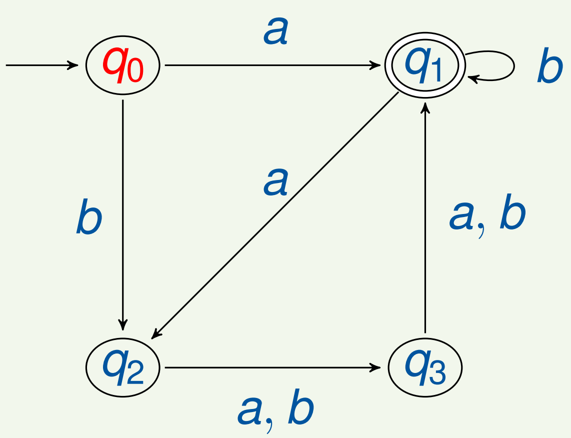

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E.g., $\mathfrak{A}$'s run on input $baab \in L(\mathfrak{A})$ is simulated by the following derivation in $G_{\mathfrak{A}}$:

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From Regular to Context-Free Languages

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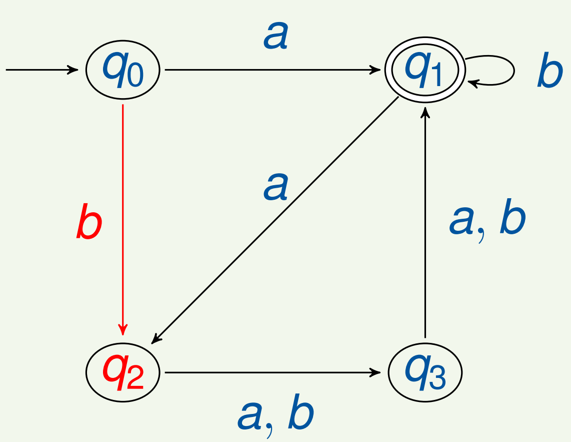

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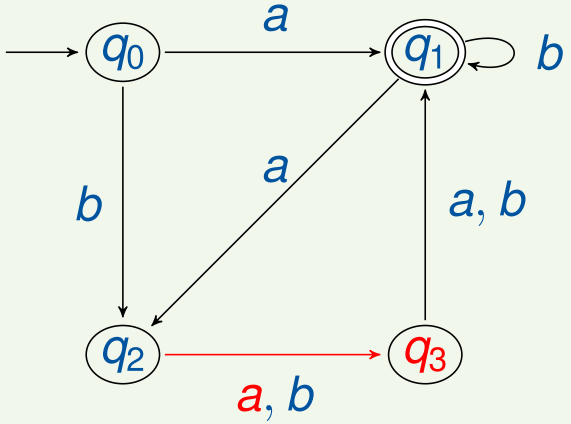

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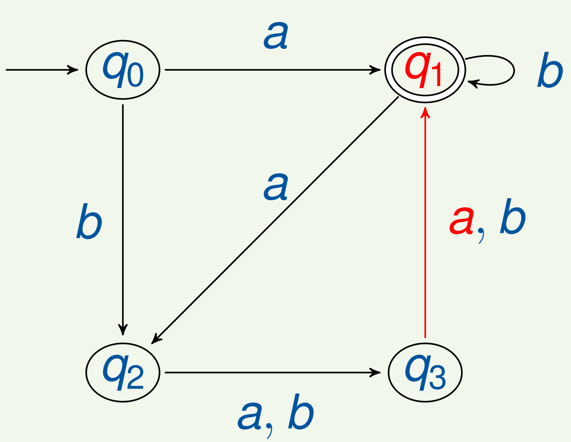

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From Regular to Context-Free Languages

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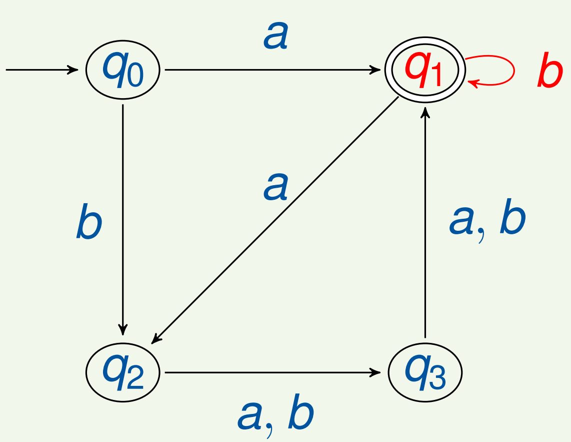

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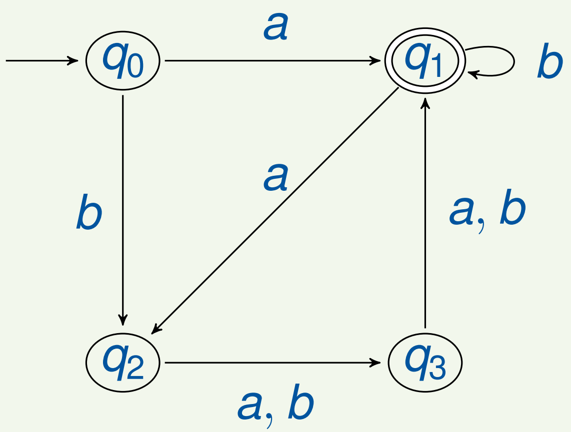

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- CFLs are more expressive than regular languages

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Next:

- Decidability of word problem

Outline of Part C

The Word Problem for CFL

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- For regular languages this was easy: just let the corresponding DFA run on w.
- But here: how to decide when to stop a derivation?
- **Solution:** establish normal form for grammars which guarantees that each nonterminal produces at least one terminal symbol

$\Rightarrow$ Only finitely many combinations to be inspected

Chomsky Normal Form

Definition C.8

A CFG is in Chomsky Normal Form (Chomsky NF) if every of its productions is of the form

$$A \to BC \quad \text{or} \quad A \to a$$

Chomsky Normal Form

Definition C.8

A CFG is in Chomsky Normal Form (Chomsky NF) if every of its productions is of the form

$$A \rightarrow BC \quad \text{or} \quad A \rightarrow a$$

Example C.9

Consider the grammar $S \rightarrow ab \mid aSb$, which generates $L := \{a^n b^n \mid n \geq 1\}$. An equivalent grammar in Chomsky NF is

$$\begin{aligned}
S &\rightarrow AB \mid AC & &(\text{generates } L) \\
A &\rightarrow a & &(\text{generates } \{a\}) \\
B &\rightarrow b & &(\text{generates } \{b\}) \\
C &\rightarrow SB & &(\text{generates } \{a^n b^{n+1} \mid n \geq 1\})
\end{aligned}$$

Conversion to Chomsky Normal Form

Theorem C.10

Every CFL L (without ε-productions) can be generated by a CFG in Chomsky NF.

Conversion to Chomsky Normal Form

Theorem C.10

Every CFL L (without ε-productions) can be generated by a CFG in Chomsky NF.

Proof.

Let L be a CFL, and let $G = \langle N, \Sigma, P, S \rangle$ be some CFG which generates L. The transformation of P into rules of the form $A \to BC$ and $A \to a$ proceeds in three steps:

1. terminal symbols only in rules of the form $A \to a$
   (thus all other rules have the shape $A \to A_1 \dots A_n$)
2. elimination of "chain rules" of the form $A \to B$
3. elimination of rules of the form $A \to A_1 \dots A_n$ where $n > 2$

(see following slides for details) □

Step 1: Only $A \to a$

Procedure

1. For every terminal symbol $a \in \Sigma$, introduce a new nonterminal symbol $B_a \in N$.
2. Add corresponding productions $B_a \to a$ to P.
3. In each original production $A \to \alpha$, replace every $a \in \Sigma$ with B_a.

This yields G'.

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3. In each original production $A \to \alpha$, replace every $a \in \Sigma$ with B_a.

This yields G'.

Example C.11

$G: \; S \to ab \mid aSb$ is converted to

$$\begin{aligned} G': \; S &\to AB \mid ASB \\ A &\to a \\ B &\to b \end{aligned}$$

Step 2: Elimination of Chain Rules $A \to B$

Procedure

1. Determine all derivations $A_1 \Rightarrow \ldots \Rightarrow A_n$ with rules of the form $A \to B$ without repetition of nonterminals ($\Longrightarrow$ only finitely many!).
2. Determine all productions $A_n \to \alpha$ with $\alpha \notin N$.
3. Add corresponding productions $A_1 \to \alpha$ to P.
4. Remove all chain rules from P.

This yields G''.

Step 2: Elimination of Chain Rules $A \to B$

Procedure

1. Determine all derivations $A_1 \Rightarrow \ldots \Rightarrow A_n$ with rules of the form $A \to B$ without repetition of nonterminals ($\implies$ only finitely many!).
2. Determine all productions $A_n \to \alpha$ with $\alpha \notin N$.
3. Add corresponding productions $A_1 \to \alpha$ to P.
4. Remove all chain rules from P.

This yields G''.

Example C.12

$$
\begin{aligned}
G' : \quad S &\to A \\
A &\to B \mid C \\
B &\to A \mid DA \\
C &\to c \\
D &\to d
\end{aligned}
$$

Step 2: Elimination of Chain Rules $A \to B$

Procedure

1. Determine all derivations $A_1 \Rightarrow \ldots \Rightarrow A_n$ with rules of the form $A \to B$ without repetition of nonterminals ($\Longrightarrow$ only finitely many!).
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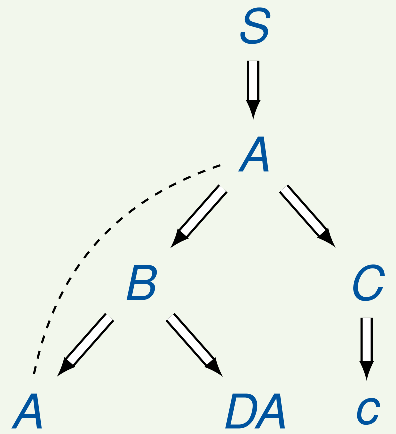
Example C.12

$$
\begin{aligned}
G' :\ S &\to A \\
A &\to B \mid C \\
B &\to A \mid DA \\
C &\to c \\
D &\to d
\end{aligned}
$$



Step 2: Elimination of Chain Rules $A \to B$

Procedure

1. Determine all derivations $A_1 \Rightarrow \ldots \Rightarrow A_n$ with rules of the form $A \to B$ without repetition of nonterminals ($\Longrightarrow$ only finitely many!).
2. Determine all productions $A_n \to \alpha$ with $\alpha \notin N$.
3. Add corresponding productions $A_1 \to \alpha$ to P.
4. Remove all chain rules from P.

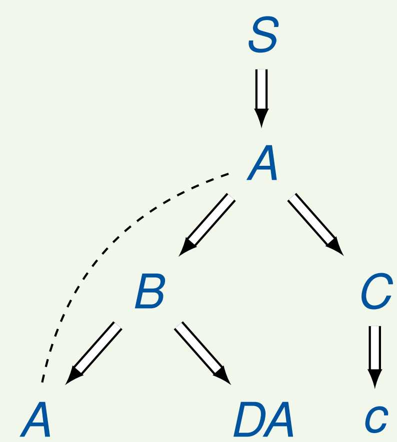
This yields G''.

Example C.12

$$G' : \begin{array}{rcl} S & \to & A \\ A & \to & B \mid C \\ B & \to & A \mid DA \\ C & \to & c \\ D & \to & d \end{array}$$

is converted to

$$G'' : \begin{array}{rcl} S & \to & DA \mid c \\ A & \to & DA \mid c \\ B & \to & DA \mid c \\ C & \to & c \\ D & \to & d \end{array}$$



Step 3: Elimination of Rules $A \to A_1 \dots A_n$ with $n > 2$

Procedure

Iteratively apply the following transformation:

1. For every $A \to A_1 \dots A_n$ with $n > 2$, introduce a new nonterminal symbol $B \in N$.
2. Replace original production by $A \to A_1 B$.
3. Add new production $B \to A_2 \dots A_n$.

This yields G'''.

Step 3: Elimination of Rules $A \to A_1 \dots A_n$ with $n > 2$

Procedure

Iteratively apply the following transformation:

1. For every $A \to A_1 \dots A_n$ with $n > 2$, introduce a new nonterminal symbol $B \in N$.
2. Replace original production by $A \to A_1 B$.
3. Add new production $B \to A_2 \dots A_n$.

This yields G'''.

Example C.13

$$G'' : \begin{array}{lcl} S & \to & AB \mid ASB \\ A & \to & a \\ B & \to & b \end{array}$$

is converted to

$$G''' : \begin{array}{lcl} S & \to & AB \mid AC \\ A & \to & a \\ B & \to & b \\ C & \to & SB \end{array}$$

Summary: Chomsky Normal Form

Seen:

- Chomsky NF: all productions of the form $A \to BC$ or $A \to a$

Summary: Chomsky Normal Form

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- Chomsky NF: all productions of the form $A \to BC$ or $A \to a$

Next:

- Exploit Chomsky Normal Form to solve word problem for CFL

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Outline of Part C

The Word Problem for CFL

Word Problem for ε-free CFL

Given CFG $G = \langle N, \Sigma, P, S \rangle$ such that $\varepsilon \notin L(G)$ and $w \in \Sigma^+$, decide whether $w \in L(G)$ or not.

(If $w = \varepsilon$, then $w \in L(G)$ easily decidable for arbitrary G)

Word Problem for ε-free CFL

Given CFG $G = \langle N, \Sigma, P, S \rangle$ such that $\varepsilon \notin L(G)$ and $w \in \Sigma^+$, decide whether $w \in L(G)$ or not.

(If $w = \varepsilon$, then $w \in L(G)$ easily decidable for arbitrary G)

Algorithm C.14 (by Cocke, Younger, Kasami – CYK algorithm)

1. *Transform G into Chomsky NF*
2. *Let* $w = a_1 \ldots a_n$ *(*$n \geq 1$*)*
3. *Let* $w[i,j] := a_i \ldots a_j$ *for every* $1 \leq i \leq j \leq n$
4. *Consider segments* $w[i,j]$ *in order of increasing length, starting with* $w[i,i] = a_i$ *(i.e., letters)*
5. *In each case, determine* $N_{i,j} := \{A \in N \mid A \Rightarrow^* w[i,j]\}$ *using a "dynamic programming" approach:*
   - $i = j$: $N_{i,i} = \{A \in N \mid A \to a_i \in P\}$
   - $i < j$: $N_{i,j} = \{A \in N \mid \exists B, C \in N, k \in \{i, \ldots, j-1\} : A \to BC \in P, B \in N_{i,k}, C \in N_{k+1,j}\}$
6. *Test whether* $S \in N_{1,n}$ *(and thus, whether* $S \Rightarrow^* w[1,n] = w$*)*

Matrix Representation of CYK Algorithm

	a_1	a_2	a_3	$\cdots$	a_n
$i \backslash j$	1	2	3	$\cdots$	n
1	$N_{1,1}$	$N_{1,2}$	$N_{1,3}$	$\cdots$	$N_{1,n}$
2	X	$N_{2,2}$	$N_{2,3}$	$\cdots$	$N_{2,n}$
3	X	X	$N_{3,3}$	$\cdots$	$N_{3,n}$
$\vdots$	$\vdots$	$\vdots$		$\cdots$	$\vdots$
n	X	X	$\cdots$	$\cdots$	$N_{n,n}$

Matrix Representation of CYK Algorithm

	a_1	a_2	a_3	$\cdots$	a_n
$i \backslash j$	1	2	3	$\cdots$	n
1	$N_{1,1}$	$N_{1,2}$	$N_{1,3}$	$\cdots$	$N_{1,n}$
2	X	$N_{2,2}$	$N_{2,3}$	$\cdots$	$N_{2,n}$
3	X	X	$N_{3,3}$	$\cdots$	$N_{3,n}$
$\vdots$	$\vdots$	$\vdots$		$\cdots$	$\vdots$
n	X	X	$\cdots$	$\cdots$	$N_{n,n}$

$$
\begin{aligned}
N_{1,1} &= \{A \in N \mid A \to a_1 \in P\} \\
N_{2,2} &= \{A \in N \mid A \to a_2 \in P\} \\
&\vdots
\end{aligned}
$$

Matrix Representation of CYK Algorithm

	a_1	a_2	a_3	$\cdots$	a_n
$i \backslash j$	1	2	3	$\cdots$	n
1	$N_{1,1}$	$N_{1,2}$	$N_{1,3}$	$\cdots$	$N_{1,n}$
2	X	$N_{2,2}$	$N_{2,3}$	$\cdots$	$N_{2,n}$
3	X	X	$N_{3,3}$	$\cdots$	$N_{3,n}$
$\vdots$	$\vdots$	$\vdots$		$\cdots$	$\vdots$
n	X	X	$\cdots$	$\cdots$	$N_{n,n}$

$$
\begin{aligned}
N_{1,1} &= \{A \in N \mid A \to a_1 \in P\} \\
N_{2,2} &= \{A \in N \mid A \to a_2 \in P\} \\
&\vdots \\
N_{1,2} &= \{A \in N \mid \exists B, C \in N : A \to BC \in P, B \in N_{1,1}, C \in N_{2,2}\} \\
N_{2,3} &= \{A \in N \mid \exists B, C \in N : A \to BC \in P, B \in N_{2,2}, C \in N_{3,3}\} \\
&\vdots
\end{aligned}
$$

Matrix Representation of CYK Algorithm

	a_1	a_2	a_3	$\cdots$	a_n
$i \backslash j$	1	2	3	$\cdots$	n
1	$N_{1,1}$	$N_{1,2}$	$N_{1,3}$	$\cdots$	$N_{1,n}$
2	X	$N_{2,2}$	$N_{2,3}$	$\cdots$	$N_{2,n}$
3	X	X	$N_{3,3}$	$\cdots$	$N_{3,n}$
$\vdots$	$\vdots$	$\vdots$		$\cdots$	$\vdots$
n	X	X	$\cdots$	$\cdots$	$N_{n,n}$

$$
\begin{aligned}
N_{1,1} &= \{A \in N \mid A \to a_1 \in P\} \\
N_{2,2} &= \{A \in N \mid A \to a_2 \in P\} \\
&\vdots \\
N_{1,2} &= \{A \in N \mid \exists B, C \in N : A \to BC \in P, B \in N_{1,1}, C \in N_{2,2}\} \\
N_{2,3} &= \{A \in N \mid \exists B, C \in N : A \to BC \in P, B \in N_{2,2}, C \in N_{3,3}\} \\
&\vdots \\
N_{1,3} &= \{A \in N \mid \exists B, C \in N : A \to BC \in P, B \in N_{1,1}, C \in N_{2,3}\} \\
&\cup \{A \in N \mid \exists B, C \in N : A \to BC \in P, B \in N_{1,2}, C \in N_{3,3}\} \\
N_{2,4} &= \{A \in N \mid \exists B, C \in N : A \to BC \in P, B \in N_{2,2}, C \in N_{3,4}\} \\
&\cup \{A \in N \mid \exists B, C \in N : A \to BC \in P, B \in N_{2,3}, C \in N_{4,4}\} \\
&\vdots
\end{aligned}
$$

Applying the CYK Algorithm

Example C.15

- $G: \; S \to SA \mid a$
  $A \to BS$
  $B \to BB \mid BS \mid b \mid c$
- $w = abaaba$

Applying the CYK Algorithm

Example C.15

- $G: \quad S \rightarrow SA \mid a$
  $\quad A \rightarrow BS$
  $\quad B \rightarrow BB \mid BS \mid b \mid c$
- $w = abaaba$

	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1						
2	X					
3	X	X				
4	X	X	X			
5	X	X	X	X		
6	X	X	X	X	X	

Applying the CYK Algorithm

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- $G: \; S \to SA \mid a$
  $A \to BS$
  $B \to BB \mid BS \mid b \mid c$
- $w = abaaba$

	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$					
2	X					
3	X	X	$\{S\}$			
4	X	X	X	$\{S\}$		
5	X	X	X	X		
6	X	X	X	X	X	$\{S\}$

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	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$					
2	X	$\{B\}$				
3	X	X	$\{S\}$			
4	X	X	X	$\{S\}$		
5	X	X	X	X	$\{B\}$	
6	X	X	X	X	X	$\{S\}$

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	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$				
2	X	$\{B\}$				
3	X	X	$\{S\}$	$\emptyset$		
4	X	X	X	$\{S\}$	$\emptyset$	
5	X	X	X	X	$\{B\}$	
6	X	X	X	X	X	$\{S\}$

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	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$				
2	X	$\{B\}$	$\{A \quad\}$			
3	X	X	$\{S\}$	$\emptyset$		
4	X	X	X	$\{S\}$	$\emptyset$	
5	X	X	X	X	$\{B\}$	$\{A \quad\}$
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

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  $\quad\quad A \to BS$
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- $w = abaaba$

	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$				
2	X	$\{B\}$	$\{A, B\}$			
3	X	X	$\{S\}$	$\emptyset$		
4	X	X	X	$\{S\}$	$\emptyset$	
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

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- $G: \quad S \to SA \mid a$
  $A \to BS$
  $B \to BB \mid BS \mid b \mid c$
- $w = abaaba$

	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$			
2	X	$\{B\}$	$\{A, B\}$			
3	X	X	$\{S\}$	$\emptyset$		
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

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	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$			
2	X	$\{B\}$	$\{A, B\}$	$\{A \quad\}$		
3	X	X	$\{S\}$	$\emptyset$		
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

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$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$			
2	X	$\{B\}$	$\{A, B\}$	$\{A, B\}$		
3	X	X	$\{S\}$	$\emptyset$		
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

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  $A \rightarrow BS$
  $B \rightarrow BB \mid BS \mid b \mid c$
- $w = abaaba$

	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$			
2	X	$\{B\}$	$\{A, B\}$	$\{A, B\}$		
3	X	X	$\{S\}$	$\emptyset$	$\emptyset$	
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

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	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$	$\{S\}$		
2	X	$\{B\}$	$\{A, B\}$	$\{A, B\}$		
3	X	X	$\{S\}$	$\emptyset$	$\emptyset$	
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

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  $A \rightarrow BS$
  $B \rightarrow BB \mid BS \mid b \mid c$
- $w = abaaba$

	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$	$\{S\}$		
2	X	$\{B\}$	$\{A, B\}$	$\{A, B\}$	$\{B\}$	
3	X	X	$\{S\}$	$\emptyset$	$\emptyset$	
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

Example C.15

- $G: \; S \to SA \mid a$
  $A \to BS$
  $B \to BB \mid BS \mid b \mid c$
- $w = abaaba$

	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$	$\{S\}$		
2	X	$\{B\}$	$\{A, B\}$	$\{A, B\}$	$\{B\}$	
3	X	X	$\{S\}$	$\emptyset$	$\emptyset$	$\emptyset$
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

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- $G: \; S \to SA \mid a$
  $A \to BS$
  $B \to BB \mid BS \mid b \mid c$
- $w = abaaba$

	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$	$\{S\}$	$\emptyset$	
2	X	$\{B\}$	$\{A,B\}$	$\{A,B\}$	$\{B\}$	
3	X	X	$\{S\}$	$\emptyset$	$\emptyset$	$\emptyset$
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A,B\}$
6	X	X	X	X	X	$\{S\}$

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  $A \to BS$
  $B \to BB \mid BS \mid b \mid c$
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	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$	$\{S\}$	$\emptyset$	
2	X	$\{B\}$	$\{A, B\}$	$\{A, B\}$	$\{B\}$	$\{A \quad\}$
3	X	X	$\{S\}$	$\emptyset$	$\emptyset$	$\emptyset$
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

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	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$	$\{S\}$	$\emptyset$	
2	X	$\{B\}$	$\{A, B\}$	$\{A, B\}$	$\{B\}$	$\{A, B\}$
3	X	X	$\{S\}$	$\emptyset$	$\emptyset$	$\emptyset$
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

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- $G: \quad S \to SA \mid a$
  $\quad\quad A \to BS$
  $\quad\quad B \to BB \mid BS \mid b \mid c$
- $w = abaaba$

	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$	$\{S\}$	$\emptyset$	$\{S\}$
2	X	$\{B\}$	$\{A, B\}$	$\{A, B\}$	$\{B\}$	$\{A, B\}$
3	X	X	$\{S\}$	$\emptyset$	$\emptyset$	$\emptyset$
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

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  $A \to BS$
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- $w = abaaba$

	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$	$\{S\}$	$\emptyset$	$\{S\}$
2	X	$\{B\}$	$\{A, B\}$	$\{A, B\}$	$\{B\}$	$\{A, B\}$
3	X	X	$\{S\}$	$\emptyset$	$\emptyset$	$\emptyset$
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

$$S \in N_{1,6} \implies w = abaaba \in L(G)$$

Summary: The Word Problem for Context-Free Languages

Seen:

- Given CFG G and $w \in \Sigma^*$, decide whether $w \in L(G)$ or not
- Decidable using CYK algorithm (based on dynamic programming)
- Cubic complexity

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Next:

- Emptiness problem

Outline of Part C

The Emptiness Problem

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- Important problem with many applications
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- For regular languages this was easy: check in the corresponding DFA whether some final state is reachable from the initial state.
- Here: test whether start symbol is productive, i.e., whether it generates a terminal word

The Emptiness Test

Algorithm C.16 (Emptiness Test)

Input: $G = \langle N, \Sigma, P, S \rangle$

Question: $L(G) = \emptyset$?

Procedure: *mark every* $a \in \Sigma$ *as productive*;

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repeat
   if there is A → α ∈ P such that all symbols in α productive then
      mark A as productive
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until no further productive symbols found;
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Seen:

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Next:

- Closure properties of CFLs

Outline of Part C

Positive Results

Theorem C.18

The set of CFLs is closed under concatenation, union, and iteration.

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For $i = 1, 2$, let $G_i = \langle N_i, \Sigma, P_i, S_i \rangle$ with $L_i := L(G_i)$ and $N_1 \cap N_2 = \emptyset$, and let $S \notin N_1 \cup N_2$ be a fresh nonterminal. Then

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- L_1^* is generated by $G := \langle N, \Sigma, P, S \rangle$ with $N := \{S\} \cup N_1$ and
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Negative Results

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The set of CFLs is not closed under intersection and complement.

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- Intersection: both

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  and

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- Complement: if CFLs were closed under complement, then also under intersection (as $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$).

□

Overview of Decidability and Closure Results

Decidability Results			
Class	$w \in L$	$L = \emptyset$	$L_1 = L_2$
Reg	+	+	+
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Closure Results					
Class	$L_1 \cdot L_2$	$L_1 \cup L_2$	$L_1 \cap L_2$	$\overline{L}$	L^*
Reg	+	+	+	+	+
CFL	+	+	−	−	+

Summary: Closure Properties of Context-Free Languages

Seen:

- Closure under concatenation, union and iteration
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Next:

- Automata model for CFLs

Outline of Part C

Pushdown Automata I

- **Goal:** introduce an automata model which exactly accepts CFLs
- **Clear:** DFA not sufficient
  (missing "counting capability", e.g. for $\{a^n b^n \mid n \geq 1\}$)

Pushdown Automata I

- **Goal:** introduce an automata model which exactly accepts CFLs
- **Clear:** DFA not sufficient
  (missing "counting capability", e.g. for $\{a^n b^n \mid n \geq 1\}$)
- DFA will be extended to pushdown automata by
  - adding a pushdown store which stores symbols from a pushdown alphabet and uses a special bottom symbol
  - adding push and pop operations to transitions

Definition C.20

A pushdown automaton (PDA) is of the form $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ where

- Q is a finite set of states
- Σ is the (finite) input alphabet
- Γ is the (finite) pushdown alphabet
- $\Delta \subseteq (Q \times \Gamma \times \Sigma_\varepsilon) \times (Q \times \Gamma^*)$ is a finite set of transitions
- $q_0 \in Q$ is the initial state
- Z_0 is the (pushdown) bottom symbol
- $F \subseteq Q$ is a set of final states

Interpretation of $((q, Z, x), (q', \delta)) \in \Delta$: if the PDA $\mathfrak{A}$ is in state q where Z is on top of the stack and x is the next input symbol (or empty), then $\mathfrak{A}$ reads x, replaces Z by δ, and changes into the state q'.

Configurations, Runs, Acceptance

Definition C.21

Let $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ be a PDA.

- An element of $Q \times \Gamma^* \times \Sigma^*$ is called a configuration of $\mathfrak{A}$.
- The initial configuration for input $w \in \Sigma^*$ is given by (q_0, Z_0, w).
- The set of final configurations is given by $F \times \{\varepsilon\} \times \{\varepsilon\}$.
- If $((q, Z, x), (q', \delta)) \in \Delta$, then $(q, Z\gamma, xw) \vdash (q', \delta\gamma, w)$ for every $\gamma \in \Gamma^*$, $w \in \Sigma^*$.

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- $\mathfrak{A}$ accepts $w \in \Sigma^*$ if $(q_0, Z_0, w) \vdash^* (q, \varepsilon, \varepsilon)$ for some $q \in F$.
- The language accepted by $\mathfrak{A}$ is $L(\mathfrak{A}) := \{w \in \Sigma^* \mid \mathfrak{A} \text{ accepts } w\}$.
- A language L is called PDA-recognisable if $L = L(\mathfrak{A})$ for some PDA $\mathfrak{A}$.
- Two PDA $\mathfrak{A}_1, \mathfrak{A}_2$ are called equivalent if $L(\mathfrak{A}_1) = L(\mathfrak{A}_2)$.

Examples of PDA I

Example C.22 (PDA for $L = \{a^n b^n \mid n \geq 1\}$)

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- $Q = \{q_0, q_1, q_2\}$
  - q_0: construction of PD while reading a's
  - q_1: deconstruction while reading b's
  - q_2: accepting state
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, Z\}$
  - Z_0 = bottom
  - $\#Z$ on PD = $\#a - \#b$ read so far
- $F = \{q_2\}$

- Δ:

$((q_0, Z_0, a), (q_0, ZZ_0))$	read first a
$((q_0, Z, a), (q_0, ZZ))$	read following a's
$((q_0, Z, b), (q_1, \varepsilon))$	read first b
$((q_1, Z, b), (q_1, \varepsilon))$	read following b's
$((q_1, Z_0, \varepsilon), (q_2, \varepsilon))$	change to final state

- Observation: no transitions for
  - (q_0, Z_0, b): input must start with a
  - (q_1, Z, a): no a's following b's
  - (q_1, Z_0, b): more b's than a's
  - ...

Accepting run of PDA for input $w = aabb$:

(remember: if $((q, Z, x), (q', \delta)) \in \Delta$, then $(q, Z\gamma, xw) \vdash (q', \delta\gamma, w)$)

$$(q_0, Z_0, aabb) \vdash (q_0, ZZ_0, abb) \vdash (q_0, ZZZ_0, bb) \vdash (q_1, ZZ_0, b) \vdash (q_1, Z_0, \varepsilon)$$

Example C.22 (PDA for $L = \{a^n b^n \mid n \geq 1\}$)

$\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

- $Q = \{q_0, q_1, q_2\}$
  - q_0: construction of PD while reading a's
  - q_1: deconstruction while reading b's
  - q_2: accepting state
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, Z\}$
  - Z_0 = bottom
  - $\#Z$ on PD = $\#a - \#b$ read so far
- $F = \{q_2\}$

- Δ: $((q_0, Z_0, a),(q_0, ZZ_0))$ read first a
  $((q_0, Z, a),(q_0, ZZ))$ read following a's
  $((q_0, Z, b),(q_1, \varepsilon))$ read first b
  $((q_1, Z, b),(q_1, \varepsilon))$ read following b's
  $((q_1, Z_0, \varepsilon),(q_2, \varepsilon))$ change to final state
- Observation: no transitions for
  - (q_0, Z_0, b): input must start with a
  - (q_1, Z, a): no a's following b's
  - (q_1, Z_0, b): more b's than a's
  - ...

Accepting run of PDA for input $w = aabb$:

(remember: if $((q, Z, x), (q', \delta)) \in \Delta$, then $(q, Z\gamma, xw) \vdash (q', \delta\gamma, w)$)

$$(q_0, Z_0, aabb) \vdash (q_0, ZZ_0, abb) \vdash (q_0, ZZZ_0, bb) \vdash (q_1, ZZ_0, b) \vdash (q_1, Z_0, \varepsilon) \vdash (q_2, \varepsilon, \varepsilon)$$

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a,b\}^*\}$ (palindromes of even length))

Idea:
1. $\mathfrak{A}$ pushes input w
2. switches nondeterministically to the w^R recognition phase
3. compares w and w^R symbol-wise by matching steps
4. accepts with empty pushdown

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a,b\}^*\}$ (palindromes of even length))

Idea:
1. $\mathfrak{A}$ pushes input w
2. switches nondeterministically to the w^R recognition phase
3. compares w and w^R symbol-wise by matching steps
4. accepts with empty pushdown

Formally: $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$

- $Q = \{q_0, q_1, q_2\}$

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a,b\}^*\}$ (palindromes of even length))

Idea:
1. $\mathfrak{A}$ pushes input w
2. switches nondeterministically to the w^R recognition phase
3. compares w and w^R symbol-wise by matching steps
4. accepts with empty pushdown

Formally: $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{a, b\}$

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a, b\}^*\}$ (palindromes of even length))

Idea:
1. $\mathfrak{A}$ pushes input w
2. switches nondeterministically to the w^R recognition phase
3. compares w and w^R symbol-wise by matching steps
4. accepts with empty pushdown

Formally: $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F\rangle$

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, a, b\}$
- $F = \{q_2\}$

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a,b\}^*\}$ (palindromes of even length))

Idea:
1. $\mathfrak{A}$ pushes input w
2. switches nondeterministically to the w^R recognition phase
3. compares w and w^R symbol-wise by matching steps
4. accepts with empty pushdown

Formally: $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, a, b\}$
- $F = \{q_2\}$
- Δ:

$((q_0, Z, c), (q_0, cZ))$	for $Z \in \Gamma, c \in \Sigma$	(1)
$((q_0, c, c), (q_1, \varepsilon))$	for $c \in \Sigma$	(2)
$((q_0, Z_0, \varepsilon), (q_1, Z_0))$		(2)
$((q_1, c, c), (q_1, \varepsilon))$	for $c \in \Sigma$	(3)
$((q_1, Z_0, \varepsilon), (q_2, \varepsilon))$		(4)

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a,b\}^*\}$ (palindromes of even length))

Idea:
1. $\mathfrak{A}$ pushes input w
2. switches nondeterministically to the w^R recognition phase
3. compares w and w^R symbol-wise by matching steps
4. accepts with empty pushdown

Formally: $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F\rangle$

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, a, b\}$
- $F = \{q_2\}$
- Δ:

$((q_0, Z, c), (q_0, cZ))$	for $Z \in \Gamma, c \in \Sigma$	(1)
$((q_0, c, c), (q_1, \varepsilon))$	for $c \in \Sigma$	(2)
$((q_0, Z_0, \varepsilon), (q_1, Z_0))$		(2)
$((q_1, c, c), (q_1, \varepsilon))$	for $c \in \Sigma$	(3)
$((q_1, Z_0, \varepsilon), (q_2, \varepsilon))$		(4)

Accepting run of PDA for input $w = abba$:

$(q_0, Z_0, abba)$

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a,b\}^*\}$ (palindromes of even length))

Idea:
1. $\mathfrak{A}$ pushes input w
2. switches nondeterministically to the w^R recognition phase
3. compares w and w^R symbol-wise by matching steps
4. accepts with empty pushdown

Formally: $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F\rangle$

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, a, b\}$
- $F = \{q_2\}$

- Δ:

$((q_0, Z, c), (q_0, cZ))$	for $Z \in \Gamma, c \in \Sigma$	(1)
$((q_0, c, c), (q_1, \varepsilon))$	for $c \in \Sigma$	(2)
$((q_0, Z_0, \varepsilon), (q_1, Z_0))$		(2)
$((q_1, c, c), (q_1, \varepsilon))$	for $c \in \Sigma$	(3)
$((q_1, Z_0, \varepsilon), (q_2, \varepsilon))$		(4)

Accepting run of PDA for input $w = abba$:

$$(q_0, Z_0, abba) \vdash (q_0, aZ_0, bba)$$

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a,b\}^*\}$ (palindromes of even length))

Idea:
1. $\mathfrak{A}$ pushes input w
2. switches nondeterministically to the w^R recognition phase
3. compares w and w^R symbol-wise by matching steps
4. accepts with empty pushdown

Formally: $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F\rangle$

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, a, b\}$
- $F = \{q_2\}$
- Δ:
  - $((q_0, Z, c), (q_0, cZ))$ for $Z \in \Gamma, c \in \Sigma$ (1)
  - $((q_0, c, c), (q_1, \varepsilon))$ for $c \in \Sigma$ (2)
  - $((q_0, Z_0, \varepsilon), (q_1, Z_0))$ (2)
  - $((q_1, c, c), (q_1, \varepsilon))$ for $c \in \Sigma$ (3)
  - $((q_1, Z_0, \varepsilon), (q_2, \varepsilon))$ (4)

Accepting run of PDA for input $w = abba$:

$$(q_0, Z_0, abba) \vdash (q_0, aZ_0, bba) \vdash (q_0, baZ_0, ba)$$

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a,b\}^*\}$ (palindromes of even length))

Idea:
1. $\mathfrak{A}$ pushes input w
2. switches nondeterministically to the w^R recognition phase
3. compares w and w^R symbol-wise by matching steps
4. accepts with empty pushdown

Formally: $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, a, b\}$
- $F = \{q_2\}$
- Δ:

$((q_0, Z, c), (q_0, cZ))$	for $Z \in \Gamma, c \in \Sigma$	(1)
$((q_0, c, c), (q_1, \varepsilon))$	for $c \in \Sigma$	(2)
$((q_0, Z_0, \varepsilon), (q_1, Z_0))$		(2)
$((q_1, c, c), (q_1, \varepsilon))$	for $c \in \Sigma$	(3)
$((q_1, Z_0, \varepsilon), (q_2, \varepsilon))$		(4)

Accepting run of PDA for input $w = abba$:

$$(q_0, Z_0, abba) \vdash (q_0, aZ_0, bba) \vdash (q_0, baZ_0, ba) \vdash (q_1, aZ_0, a)$$

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a,b\}^*\}$ (palindromes of even length))

Idea:
1. $\mathfrak{A}$ pushes input w
2. switches nondeterministically to the w^R recognition phase
3. compares w and w^R symbol-wise by matching steps
4. accepts with empty pushdown

Formally: $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, a, b\}$
- $F = \{q_2\}$

- Δ:
  $((q_0, Z, c), (q_0, cZ))$ for $Z \in \Gamma, c \in \Sigma$ (1)
  $((q_0, c, c), (q_1, \varepsilon))$ for $c \in \Sigma$ (2)
  $((q_0, Z_0, \varepsilon), (q_1, Z_0))$ (2)
  $((q_1, c, c), (q_1, \varepsilon))$ for $c \in \Sigma$ (3)
  $((q_1, Z_0, \varepsilon), (q_2, \varepsilon))$ (4)

Accepting run of PDA for input $w = abba$:

$$(q_0, Z_0, abba) \vdash (q_0, aZ_0, bba) \vdash (q_0, baZ_0, ba) \vdash (q_1, aZ_0, a) \vdash (q_1, Z_0, \varepsilon)$$

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a,b\}^*\}$ (palindromes of even length))

Idea:
1. $\mathfrak{A}$ pushes input w
2. switches nondeterministically to the w^R recognition phase
3. compares w and w^R symbol-wise by matching steps
4. accepts with empty pushdown

Formally: $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F\rangle$

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, a, b\}$
- $F = \{q_2\}$
- Δ:

$$
\begin{array}{lll}
((q_0, Z, c), (q_0, cZ)) & \text{for } Z \in \Gamma, c \in \Sigma & (1)\\
((q_0, c, c), (q_1, \varepsilon)) & \text{for } c \in \Sigma & (2)\\
((q_0, Z_0, \varepsilon), (q_1, Z_0)) & & (2)\\
((q_1, c, c), (q_1, \varepsilon)) & \text{for } c \in \Sigma & (3)\\
((q_1, Z_0, \varepsilon), (q_2, \varepsilon)) & & (4)
\end{array}
$$

Accepting run of PDA for input $w = abba$:

$$(q_0, Z_0, abba) \vdash (q_0, aZ_0, bba) \vdash (q_0, baZ_0, ba) \vdash (q_1, aZ_0, a) \vdash (q_1, Z_0, \varepsilon) \vdash (q_2, \varepsilon, \varepsilon)$$

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a, b\}^*\}$ (palindromes of even length))

Idea:
1. $\mathfrak{A}$ pushes input w
2. switches nondeterministically to the w^R recognition phase
3. compares w and w^R symbol-wise by matching steps
4. accepts with empty pushdown

Formally: $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F\rangle$

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, a, b\}$
- $F = \{q_2\}$
- Δ:

$((q_0, Z, c), (q_0, cZ))$	for $Z \in \Gamma, c \in \Sigma$	(1)
$((q_0, c, c), (q_1, \varepsilon))$	for $c \in \Sigma$	(2)
$((q_0, Z_0, \varepsilon), (q_1, Z_0))$		(2)
$((q_1, c, c), (q_1, \varepsilon))$	for $c \in \Sigma$	(3)
$((q_1, Z_0, \varepsilon), (q_2, \varepsilon))$		(4)

Accepting run of PDA for input $w = abba$:

$$(q_0, Z_0, abba) \vdash (q_0, aZ_0, bba) \vdash (q_0, baZ_0, ba) \vdash (q_1, aZ_0, a) \vdash (q_1, Z_0, \varepsilon) \vdash (q_2, \varepsilon, \varepsilon)$$

Observation: $\mathfrak{A}$ is nondeterministic – in a configuration of the form (q_0, cv, cw) ($c \in \Sigma$, $v, w \in \Sigma^*$), both (1) and (2) are applicable.

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a,b\}^*\}$ (palindromes of even length))

Idea:
1. $\mathfrak{A}$ pushes input w
2. switches nondeterministically to the w^R recognition phase
3. compares w and w^R symbol-wise by matching steps
4. accepts with empty pushdown

Formally: $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, a, b\}$
- $F = \{q_2\}$
- Δ:

$$
\begin{array}{lll}
((q_0, Z, c), (q_0, cZ)) & \text{for } Z \in \Gamma, c \in \Sigma & (1) \\
((q_0, c, c), (q_1, \varepsilon)) & \text{for } c \in \Sigma & (2) \\
((q_0, Z_0, \varepsilon), (q_1, Z_0)) & & (2) \\
((q_1, c, c), (q_1, \varepsilon)) & \text{for } c \in \Sigma & (3) \\
((q_1, Z_0, \varepsilon), (q_2, \varepsilon)) & & (4)
\end{array}
$$

Accepting run of PDA for input $w = abba$:

$$(q_0, Z_0, abba) \vdash (q_0, aZ_0, bba) \vdash (q_0, baZ_0, ba) \vdash (q_1, aZ_0, a) \vdash (q_1, Z_0, \varepsilon) \vdash (q_2, \varepsilon, \varepsilon)$$

Observation: $\mathfrak{A}$ is nondeterministic – in a configuration of the form (q_0, cv, cw) ($c \in \Sigma$, $v, w \in \Sigma^*$), both (1) and (2) are applicable. This yields rejecting runs, e.g.,

$$(q_0, Z_0, abba) \vdash (q_0, aZ_0, bba) \vdash (q_0, baZ_0, ba) \vdash (q_0, bbaZ_0, a) \vdash (q_0, abbaZ_0, \varepsilon) \not\vdash$$

Deterministic PDA

Definition C.24

A PDA $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is called deterministic (DPDA) if for every $q \in Q, Z \in \Gamma$,

1. for every $x \in \Sigma_\varepsilon$, there is at most one (q, Z, x)-transition in Δ and
2. if there is a (q, Z, a)-transition in Δ for some $a \in \Sigma$, then there is no (q, Z, ε)-transition in Δ.

Remark: this excludes two types of nondeterminism:

1. if $((q, Z, x), (q'_1, \delta_1)), ((q, Z, x), (q'_2, \delta_2)) \in \Delta$:
$$(q'_1, \delta_1\gamma, w) \dashv (q, Z\gamma, xw) \vdash (q'_2, \delta_2\gamma, w)$$
2. if $((q, Z, a), (q'_1, \delta_1)), ((q, Z, \varepsilon), (q'_2, \delta_2)) \in \Delta$:
$$(q'_1, \delta_1\gamma, w) \dashv (q, Z\gamma, aw) \vdash (q'_2, \delta_2\gamma, aw)$$

Deterministic PDA

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1. for every $x \in \Sigma_\varepsilon$, there is at most one (q, Z, x)-transition in Δ and
2. if there is a (q, Z, a)-transition in Δ for some $a \in \Sigma$, then there is no (q, Z, ε)-transition in Δ.

Remark: this excludes two types of nondeterminism:

1. if $((q, Z, x), (q'_1, \delta_1)), ((q, Z, x), (q'_2, \delta_2)) \in \Delta$:
$$(q'_1, \delta_1\gamma, w) \dashv (q, Z\gamma, xw) \vdash (q'_2, \delta_2\gamma, w)$$
2. if $((q, Z, a), (q'_1, \delta_1)), ((q, Z, \varepsilon), (q'_2, \delta_2)) \in \Delta$:
$$(q'_1, \delta_1\gamma, w) \dashv (q, Z\gamma, aw) \vdash (q'_2, \delta_2\gamma, aw)$$

Corollary C.25

In a DPDA, every configuration has at most one $\vdash$-successor.

Expressiveness of DPDA

One can show: determinism restricts the set of acceptable languages (DPDA-recognisable languages are closed under complement, which is generally not true for PDA-recognisable languages)

Expressiveness of DPDA

One can show: determinism restricts the set of acceptable languages (DPDA-recognisable languages are closed under complement, which is generally not true for PDA-recognisable languages)

Example C.26

The set of palindromes of even length is PDA-recognisable, but not DPDA-recognisable (without proof).

Summary: Pushdown Automata

Seen:

- Extension of finite automata by pushdown store
- Enables “counting” (e.g., $\{a^n b^n \mid n \geq 1\}$)
- Determinism restricts expressivity (in contrast to finite automata)

Summary: Pushdown Automata

Seen:

- Extension of finite automata by pushdown store
- Enables “counting” (e.g., $\{a^n b^n \mid n \geq 1\}$)
- Determinism restricts expressivity (in contrast to finite automata)

Next:

- Relation between PDA and context-free languages

Outline of Part C

PDA and Context-Free Languages I

Theorem C.27

A language is context-free iff it is PDA-recognisable.

PDA and Context-Free Languages I

Theorem C.27

A language is context-free iff it is PDA-recognisable.

Proof.

“$\Longleftarrow$”: omitted

“$\Longrightarrow$”: let $G = \langle N, \Sigma, P, S\rangle$ be a CFG. Construction of PDA $\mathfrak{A}_G$ recognising $L(G)$:

- $\mathfrak{A}_G$ simulates a derivation of G where always the leftmost nonterminal of a sentence is replaced (“leftmost derivation”)
- begin with S on pushdown
- if nonterminal on top: apply a corresponding production rule
- if terminal on top: match with next input symbol

(cf. formal construction on following slide)

□

PDA and Context-Free Languages II

Proof of Theorem C.27 (continued).

"$\Rightarrow$": Formally, $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

- $Q := \{q_0\}$
- $\Gamma := N \cup \Sigma$
- $Z_0 := S$
- for each $A \to \alpha \in P$: $((q_0, A, \varepsilon), (q_0, \alpha)) \in \Delta$ ("expansion")
- for each $a \in \Sigma$: $((q_0, a, a), (q_0, \varepsilon)) \in \Delta$ ("matching")
- $F := Q$

□

Proof of Theorem C.27 (continued).

"$\Rightarrow$": Formally, $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

- $Q := \{q_0\}$
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- for each $A \to \alpha \in P$: $((q_0, A, \varepsilon), (q_0, \alpha)) \in \Delta$ ("expansion")
- for each $a \in \Sigma$: $((q_0, a, a), (q_0, \varepsilon)) \in \Delta$ ("matching")
- $F := Q$

□

Example C.28 ("Bracket language" given by $G : S \to \langle\rangle \mid \langle S \rangle \mid SS$)

$\mathfrak{A}_G = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ with

- $Q = F = \{q_0\}$
- $\Sigma = \{\langle, \rangle\}$, $\Gamma = \{S, \langle, \rangle\}$
- $Z_0 = S$
- Δ: $((q_0, S, \varepsilon), (q_0, \langle\rangle))$ $((q_0, \langle, \langle), (q_0, \varepsilon))$
  $((q_0, S, \varepsilon), (q_0, \langle S \rangle))$ $((q_0, \rangle, \rangle), (q_0, \varepsilon))$
  $((q_0, S, \varepsilon), (q_0, SS))$

Proof of Theorem C.27 (continued).

"$\Rightarrow$": Formally, $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F\rangle$ is given by

- $Q := \{q_0\}$
- $\Gamma := N \cup \Sigma$
- $Z_0 := S$
- for each $A \to \alpha \in P$: $((q_0, A, \varepsilon), (q_0, \alpha)) \in \Delta$ ("expansion")
- for each $a \in \Sigma$: $((q_0, a, a), (q_0, \varepsilon)) \in \Delta$ ("matching")
- $F := Q$

$\square$

Example C.28 ("Bracket language" given by $G : S \to \langle\rangle \mid \langle S\rangle \mid SS$)

$\mathfrak{A}_G = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F\rangle$ with

- $Q = F = \{q_0\}$
- $\Sigma = \{\langle, \rangle\}$, $\Gamma = \{S, \langle, \rangle\}$
- $Z_0 = S$
- Δ: $((q_0, S, \varepsilon),(q_0, \langle\rangle))$ $\quad((q_0, \langle, \langle),(q_0, \varepsilon))$
  $((q_0, S, \varepsilon),(q_0, \langle S\rangle))$ $\quad((q_0, \rangle, \rangle),(q_0, \varepsilon))$
  $((q_0, S, \varepsilon),(q_0, SS))$

Accepting run for input $w = \langle\langle\rangle\rangle\langle\rangle$:

$$(q_0, S, \langle\langle\rangle\rangle\langle\rangle)$$

PDA and Context-Free Languages II

Proof of Theorem C.27 (continued).

"$\Rightarrow$": Formally, $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

- $Q := \{q_0\}$
- $\Gamma := N \cup \Sigma$
- $Z_0 := S$
- for each $A \to \alpha \in P$: $((q_0, A, \varepsilon), (q_0, \alpha)) \in \Delta$ ("expansion")
- for each $a \in \Sigma$: $((q_0, a, a), (q_0, \varepsilon)) \in \Delta$ ("matching")
- $F := Q$

$\square$

Example C.28 ("Bracket language" given by $G : S \to \langle\rangle \mid \langle S \rangle \mid SS$)

$\mathfrak{A}_G = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ with

- $Q = F = \{q_0\}$
- $\Sigma = \{\langle, \rangle\}$, $\Gamma = \{S, \langle, \rangle\}$
- $Z_0 = S$
- Δ: $((q_0, S, \varepsilon),(q_0, \langle\rangle))$ $((q_0, \langle, \langle),(q_0, \varepsilon))$
  $((q_0, S, \varepsilon),(q_0, \langle S \rangle))$ $((q_0, \rangle, \rangle),(q_0, \varepsilon))$
  $((q_0, S, \varepsilon),(q_0, SS))$

Accepting run for input $w = \langle\langle\rangle\rangle\langle\rangle$:

$$(q_0, S, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, SS, \langle\langle\rangle\rangle\langle\rangle)$$

PDA and Context-Free Languages II

Proof of Theorem C.27 (continued).

"$\Rightarrow$": Formally, $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F\rangle$ is given by

- $Q := \{q_0\}$
- $\Gamma := N \cup \Sigma$
- $Z_0 := S$
- for each $A \to \alpha \in P$: $((q_0, A, \varepsilon), (q_0, \alpha)) \in \Delta$ ("expansion")
- for each $a \in \Sigma$: $((q_0, a, a), (q_0, \varepsilon)) \in \Delta$ ("matching")
- $F := Q$

$\square$

Example C.28 ("Bracket language" given by $G : S \to \langle\rangle \mid \langle S\rangle \mid SS$)

$\mathfrak{A}_G = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F\rangle$ with

- $Q = F = \{q_0\}$
- $\Sigma = \{\langle, \rangle\}$, $\Gamma = \{S, \langle, \rangle\}$
- $Z_0 = S$
- Δ: $((q_0, S, \varepsilon),(q_0, \langle\rangle))$ $((q_0, \langle, \langle),(q_0, \varepsilon))$
  $((q_0, S, \varepsilon),(q_0, \langle S\rangle))$ $((q_0, \rangle, \rangle),(q_0, \varepsilon))$
  $((q_0, S, \varepsilon),(q_0, SS))$

Accepting run for input $w = \langle\langle\rangle\rangle\langle\rangle$:

$$(q_0, S, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, SS, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, \langle S\rangle S, \langle\langle\rangle\rangle\langle\rangle)$$

PDA and Context-Free Languages II

Proof of Theorem C.27 (continued).

"$\Rightarrow$": Formally, $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F\rangle$ is given by

- $Q := \{q_0\}$
- $\Gamma := N \cup \Sigma$
- $Z_0 := S$
- for each $A \to \alpha \in P$: $((q_0, A, \varepsilon), (q_0, \alpha)) \in \Delta$ ("expansion")
- for each $a \in \Sigma$: $((q_0, a, a), (q_0, \varepsilon)) \in \Delta$ ("matching")
- $F := Q$

□

Example C.28 ("Bracket language" given by $G : S \to \langle\rangle \mid \langle S\rangle \mid SS$)

$\mathfrak{A}_G = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F\rangle$ with

- $Q = F = \{q_0\}$
- $\Sigma = \{\langle, \rangle\}$, $\Gamma = \{S, \langle, \rangle\}$
- $Z_0 = S$
- Δ: $((q_0, S, \varepsilon),(q_0, \langle\rangle))$, $((q_0, S, \varepsilon),(q_0, \langle S\rangle))$, $((q_0, S, \varepsilon),(q_0, SS))$, $((q_0, \langle, \langle),(q_0, \varepsilon))$, $((q_0, \rangle, \rangle),(q_0, \varepsilon))$

Accepting run for input $w = \langle\langle\rangle\rangle\langle\rangle$:

$$(q_0, S, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, SS, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, \langle S\rangle S, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, S\rangle S, \langle\rangle\rangle\langle\rangle)$$

PDA and Context-Free Languages II

Proof of Theorem C.27 (continued).

"$\Rightarrow$": Formally, $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

- $Q := \{q_0\}$
- $\Gamma := N \cup \Sigma$
- $Z_0 := S$
- for each $A \to \alpha \in P$: $((q_0, A, \varepsilon), (q_0, \alpha)) \in \Delta$ ("expansion")
- for each $a \in \Sigma$: $((q_0, a, a), (q_0, \varepsilon)) \in \Delta$ ("matching")
- $F := Q$

$\square$

Example C.28 ("Bracket language" given by $G : S \to \langle\rangle \mid \langle S\rangle \mid SS$)

$\mathfrak{A}_G = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ with

- $Q = F = \{q_0\}$
- $\Sigma = \{\langle, \rangle\}$, $\Gamma = \{S, \langle, \rangle\}$
- $Z_0 = S$
- Δ: $((q_0, S, \varepsilon), (q_0, \langle\rangle))$, $((q_0, S, \varepsilon), (q_0, \langle S\rangle))$, $((q_0, S, \varepsilon), (q_0, SS))$, $((q_0, \langle, \langle), (q_0, \varepsilon))$, $((q_0, \rangle, \rangle), (q_0, \varepsilon))$

Accepting run for input $w = \langle\langle\rangle\rangle\langle\rangle$:

$$(q_0, S, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, SS, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, \langle S\rangle S, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, S\rangle S, \langle\rangle\rangle\langle\rangle)$$
$$\vdash (q_0, \langle\rangle\rangle S, \langle\rangle\rangle\langle\rangle)$$

PDA and Context-Free Languages II

Proof of Theorem C.27 (continued).

"$\Rightarrow$": Formally, $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F\rangle$ is given by

- $Q := \{q_0\}$
- $\Gamma := N \cup \Sigma$
- $Z_0 := S$
- for each $A \to \alpha \in P$: $((q_0, A, \varepsilon), (q_0, \alpha)) \in \Delta$ ("expansion")
- for each $a \in \Sigma$: $((q_0, a, a), (q_0, \varepsilon)) \in \Delta$ ("matching")
- $F := Q$

□

Example C.28 ("Bracket language" given by $G : S \to \langle\rangle \mid \langle S\rangle \mid SS$)

$\mathfrak{A}_G = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F\rangle$ with

- $Q = F = \{q_0\}$
- $\Sigma = \{\langle, \rangle\}$, $\Gamma = \{S, \langle, \rangle\}$
- $Z_0 = S$
- Δ: $((q_0, S, \varepsilon),(q_0, \langle\rangle))$ $\quad ((q_0, \langle, \langle),(q_0, \varepsilon))$
  $((q_0, S, \varepsilon),(q_0, \langle S\rangle))$ $\quad ((q_0, \rangle, \rangle),(q_0, \varepsilon))$
  $((q_0, S, \varepsilon),(q_0, SS))$

Accepting run for input $w = \langle\langle\rangle\rangle\langle\rangle$:

$$(q_0, S, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, SS, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, \langle S\rangle S, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, S\rangle S, \langle\rangle\rangle\langle\rangle)$$
$$\vdash (q_0, \langle\rangle\rangle S, \langle\rangle\rangle\langle\rangle) \vdash (q_0, \rangle\rangle S, \rangle\rangle\langle\rangle)$$

PDA and Context-Free Languages II

Proof of Theorem C.27 (continued).

"$\Rightarrow$": Formally, $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F\rangle$ is given by

- $Q := \{q_0\}$
- $\Gamma := N \cup \Sigma$
- $Z_0 := S$
- for each $A \to \alpha \in P$: $((q_0, A, \varepsilon), (q_0, \alpha)) \in \Delta$ ("expansion")
- for each $a \in \Sigma$: $((q_0, a, a), (q_0, \varepsilon)) \in \Delta$ ("matching")
- $F := Q$

$\square$

Example C.28 ("Bracket language" given by $G : S \to \langle\rangle \mid \langle S\rangle \mid SS$)

$\mathfrak{A}_G = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F\rangle$ with

- $Q = F = \{q_0\}$
- $\Sigma = \{\langle, \rangle\}$, $\Gamma = \{S, \langle, \rangle\}$
- $Z_0 = S$
- Δ: $((q_0, S, \varepsilon), (q_0, \langle\rangle))$, $((q_0, S, \varepsilon), (q_0, \langle S\rangle))$, $((q_0, S, \varepsilon), (q_0, SS))$, $((q_0, \langle, \langle), (q_0, \varepsilon))$, $((q_0, \rangle, \rangle), (q_0, \varepsilon))$

Accepting run for input $w = \langle\langle\rangle\rangle\langle\rangle$:

$$(q_0, S, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, SS, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, \langle S\rangle S, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, S\rangle S, \langle\rangle\rangle\langle\rangle)$$
$$\vdash (q_0, \langle\rangle\rangle S, \langle\rangle\rangle\langle\rangle) \vdash (q_0, \rangle\rangle S, \rangle\rangle\langle\rangle) \vdash (q_0, \rangle S, \rangle\langle\rangle)$$

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Proof of Theorem C.27 (continued).

"$\Rightarrow$": Formally, $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

- $Q := \{q_0\}$
- $\Gamma := N \cup \Sigma$
- $Z_0 := S$
- for each $A \to \alpha \in P$: $((q_0, A, \varepsilon), (q_0, \alpha)) \in \Delta$ ("expansion")
- for each $a \in \Sigma$: $((q_0, a, a), (q_0, \varepsilon)) \in \Delta$ ("matching")
- $F := Q$

$\square$

Example C.28 ("Bracket language" given by $G : S \to \langle\rangle \mid \langle S \rangle \mid SS$)

$\mathfrak{A}_G = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ with

- $Q = F = \{q_0\}$
- $\Sigma = \{\langle, \rangle\}$, $\Gamma = \{S, \langle, \rangle\}$
- $Z_0 = S$
- Δ: $((q_0, S, \varepsilon), (q_0, \langle\rangle))$, $((q_0, S, \varepsilon), (q_0, \langle S \rangle))$, $((q_0, S, \varepsilon), (q_0, SS))$, $((q_0, \langle, \langle), (q_0, \varepsilon))$, $((q_0, \rangle, \rangle), (q_0, \varepsilon))$

Accepting run for input $w = \langle\langle\rangle\rangle\langle\rangle$:

$$(q_0, S, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, SS, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, \langle S \rangle S, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, S\rangle S, \langle\rangle\rangle\langle\rangle)$$
$$\vdash (q_0, \langle\rangle\rangle S, \langle\rangle\rangle\langle\rangle) \vdash (q_0, \rangle\rangle S, \rangle\rangle\langle\rangle) \vdash (q_0, \rangle S, \rangle\langle\rangle) \vdash (q_0, S, \langle\rangle)$$

PDA and Context-Free Languages II

Proof of Theorem C.27 (continued).

"$\Rightarrow$": Formally, $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

- $Q := \{q_0\}$
- $\Gamma := N \cup \Sigma$
- $Z_0 := S$
- for each $A \to \alpha \in P$: $((q_0, A, \varepsilon), (q_0, \alpha)) \in \Delta$ ("expansion")
- for each $a \in \Sigma$: $((q_0, a, a), (q_0, \varepsilon)) \in \Delta$ ("matching")
- $F := Q$

$\square$

Example C.28 ("Bracket language" given by $G : S \to \langle\rangle \mid \langle S \rangle \mid SS$)

$\mathfrak{A}_G = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ with

- $Q = F = \{q_0\}$
- $\Sigma = \{\langle, \rangle\}$, $\Gamma = \{S, \langle, \rangle\}$
- $Z_0 = S$
- Δ: $((q_0, S, \varepsilon),(q_0, \langle\rangle))$ $\quad((q_0, \langle, \langle),(q_0, \varepsilon))$
  $((q_0, S, \varepsilon),(q_0, \langle S \rangle))$ $\quad((q_0, \rangle, \rangle),(q_0, \varepsilon))$
  $((q_0, S, \varepsilon),(q_0, SS))$

Accepting run for input $w = \langle\langle\rangle\rangle\langle\rangle$:

$$(q_0, S, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, SS, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, \langle S \rangle S, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, S\rangle S, \langle\rangle\rangle\langle\rangle)$$
$$\vdash (q_0, \langle\rangle\rangle S, \langle\rangle\rangle\langle\rangle) \vdash (q_0, \rangle\rangle S, \rangle\rangle\langle\rangle) \vdash (q_0, \rangle S, \rangle\langle\rangle) \vdash (q_0, S, \langle\rangle)$$
$$\vdash (q_0, \langle\rangle, \langle\rangle)$$

PDA and Context-Free Languages II

Proof of Theorem C.27 (continued).

"$\Rightarrow$": Formally, $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

- $Q := \{q_0\}$
- $\Gamma := N \cup \Sigma$
- $Z_0 := S$
- for each $A \to \alpha \in P$: $((q_0, A, \varepsilon), (q_0, \alpha)) \in \Delta$ ("expansion")
- for each $a \in \Sigma$: $((q_0, a, a), (q_0, \varepsilon)) \in \Delta$ ("matching")
- $F := Q$

□

Example C.28 ("Bracket language" given by $G : S \to \langle\rangle \mid \langle S \rangle \mid SS$)

$\mathfrak{A}_G = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ with

- $Q = F = \{q_0\}$
- $\Sigma = \{\langle, \rangle\}$, $\Gamma = \{S, \langle, \rangle\}$
- $Z_0 = S$
- Δ: $((q_0, S, \varepsilon), (q_0, \langle\rangle))$, $((q_0, S, \varepsilon), (q_0, \langle S \rangle))$, $((q_0, S, \varepsilon), (q_0, SS))$, $((q_0, \langle, \langle), (q_0, \varepsilon))$, $((q_0, \rangle, \rangle), (q_0, \varepsilon))$

Accepting run for input $w = \langle\langle\rangle\rangle\langle\rangle$:

$$
\begin{aligned}
& (q_0, S, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, SS, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, \langle S \rangle S, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, S\rangle S, \langle\rangle\rangle\langle\rangle) \\
\vdash\ & (q_0, \langle\rangle\rangle S, \langle\rangle\rangle\langle\rangle) \vdash (q_0, \rangle\rangle S, \rangle\rangle\langle\rangle) \vdash (q_0, \rangle S, \rangle\langle\rangle) \vdash (q_0, S, \langle\rangle) \\
\vdash\ & (q_0, \langle\rangle, \langle\rangle) \vdash (q_0, \rangle, \rangle)
\end{aligned}
$$

PDA and Context-Free Languages II

Proof of Theorem C.27 (continued).

"$\Rightarrow$": Formally, $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F\rangle$ is given by

- $Q := \{q_0\}$
- $\Gamma := N \cup \Sigma$
- $Z_0 := S$
- for each $A \to \alpha \in P$: $((q_0, A, \varepsilon), (q_0, \alpha)) \in \Delta$ ("expansion")
- for each $a \in \Sigma$: $((q_0, a, a), (q_0, \varepsilon)) \in \Delta$ ("matching")
- $F := Q$

□

Example C.28 ("Bracket language" given by $G : S \to \langle\rangle \mid \langle S\rangle \mid SS$)

$\mathfrak{A}_G = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F\rangle$ with

- $Q = F = \{q_0\}$
- $\Sigma = \{\langle, \rangle\}$, $\Gamma = \{S, \langle, \rangle\}$
- $Z_0 = S$
- Δ: $((q_0, S, \varepsilon),(q_0, \langle\rangle))$ $((q_0, \langle, \langle),(q_0, \varepsilon))$
  $((q_0, S, \varepsilon),(q_0, \langle S\rangle))$ $((q_0, \rangle, \rangle),(q_0, \varepsilon))$
  $((q_0, S, \varepsilon),(q_0, SS))$

Accepting run for input $w = \langle\langle\rangle\rangle\langle\rangle$:

$$\begin{aligned}
&(q_0, S, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, SS, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, \langle S\rangle S, \langle\langle\rangle\rangle\langle\rangle) \vdash (q_0, S\rangle S, \langle\rangle\rangle\langle\rangle)\\
\vdash\ &(q_0, \langle\rangle\rangle S, \langle\rangle\rangle\langle\rangle) \vdash (q_0, \rangle\rangle S, \rangle\rangle\langle\rangle) \vdash (q_0, \rangle S, \rangle\langle\rangle) \vdash (q_0, S, \langle\rangle)\\
\vdash\ &(q_0, \langle\rangle, \langle\rangle) \vdash (q_0, \rangle, \rangle) \vdash (q_0, \varepsilon, \varepsilon)
\end{aligned}$$

Summary: Pushdown Automata and Context-Free Languages

Seen:

- Construction of PDA for given CFG ($\Rightarrow$ parser generation!)
- Reverse direction also possible
- Thus: PDA and CFG equivalent

Summary: Pushdown Automata and Context-Free Languages

Seen:

- Construction of PDA for given CFG ($\Rightarrow$ parser generation!)
- Reverse direction also possible
- Thus: PDA and CFG equivalent

Outlook:

- Equivalence problem for CFG and PDA ("$L(X_1) = L(X_2)$?"): generally undecidable, but decidable for DPDA
- Pumping Lemma for CFL (e.g., to prove that $\{a^n b^n c^n \mid n \geq 1\}$ not context-free)
- Greibach Normal Form for CFG
- Systematic construction of deterministic and efficient parsers for compilers (LL/LR grammars)
- Non-context-free grammars and languages (e.g., context-sensitive languages such as $\{a^n b^n c^n \mid n \geq 1\}$)