Latticed *k*-Induction

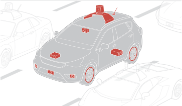
with an Application to Probabilistic Programs

Mingshuai Chen

—Joint work with K. Batz, B. L. Kaminski, J.-P. Katoen, C. Matheja, P. Schröer—

Cyber-Physical Systems (CPS)

An open, interconnected form of embedded systems that integrates capabilities of *computing*, *communication*, and *control*, among which many are safety-critical.



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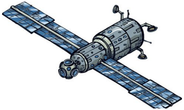

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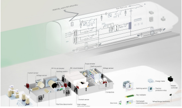

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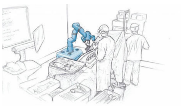

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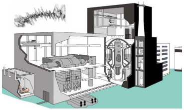

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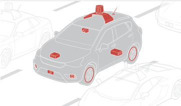

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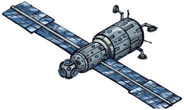

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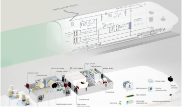

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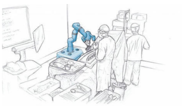

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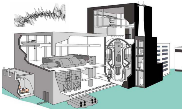

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"How can we provide people with CPS they can bet their lives on?"

— Jeannette Wing

Formal Methods

"By a formal method we shall understand a method whose techniques and tools can be explained in mathematics."

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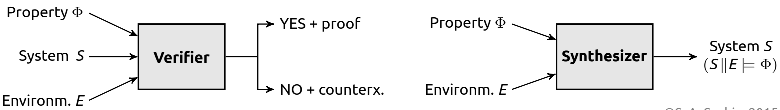

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Develop mathematically rigorous techniques for designing safety-critical CPS while *pushing the limits of automation* as far as possible.

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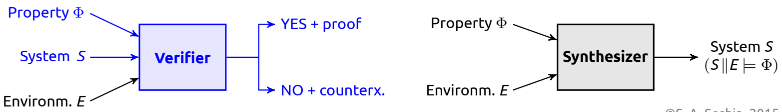

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Is k-induction applicable to verifying infinite-state probabilistic programs?

$\Downarrow$ **Latticed k-Induction**

Yes. It enables fully automatic verification of non-trivial properties.

k-Induction for Probabilistic Programs — Intuition

For a probabilistic loop C:

$$\texttt{while}\,(\,c = 1\,)\,\{\,c \coloneqq 0\,[\,1/2\,]\,x \coloneqq x + 1\,\}\,,$$

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is not inductive but 2-inductive.

k-Induction for Transition Systems

Given : TS $= (S, I, T)$, invariant property $P \subseteq S$.

Goal : Prove that P covers all *reachable states* of TS .

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Let $k \geq 1$. If the following two formulae are valid

$$\underbrace{I(s_1) \wedge T(s_1, s_2) \wedge \ldots \wedge T(s_{k-1}, s_k)}_{\text{all states reachable within } k-1 \text{ steps}} \implies \underbrace{P(s_1) \wedge \ldots \wedge P(s_k)}_{\text{are } P\text{-states}} \qquad \text{[base case]}$$

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- leave the Boolean domain and reason about quantities ;
- reason about sets of paths rather than individual paths.

Idea Sketch

k-induction for TS
in terms of a **SAT problem**

Idea Sketch

k-induction for TS
in terms of **fixed points**

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Idea Sketch

Latticed k-induction for proving $\mathrm{lfp}\ \Phi \sqsubseteq f$

$\uparrow$ abstract

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Whenever your problem boils down to verifying an upper bound on a lfp ,
latticed *k*-induction provides you with **inductive proof rules**!

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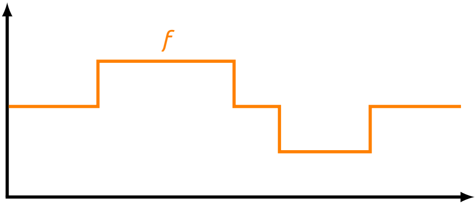

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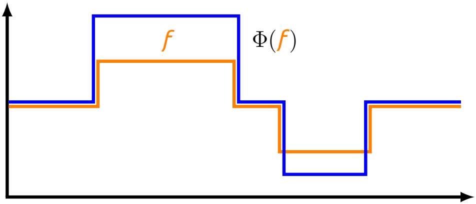

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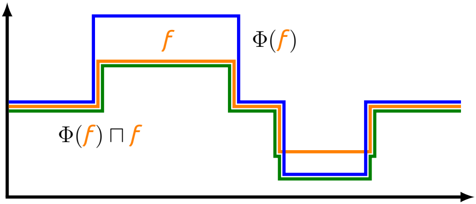

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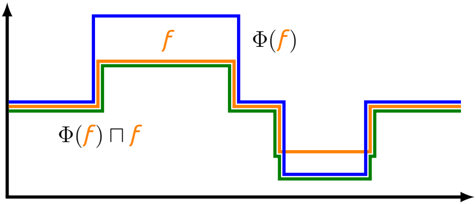

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For every $k \geq 1$,

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We call such f k-inductive invariant.

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Can be further generalized to *transfinite κ-induction* (not in this talk).

Key Insights of Soundness

Lemma (Descending chain)

Iterating Ψ_f on f yields a descending chain, i.e.,

$$f \sqsupseteq \Psi_f^1(f) \sqsupseteq \Psi_f^2(f) \sqsupseteq \Psi_f^3(f) \sqsupseteq \ldots .$$

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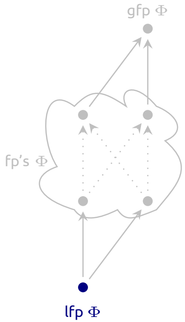

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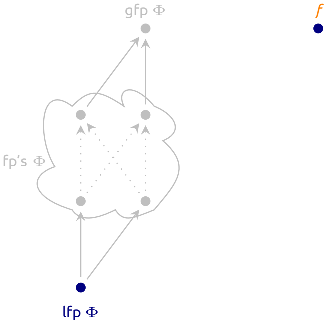

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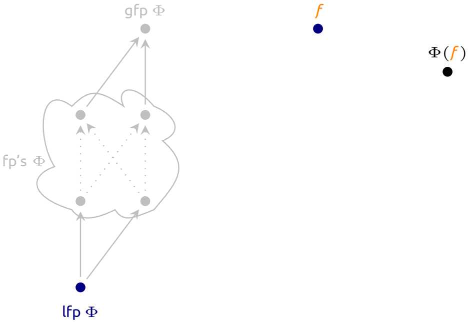

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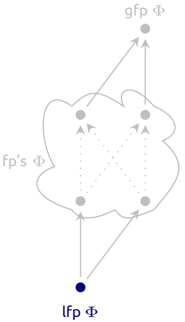

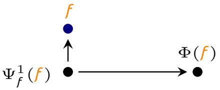

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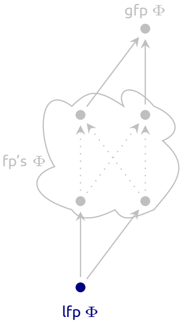

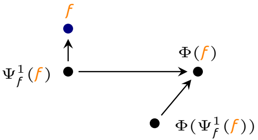

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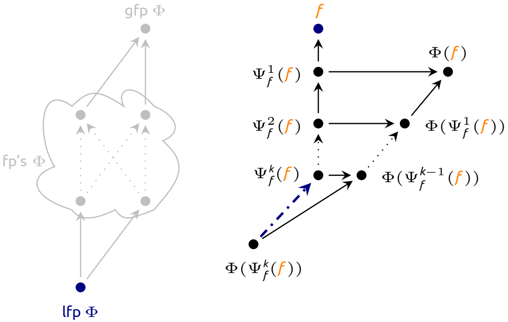

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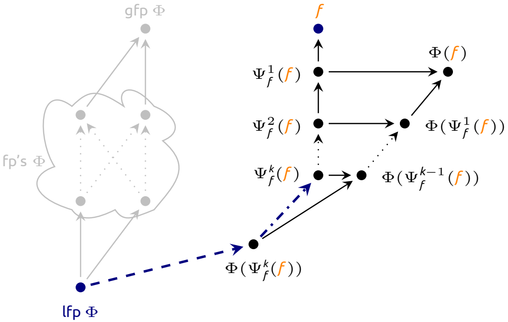

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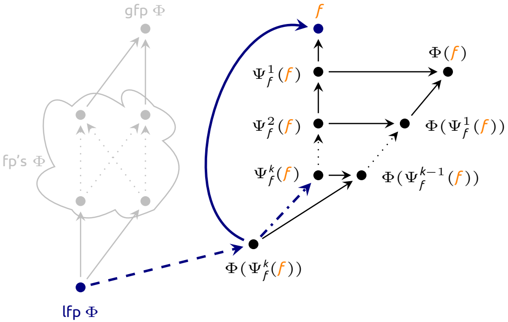

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f is a k-inductive invariant $\iff$ $\Psi_f^{k-1}(f)$ is an inductive invariant *stronger than* f.

Idea Sketch

Latticed k-induction for proving $\mathrm{lfp}\ \Phi \sqsubseteq f$

abstract

instantiate

k-induction for TS
in terms of **fixed points**

k-induction for **probabilistic programs**
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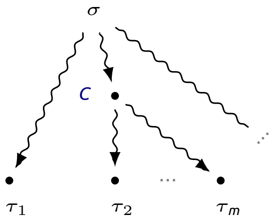

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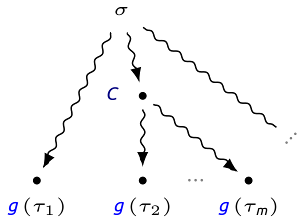

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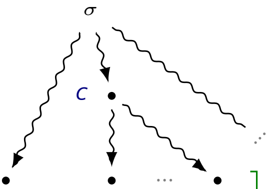

$$\mathrm{wp}[\![C]\!](g)(\sigma) \;\widehat{=}\; \mathrm{Exp}\left[\; g(\tau_1) \qquad g(\tau_2) \quad \cdots \quad g(\tau_m) \;\right]$$

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Given : Loop $C = \texttt{while}\,(\,\varphi\,)\,\{\,C'\,\}$, postexpectation $g \in \mathbb{E}$ and candidate $f \in \mathbb{E}$.

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Here

$$\Psi_f(h) = \Phi\,(h) \sqcap f \qquad \text{where for } h, h' \in \mathbb{E}, \quad h \sqcap h' = \lambda\sigma\text{.}\ \min\{h\,(\sigma), h'\,(\sigma)\} .$$

Tool Support

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functional · Artifact · CAV · AEC · Evaluated · available · reusable

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If $\mathrm{wp}[\![C]\!]\,(g) \not\leq f$, kipro2 finds via *bounded model checking* some $\sigma \in \Sigma$ with

$$\mathrm{wp}[\![C]\!]\,(g)\,(\sigma) > f(\sigma)\,.$$

Example — Geometric Distribution

For C given by

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the property

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Example — Uniform Sampling by Fair-Coin Flips [Lumbroso, 2013]

```
while(running = 0){

    v := 2*v;
    {c := 2*c+1}[0.5]{c := 2*c};
    if(not (v<n)){
        if((not (n=c)) & (not (n<c))){ #terminate
            running := 1
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For *arbitrary* array of fixed size $n = \{2, 3, 4, 5\}$, we verify

Pr(“sample fixed element”) $\leq 1/n$.

Empirical Results (Partial)

	postexpectation	variant	result	k	#formulae	formulae_t	sat_t	total_t
geo	x	1	ind	2	18	0.01	0.00	0.08
		2	ref	11	103	0.04	0.01	0.09
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unif_sam	$[c = i]$	1	ind	2	267	0.27	0.02	0.56
		2	ind	3	1402	1.45	0.10	1.81
		3	ind	3	1402	1.48	0.11	1.86
		4	ind	5	40568	47.31	15.70	63.28
		5	TO	–	–	–	–	–

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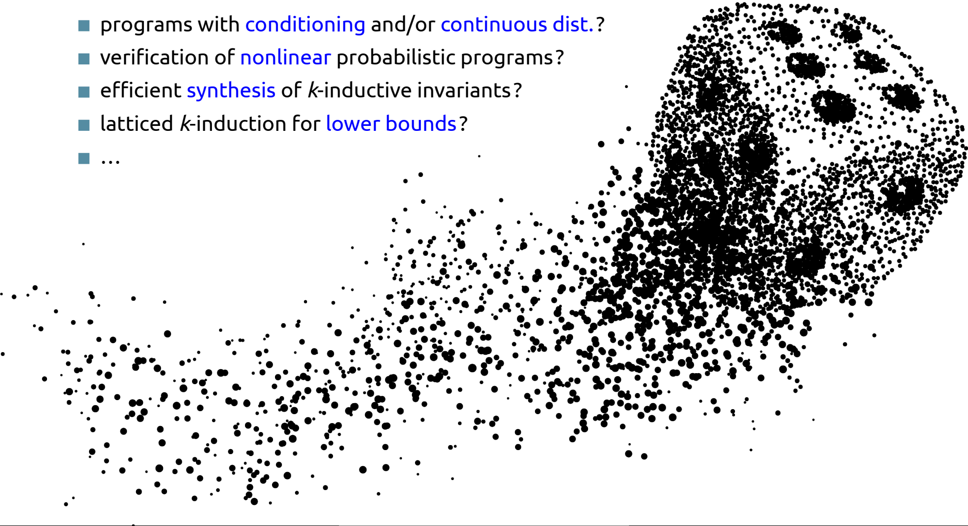
- (in)completeness of latticed k-induction;
- latticed bounded model checking for *refuting* $\mathrm{lfp}\ \Phi \sqsubseteq f$.

⇛ K. Batz, M. Chen, B. L. Kaminski, J.-P. Katoen, C. Matheja, P. Schröer : *Latticed k-Induction with an Application to Probabilistic Programs*. CAV '21.

Future Directions

- programs with conditioning and/or continuous dist.?
- verification of nonlinear probabilistic programs?
- efficient synthesis of *k*-inductive invariants?
- latticed *k*-induction for lower bounds?
- ...



In Combination with Latticed BMC

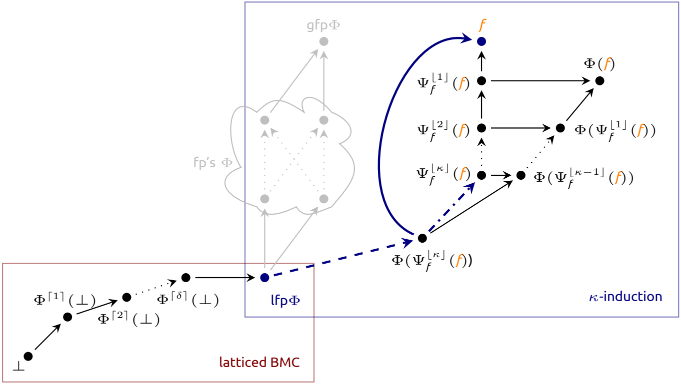


Figure – κ-induction and latticed BMC in case that lfp $\Phi \sqsubseteq f$.