Foundations of Informatics: a Bridging Course

Week 3: Formal Languages and Processes
Part B: Context-Free Languages
b-it Bonn; 02–06 March 2020

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https://moves.rwth-aachen.de/teaching/ws-19-20/foi/

Software Modeling and Verification Chair

RWTH AACHEN UNIVERSITY

Outline of Part B

Introductory Example I

Example B.1

Syntax definition of programming languages by “Backus-Naur” rules

Here: simple arithmetic expressions

$$
\begin{array}{rcl}
\langle \mathit{Expression} \rangle & ::= & 0 \\
& \mid & 1 \\
& \mid & \langle \mathit{Expression} \rangle + \langle \mathit{Expression} \rangle \\
& \mid & \langle \mathit{Expression} \rangle * \langle \mathit{Expression} \rangle \\
& \mid & (\langle \mathit{Expression} \rangle)
\end{array}
$$

Meaning:

An expression is either 0 or 1, or it is of the form $u + v$, $u * v$, *or* (u) *where* u, v *are again expressions*

Introductory Example II

Example B.1 (continued)

Here we abbreviate $\langle \mathit{Expression} \rangle$ as E, and use "$\to$" instead of "$::=$". Thus:

$$E \ \to \ 0 \mid 1 \mid E + E \mid E * E \mid (E)$$

Introductory Example II

Example B.1 (continued)

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Now expressions can be generated by replacing nonterminal symbols according to rules, beginning with the start symbol E:

$$E \Rightarrow E * E$$

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Context-Free Grammars I

Definition B.2

A context-free grammar (CFG) is a quadruple

$$G = \langle N, \Sigma, P, S \rangle$$

where

- N is a finite set of nonterminal symbols
- Σ is the (finite) alphabet of terminal symbols (disjoint from N)
- P is a finite set of production rules of the form $A \to \alpha$ where $A \in N$ and $\alpha \in (N \cup \Sigma)^*$
- $S \in N$ is a start symbol

Context-Free Grammars II

Example B.3

For the above example, we have:

- $N = \{E\}$
- $\Sigma = \{0, 1, +, *, (,)\}$
- $P = \{E \to 0, E \to 1, E \to E + E, E \to E * E, E \to (E)\}$
- $S = E$

Context-Free Grammars II

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Naming conventions:

- nonterminals start with uppercase letters
- terminals start with lowercase letters
- start symbol = symbol on LHS of first production

$\Rightarrow$ grammar completely defined by productions

Context-Free Languages I

Definition B.4

Let $G = \langle N, \Sigma, P, S \rangle$ be a CFG.

- A sentence $\gamma \in (N \cup \Sigma)^*$ is directly derivable from $\beta \in (N \cup \Sigma)^*$ if there exist $\pi = A \to \alpha \in P$ and $\delta_1, \delta_2 \in (N \cup \Sigma)^*$ such that $\beta = \delta_1 A \delta_2$ and $\gamma = \delta_1 \alpha \delta_2$ (notation: $\beta \stackrel{\pi}{\Rightarrow} \gamma$ or just $\beta \Rightarrow \gamma$).

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- A derivation (of length $n \in \mathbb{N}$) of γ from β is a sequence of direct derivations of the form $\delta_0 \Rightarrow \delta_1 \Rightarrow \ldots \Rightarrow \delta_n$ where $\delta_0 = \beta$, $\delta_n = \gamma$, and $\delta_{i-1} \Rightarrow \delta_i$ for every $i \in \{1, \ldots, n\}$ (notation: $\beta \Rightarrow^* \gamma$).

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- A word $w \in \Sigma^*$ is called derivable in G if $S \Rightarrow^* w$.
- The language generated by G is $L(G) := \{w \in \Sigma^* \mid S \Rightarrow^* w\}$.
- A language $L \subseteq \Sigma^*$ is called context-free (CFL) if it is generated by some CFG.
- Two grammars G_1, G_2 are equivalent if $L(G_1) = L(G_2)$.

Context-Free Languages II

Example B.5

The language $\{a^n b^n \mid n \in \mathbb{N}\}$ is context-free. It is generated by the grammar $G = \langle N, \Sigma, P, S \rangle$ with

- $N = \{S\}$
- $\Sigma = \{a, b\}$
- $P = \{S \to aSb \mid \varepsilon\}$

(proof: generating $a^n b^n$ requires exactly n applications of the first and one concluding application of the second rule)

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Remark: illustration of derivations by derivation trees

- root labelled by start symbol
- leaves labelled by terminal symbols
- successors of node labelled according to right-hand side of production rule
- sequence of leaf symbols = generated word

Summary: Context-Free Grammars and Languages

Seen:

- Context-free grammars
- Derivations
- Context-free languages

Summary: Context-Free Grammars and Languages

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Next:

- Relation between context-free and regular languages

Outline of Part B

Context-Free vs. Regular Languages

Theorem B.6

1. *Every regular language is context-free.*
2. *There exist CFLs which are not regular.*

(Thus: regular languages are a proper subset of CFLs.)

Context-Free vs. Regular Languages

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1. *Every regular language is context-free.*
2. *There exist CFLs which are not regular.*

(Thus: regular languages are a proper subset of CFLs.)

Proof.

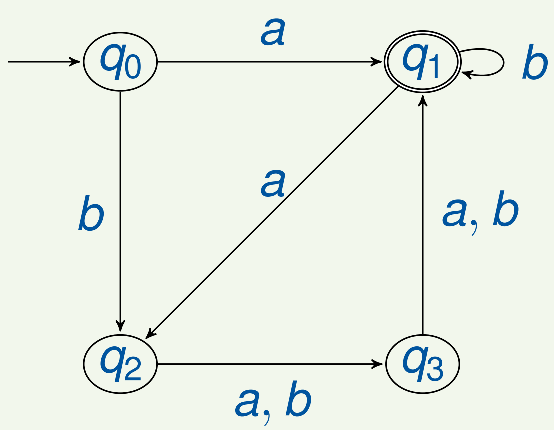
1. Let L be a regular language, and let $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$ be a DFA which recognises L.
   $G_{\mathfrak{A}} := \langle N, \Sigma, P, S \rangle$ is defined as follows:
   - $N := Q$, $S := q_0$
   - if $\delta(q, a) = q'$, then $q \to aq' \in P$
   - if $q \in F$, then $q \to \varepsilon \in P$

   Obviously a w-labelled run in $\mathfrak{A}$ from q_0 to F corresponds to a derivation of w in $G_{\mathfrak{A}}$, and vice versa. Thus $L(\mathfrak{A}) = L(G_{\mathfrak{A}})$ (example on the following slide).
2. An example is $\{a^n b^n \mid n \in \mathbb{N}\}$ (see Ex. B.5).
   Intuitive reason: recognising this language requires “unbounded counting” capability. □

From Regular to Context-Free Languages

Example B.7

DFA $\mathfrak{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$:



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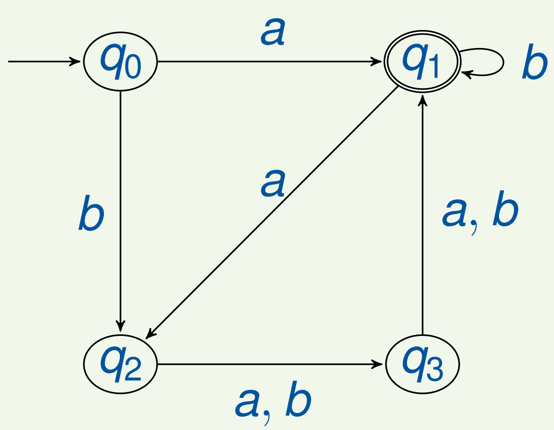

Corresponding CFG $G_{\mathfrak{A}} := \langle N, \Sigma, P, S \rangle$ with $N := Q$, $S := q_0$:

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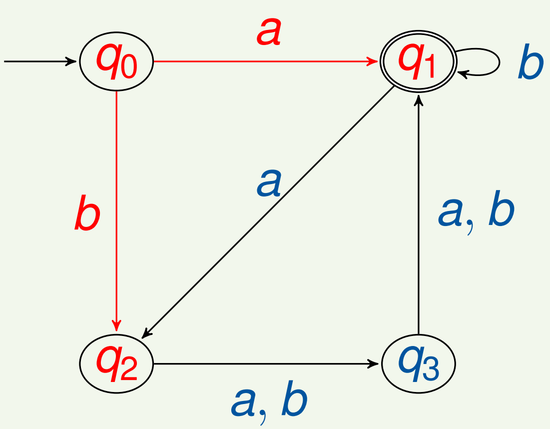

Corresponding CFG $G_{\mathfrak{A}} := \langle N, \Sigma, P, S \rangle$ with $N := Q$, $S := q_0$:

$$q_0 \rightarrow a\, q_1 \mid b\, q_2$$

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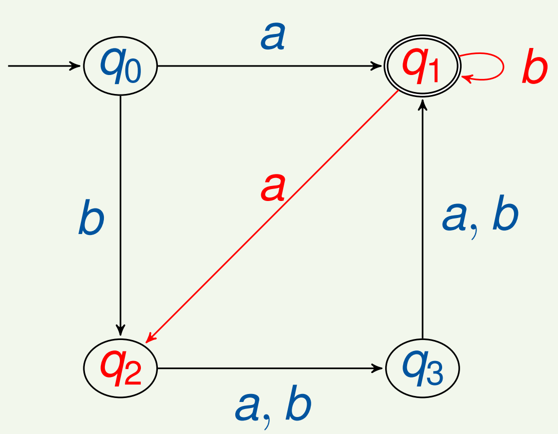

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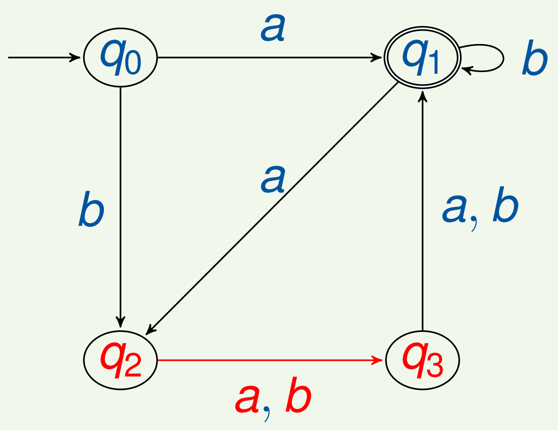

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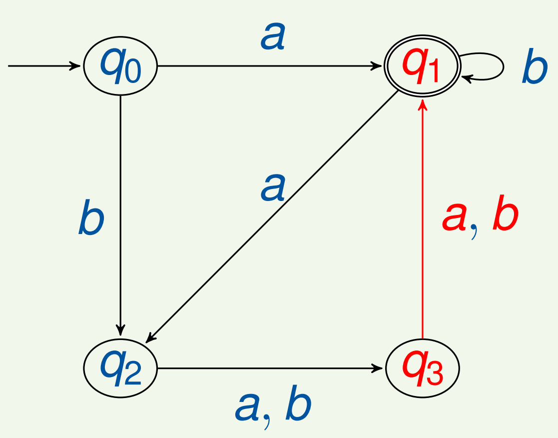

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From Regular to Context-Free Languages

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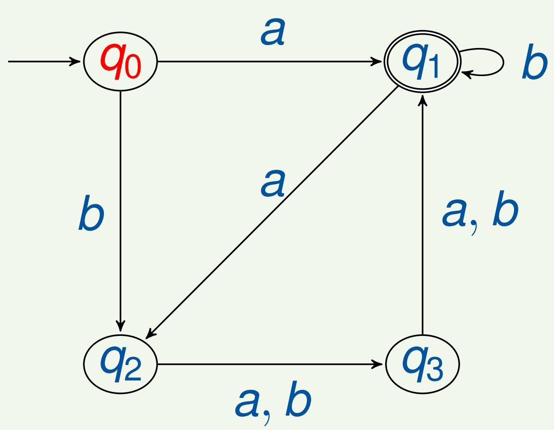

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E.g., $\mathfrak{A}$'s run on input $baab \in L(\mathfrak{A})$ is simulated by the following derivation in $G_{\mathfrak{A}}$:

$$q_0$$

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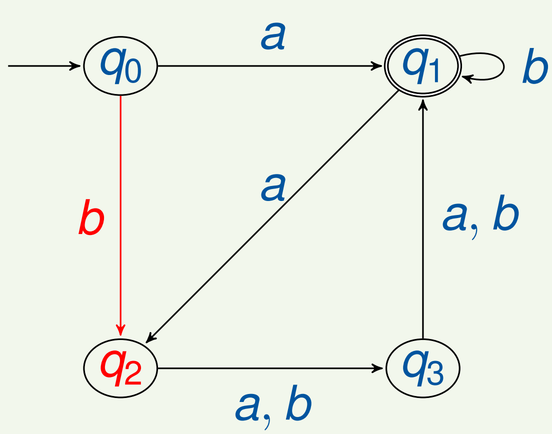

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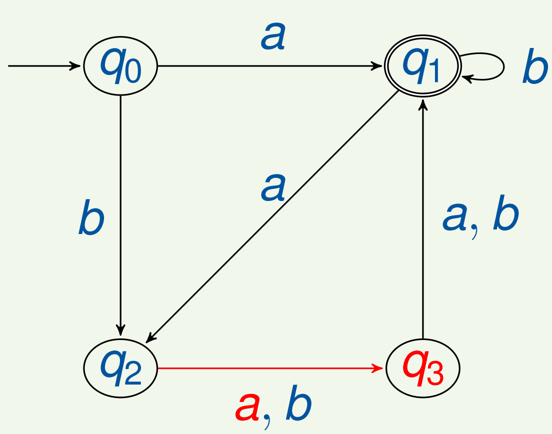

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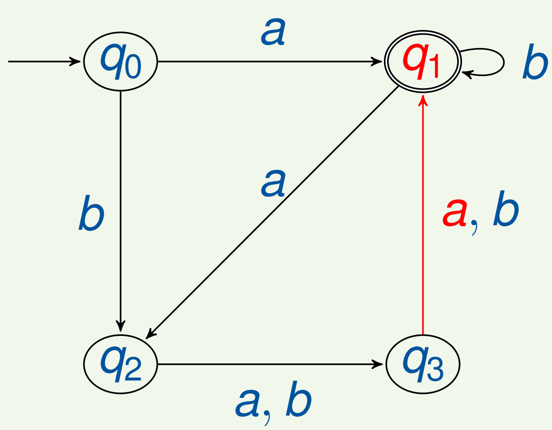

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From Regular to Context-Free Languages

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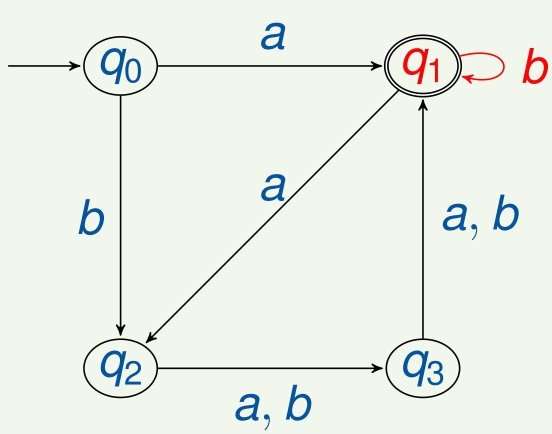

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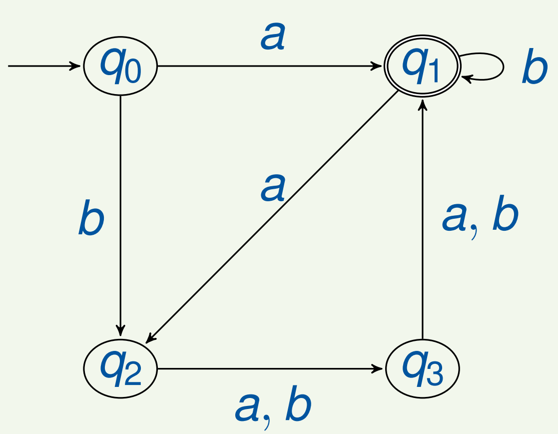

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$$q_0 \Rightarrow b\,q_2 \Rightarrow b\,a\,q_3 \Rightarrow b\,a\,a\,q_1 \Rightarrow b\,a\,a\,b\,q_1 \Rightarrow b\,a\,a\,b$$

Summary: Context-Free vs. Regular Languages

Seen:

- CFLs are more expressive than regular languages

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Next:

- Decidability of word problem

Outline of Part B

The Word Problem for CFL

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Given CFG $G = \langle N, \Sigma, P, S \rangle$ and $w \in \Sigma^*$, decide whether $w \in L(G)$ or not.

The Word Problem for CFL

Word Problem for CFL

Given CFG $G = \langle N, \Sigma, P, S \rangle$ and $w \in \Sigma^*$, decide whether $w \in L(G)$ or not.

- Important problem with many applications
  - syntax analysis of programming languages
  - HTML parsers
  - ...
- For regular languages this was easy: just let the corresponding DFA run on w.
- But here: how to decide when to stop a derivation?
- **Solution:** establish normal form for grammars which guarantees that each nonterminal produces at least one terminal symbol

$\Rightarrow$ Only finitely many combinations to be inspected

Chomsky Normal Form

Definition B.8

A CFG is in Chomsky Normal Form (Chomsky NF) if every of its productions is of the form

$$A \to BC \quad \text{or} \quad A \to a$$

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Example B.9

Let $S \to ab \mid aSb$ be the grammar which generates $L := \{a^n b^n \mid n \geq 1\}$.
An equivalent grammar in Chomsky NF is

$$\begin{aligned}
S &\to AB \mid AC && \text{(generates } L\text{)}\\
A &\to a && \text{(generates } \{a\}\text{)}\\
B &\to b && \text{(generates } \{b\}\text{)}\\
C &\to SB && \text{(generates } \{a^n b^{n+1} \mid n \geq 1\}\text{)}
\end{aligned}$$

Conversion to Chomsky Normal Form

Theorem B.10

Every CFL L (with $\varepsilon \notin L$) can be generated by a CFG in Chomsky NF.

Conversion to Chomsky Normal Form

Theorem B.10

Every CFL L (with $\varepsilon \notin L$) can be generated by a CFG in Chomsky NF.

Proof.

Let L be a CFL, and let $G = \langle N, \Sigma, P, S \rangle$ be some CFG which generates L. The transformation of P into rules of the form $A \to BC$ and $A \to a$ proceeds in three steps:

1. terminal symbols only in rules of the form $A \to a$
   (thus all other rules have the shape $A \to A_1 \ldots A_n$)
2. elimination of "chain rules" of the form $A \to B$
3. elimination of rules of the form $A \to A_1 \ldots A_n$ where $n > 2$

(see following slides for details) □

Step 1: Only $A \rightarrow a$

Procedure

1. For every terminal symbol $a \in \Sigma$, introduce a new nonterminal symbol $B_a \in N$.
2. Add corresponding productions $B_a \rightarrow a$ to P.
3. In each original production $A \rightarrow \alpha$, replace every $a \in \Sigma$ with B_a.

This yields G'.

Step 1: Only $A \to a$

Procedure

1. For every terminal symbol $a \in \Sigma$, introduce a new nonterminal symbol $B_a \in N$.
2. Add corresponding productions $B_a \to a$ to P.
3. In each original production $A \to \alpha$, replace every $a \in \Sigma$ with B_a.

This yields G'.

Example B.11

$G: S \to ab \mid aSb$ is converted to

$$\begin{aligned} G': S &\to AB \mid ASB \\ A &\to a \\ B &\to b \end{aligned}$$

Step 2: Elimination of Chain Rules $A \to B$

Procedure

1. Determine all derivations $A_1 \Rightarrow \ldots \Rightarrow A_n$ with rules of the form $A \to B$ without repetition of nonterminals ($\Longrightarrow$ only finitely many!).
2. Determine all productions $A_n \to \alpha$ with $\alpha \notin N$.
3. Add corresponding productions $A_1 \to \alpha$ to P.
4. Remove all chain rules from P.

This yields G''.

Step 2: Elimination of Chain Rules $A \to B$

Procedure

1. Determine all derivations $A_1 \Rightarrow \ldots \Rightarrow A_n$ with rules of the form $A \to B$ without repetition of nonterminals ($\Longrightarrow$ only finitely many!).
2. Determine all productions $A_n \to \alpha$ with $\alpha \notin N$.
3. Add corresponding productions $A_1 \to \alpha$ to P.
4. Remove all chain rules from P.

This yields G''.

Example B.12

$$G' : \begin{array}{rcl} S & \to & A \\ A & \to & B \mid C \\ B & \to & A \mid DA \\ C & \to & c \\ D & \to & d \end{array}$$

is converted to

$$G'' : \begin{array}{rcl} S & \to & DA \mid c \\ A & \to & DA \mid c \\ B & \to & DA \mid c \\ C & \to & c \\ D & \to & d \end{array}$$

Step 3: Elimination of Rules $A \to A_1 \dots A_n$ with $n > 2$

Procedure

Iteratively apply the following transformation:

1. For every $A \to A_1 \dots A_n$ with $n > 2$, introduce a new nonterminal symbol $B \in N$.
2. Replace original production by $A \to A_1 B$.
3. Add new production $B \to A_2 \dots A_n$.

This yields G'''.

Step 3: Elimination of Rules $A \to A_1 \dots A_n$ with $n > 2$

Procedure

Iteratively apply the following transformation:

1. For every $A \to A_1 \dots A_n$ with $n > 2$, introduce a new nonterminal symbol $B \in N$.
2. Replace original production by $A \to A_1 B$.
3. Add new production $B \to A_2 \dots A_n$.

This yields G'''.

Example B.13

$$G'' : \begin{array}{lcl} S & \to & AB \mid ASB \\ A & \to & a \\ B & \to & b \end{array}$$

is converted to

$$G''' : \begin{array}{lcl} S & \to & AB \mid AC \\ A & \to & a \\ B & \to & b \\ C & \to & SB \end{array}$$

Summary: Chomsky Normal Form

Seen:

- Chomsky NF: all productions of the form $A \to BC$ or $A \to a$

Summary: Chomsky Normal Form

Seen:

- Chomsky NF: all productions of the form $A \to BC$ or $A \to a$

Next:

- Exploit Chomsky Normal Form to solve word problem for CFL

Outline of Part B

Word Problem for ε-free CFL

Given CFG $G = \langle N, \Sigma, P, S \rangle$ such that $\varepsilon \notin L(G)$ and $w \in \Sigma^+$, decide whether $w \in L(G)$ or not.

(If $w = \varepsilon$, then $w \in L(G)$ easily decidable for arbitrary G)

The Word Problem for CFL Revisited

Word Problem for ε-free CFL

Given CFG $G = \langle N, \Sigma, P, S \rangle$ such that $\varepsilon \notin L(G)$ and $w \in \Sigma^+$, decide whether $w \in L(G)$ or not.

(If $w = \varepsilon$, then $w \in L(G)$ easily decidable for arbitrary G)

Algorithm B.14 (by Cocke, Younger, Kasami – CYK algorithm)

1. *Transform G into Chomsky NF*
2. *Let $w = a_1 \ldots a_n$ ($n \geq 1$)*
3. *Let $w[i, j] := a_i \ldots a_j$ for every $1 \leq i \leq j \leq n$*
4. *Consider segments $w[i, j]$ in order of increasing length, starting with $w[i, i] = a_i$ (i.e., letters)*
5. *In each case, determine $N_{i,j} := \{A \in N \mid A \Rightarrow^* w[i, j]\}$ using a "dynamic programming" approach:*
   - $i = j$: $N_{i,i} = \{A \in N \mid A \to a_i \in P\}$
   - $i < j$: $N_{i,j} = \{A \in N \mid \exists B, C \in N, k \in \{i, \ldots, j-1\} : A \to BC \in P, B \in N_{i,k}, C \in N_{k+1,j}\}$
6. *Test whether $S \in N_{1,n}$ (and thus, whether $S \Rightarrow^* w[1, n] = w$)*

Matrix Representation of CYK Algorithm

	a_1	a_2	a_3	$\cdots$	a_n
$i \backslash j$	1	2	3	$\cdots$	n
1	$N_{1,1}$	$N_{1,2}$	$N_{1,3}$	$\cdots$	$N_{1,n}$
2	X	$N_{2,2}$	$N_{2,3}$	$\cdots$	$N_{2,n}$
3	X	X	$N_{3,3}$	$\cdots$	$N_{3,n}$
$\vdots$	$\vdots$	$\vdots$		$\cdots$	$\vdots$
n	X	X	$\cdots$	$\cdots$	$N_{n,n}$

Matrix Representation of CYK Algorithm

	a_1	a_2	a_3	$\cdots$	a_n
$i\backslash j$	1	2	3	$\cdots$	n
1	$N_{1,1}$	$N_{1,2}$	$N_{1,3}$	$\cdots$	$N_{1,n}$
2	X	$N_{2,2}$	$N_{2,3}$	$\cdots$	$N_{2,n}$
3	X	X	$N_{3,3}$	$\cdots$	$N_{3,n}$
$\vdots$	$\vdots$	$\vdots$		$\cdots$	$\vdots$
n	X	X	$\cdots$	$\cdots$	$N_{n,n}$

$$
\begin{aligned}
N_{1,1} &= \{A \in N \mid A \to a_1 \in P\} \\
N_{2,2} &= \{A \in N \mid A \to a_2 \in P\} \\
&\vdots
\end{aligned}
$$

Matrix Representation of CYK Algorithm

	a_1	a_2	a_3	$\cdots$	a_n
$i \backslash j$	1	2	3	$\cdots$	n
1	$N_{1,1}$	$N_{1,2}$	$N_{1,3}$	$\cdots$	$N_{1,n}$
2	X	$N_{2,2}$	$N_{2,3}$	$\cdots$	$N_{2,n}$
3	X	X	$N_{3,3}$	$\cdots$	$N_{3,n}$
$\vdots$	$\vdots$	$\vdots$		$\cdots$	$\vdots$
n	X	X	$\cdots$	$\cdots$	$N_{n,n}$

$$
\begin{aligned}
N_{1,1} &= \{A \in N \mid A \to a_1 \in P\} \\
N_{2,2} &= \{A \in N \mid A \to a_2 \in P\} \\
&\vdots \\
N_{1,2} &= \{A \in N \mid \exists B, C \in N : A \to BC \in P, B \in N_{1,1}, C \in N_{2,2}\} \\
N_{2,3} &= \{A \in N \mid \exists B, C \in N : A \to BC \in P, B \in N_{2,2}, C \in N_{3,3}\} \\
&\vdots
\end{aligned}
$$

Matrix Representation of CYK Algorithm

	a_1	a_2	a_3	$\cdots$	a_n
$i \backslash j$	1	2	3	$\cdots$	n
1	$N_{1,1}$	$N_{1,2}$	$N_{1,3}$	$\cdots$	$N_{1,n}$
2	X	$N_{2,2}$	$N_{2,3}$	$\cdots$	$N_{2,n}$
3	X	X	$N_{3,3}$	$\cdots$	$N_{3,n}$
$\vdots$	$\vdots$	$\vdots$		$\cdots$	$\vdots$
n	X	X	$\cdots$	$\cdots$	$N_{n,n}$

$$
\begin{aligned}
N_{1,1} &= \{A \in N \mid A \to a_1 \in P\}\\
N_{2,2} &= \{A \in N \mid A \to a_2 \in P\}\\
&\vdots\\
N_{1,2} &= \{A \in N \mid \exists B, C \in N : A \to BC \in P, B \in N_{1,1}, C \in N_{2,2}\}\\
N_{2,3} &= \{A \in N \mid \exists B, C \in N : A \to BC \in P, B \in N_{2,2}, C \in N_{3,3}\}\\
&\vdots\\
N_{1,3} &= \{A \in N \mid \exists B, C \in N : A \to BC \in P, B \in N_{1,1}, C \in N_{2,3}\}\\
&\cup \{A \in N \mid \exists B, C \in N : A \to BC \in P, B \in N_{1,2}, C \in N_{3,3}\}\\
N_{2,4} &= \{A \in N \mid \exists B, C \in N : A \to BC \in P, B \in N_{2,2}, C \in N_{3,4}\}\\
&\cup \{A \in N \mid \exists B, C \in N : A \to BC \in P, B \in N_{2,3}, C \in N_{4,4}\}\\
&\vdots
\end{aligned}
$$

Applying the CYK Algorithm

Example B.15

- G :
  $S \to SA \mid a$
  $A \to BS$
  $B \to BB \mid BS \mid b \mid c$
- $w = abaaba$

Applying the CYK Algorithm

Example B.15

- G :
  $S \rightarrow SA \mid a$
  $A \rightarrow BS$
  $B \rightarrow BB \mid BS \mid b \mid c$
- $w = abaaba$

	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1						
2	X					
3	X	X				
4	X	X	X			
5	X	X	X	X		
6	X	X	X	X	X	

Applying the CYK Algorithm

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  $A \to BS$
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- $w = abaaba$

	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$					
2	X					
3	X	X	$\{S\}$			
4	X	X	X	$\{S\}$		
5	X	X	X	X		
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

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1	$\{S\}$					
2	X	$\{B\}$				
3	X	X	$\{S\}$			
4	X	X	X	$\{S\}$		
5	X	X	X	X	$\{B\}$	
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	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$				
2	X	$\{B\}$				
3	X	X	$\{S\}$	$\emptyset$		
4	X	X	X	$\{S\}$	$\emptyset$	
5	X	X	X	X	$\{B\}$	
6	X	X	X	X	X	$\{S\}$

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	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$				
2	X	$\{B\}$	$\{A\quad\}$			
3	X	X	$\{S\}$	$\emptyset$		
4	X	X	X	$\{S\}$	$\emptyset$	
5	X	X	X	X	$\{B\}$	$\{A\quad\}$
6	X	X	X	X	X	$\{S\}$

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  $A \to BS$
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- $w = abaaba$

	a	b	a	a	b	a
$i\backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$				
2	X	$\{B\}$	$\{A, B\}$			
3	X	X	$\{S\}$	$\emptyset$		
4	X	X	X	$\{S\}$	$\emptyset$	
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

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	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$			
2	X	$\{B\}$	$\{A, B\}$			
3	X	X	$\{S\}$	$\emptyset$		
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

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	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$			
2	X	$\{B\}$	$\{A, B\}$	$\{A \quad\}$		
3	X	X	$\{S\}$	$\emptyset$		
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

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	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$			
2	X	$\{B\}$	$\{A, B\}$	$\{A, B\}$		
3	X	X	$\{S\}$	$\emptyset$		
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

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	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$			
2	X	$\{B\}$	$\{A, B\}$	$\{A, B\}$		
3	X	X	$\{S\}$	$\emptyset$	$\emptyset$	
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
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$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$	$\{S\}$		
2	X	$\{B\}$	$\{A, B\}$	$\{A, B\}$		
3	X	X	$\{S\}$	$\emptyset$	$\emptyset$	
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

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  $S \to SA \mid a$
  $A \to BS$
  $B \to BB \mid BS \mid b \mid c$
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	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$	$\{S\}$		
2	X	$\{B\}$	$\{A, B\}$	$\{A, B\}$	$\{B\}$	
3	X	X	$\{S\}$	$\emptyset$	$\emptyset$	
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

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- G :
  $S \to SA \mid a$
  $A \to BS$
  $B \to BB \mid BS \mid b \mid c$
- $w = abaaba$

	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$	$\{S\}$		
2	X	$\{B\}$	$\{A, B\}$	$\{A, B\}$	$\{B\}$	
3	X	X	$\{S\}$	$\emptyset$	$\emptyset$	$\emptyset$
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

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	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$	$\{S\}$	$\emptyset$	
2	X	$\{B\}$	$\{A, B\}$	$\{A, B\}$	$\{B\}$	
3	X	X	$\{S\}$	$\emptyset$	$\emptyset$	$\emptyset$
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

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- $w = abaaba$

	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$	$\{S\}$	$\emptyset$	
2	X	$\{B\}$	$\{A, B\}$	$\{A, B\}$	$\{B\}$	$\{A \quad\}$
3	X	X	$\{S\}$	$\emptyset$	$\emptyset$	$\emptyset$
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

Example B.15

- G :
  $S \to SA \mid a$
  $A \to BS$
  $B \to BB \mid BS \mid b \mid c$
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	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$	$\{S\}$	$\emptyset$	
2	X	$\{B\}$	$\{A, B\}$	$\{A, B\}$	$\{B\}$	$\{A, B\}$
3	X	X	$\{S\}$	$\emptyset$	$\emptyset$	$\emptyset$
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

Example B.15

- G :
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  $A \to BS$
  $B \to BB \mid BS \mid b \mid c$
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	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$	$\{S\}$	$\emptyset$	$\{S\}$
2	X	$\{B\}$	$\{A, B\}$	$\{A, B\}$	$\{B\}$	$\{A, B\}$
3	X	X	$\{S\}$	$\emptyset$	$\emptyset$	$\emptyset$
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

Example B.15

- G :
  $S \to SA \mid a$
  $A \to BS$
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	a	b	a	a	b	a
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$	$\{S\}$	$\emptyset$	$\{S\}$
2	X	$\{B\}$	$\{A, B\}$	$\{A, B\}$	$\{B\}$	$\{A, B\}$
3	X	X	$\{S\}$	$\emptyset$	$\emptyset$	$\emptyset$
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

$$S \in N_{1,6} \implies w = abaaba \in L(G)$$

Summary: The Word Problem for Context-Free Languages

Seen:

- Given CFG G and $w \in \Sigma^*$, decide whether $w \in L(G)$ or not
- Decidable using CYK algorithm (based on dynamic programming)
- Cubic complexity

Summary: The Word Problem for Context-Free Languages

Seen:

- Given CFG G and $w \in \Sigma^*$, decide whether $w \in L(G)$ or not
- Decidable using CYK algorithm (based on dynamic programming)
- Cubic complexity

Next:

- Emptiness problem

Outline of Part B

The Emptiness Problem

Emptiness Problem for CFL

Given CFG $G = \langle N, \Sigma, P, S \rangle$, decide whether $L(G) = \emptyset$ or not.

The Emptiness Problem

Emptiness Problem for CFL

Given CFG $G = \langle N, \Sigma, P, S \rangle$, decide whether $L(G) = \emptyset$ or not.

- Important problem with many applications
  - consistency of context-free language definitions
  - correctness properties of recursive programs
  - ...
- For regular languages this was easy: check in the corresponding DFA whether some final state is reachable from the initial state.
- Here: test whether start symbol is productive, i.e., whether it generates a terminal word

The Emptiness Test

Algorithm B.16 (Emptiness Test)

Input: $G = \langle N, \Sigma, P, S \rangle$

Question: $L(G) = \emptyset$?

Procedure: *mark every* $a \in \Sigma$ *as productive*;

```
repeat
   if there is A → α ∈ P such that all symbols in α productive then
      mark A as productive;
   end;
until no further productive symbols found;
```

Output: *"no" if S productive, otherwise "yes"*

The Emptiness Test

Algorithm B.16 (Emptiness Test)

Input: $G = \langle N, \Sigma, P, S \rangle$

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```
repeat
   if there is A → α ∈ P such that all symbols in α productive then
      mark A as productive;
   end;
until no further productive symbols found;
```

Output: *"no" if S productive, otherwise "yes"*

Example B.17

$$
\begin{aligned}
G:\quad S &\to AB \mid CA \\
A &\to a \\
B &\to BC \mid AB \\
C &\to aB \mid b
\end{aligned}
$$

The Emptiness Test

Algorithm B.16 (Emptiness Test)

Input: $G = \langle N, \Sigma, P, S \rangle$

Question: $L(G) = \emptyset$?

Procedure: *mark every* $a \in \Sigma$ *as productive*;

```
repeat
  if there is A → α ∈ P such that all symbols in α productive then
    mark A as productive;
  end;
until no further productive symbols found;
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Output: "no" if S productive, otherwise "yes"

Example B.17

$$
\begin{aligned}
G:\quad S &\to AB \mid CA\\
A &\to a\\
B &\to BC \mid AB\\
C &\to aB \mid b
\end{aligned}
$$

1. Initalisation

Algorithm B.16 (Emptiness Test)

Input: $G = \langle N, \Sigma, P, S \rangle$

Question: $L(G) = \emptyset$?

Procedure: *mark every* $a \in \Sigma$ *as productive*;

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2. 1st iteration

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1. Initalisation
2. 1st iteration
3. 2nd iteration

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1. Initalisation
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S productive $\Longrightarrow L(G) \neq \emptyset$

Summary: The Emptiness Problem for CFLs

Seen:

- Emptiness problem decidable based on productivity of symbols

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Seen:

- Emptiness problem decidable based on productivity of symbols

Next:

- Closure properties of CFLs

Outline of Part B

Context-Free Grammars and Languages

Context-Free vs. Regular Languages

Chomsky Normal Form

The Word Problem for Context-Free Languages

The Emptiness Problem for CFLs

Closure Properties of CFLs

Pushdown Automata

Outlook

Positive Results

Theorem B.18

The set of CFLs is closed under concatenation, union, and iteration.

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Positive Results

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The set of CFLs is closed under concatenation, union, and iteration.

Proof.

For $i = 1, 2$, let $G_i = \langle N_i, \Sigma, P_i, S_i \rangle$ with $L_i := L(G_i)$ and $N_1 \cap N_2 = \emptyset$. Then

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- $G := \langle N, \Sigma, P, S \rangle$ with $N := \{S\} \cup N_1 \cup N_2$ and $P := \{S \to S_1 S_2\} \cup P_1 \cup P_2$ generates $L_1 \cdot L_2$;

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- $G := \langle N, \Sigma, P, S \rangle$ with $N := \{S\} \cup N_1 \cup N_2$ and $P := \{S \to S_1 \mid S_2\} \cup P_1 \cup P_2$ generates $L_1 \cup L_2$; and

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The set of CFLs is closed under concatenation, union, and iteration.

Proof.

For $i = 1, 2$, let $G_i = \langle N_i, \Sigma, P_i, S_i \rangle$ with $L_i := L(G_i)$ and $N_1 \cap N_2 = \emptyset$. Then

- $G := \langle N, \Sigma, P, S \rangle$ with $N := \{S\} \cup N_1 \cup N_2$ and $P := \{S \to S_1 S_2\} \cup P_1 \cup P_2$ generates $L_1 \cdot L_2$;
- $G := \langle N, \Sigma, P, S \rangle$ with $N := \{S\} \cup N_1 \cup N_2$ and $P := \{S \to S_1 \mid S_2\} \cup P_1 \cup P_2$ generates $L_1 \cup L_2$; and
- $G := \langle N, \Sigma, P, S \rangle$ with $N := \{S\} \cup N_1$ and $P := \{S \to \varepsilon \mid S_1 S\} \cup P_1$ generates L_1^*.

$\square$

Negative Results

Theorem B.19

The set of CFLs is not closed under intersection and complement.

Negative Results

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Proof.

- Intersection: both $L_1 := \{a^k b^k c^l \mid k, l \in \mathbb{N}\}$ and $L_2 := \{a^k b^l c^l \mid k, l \in \mathbb{N}\}$ are CFLs, but not $L_1 \cap L_2 = \{a^n b^n c^n \mid n \in \mathbb{N}\}$ (without proof).

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- Complement: if CFLs were closed under complement, then also under intersection (as $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$).

□

Overview of Decidability and Closure Results

Decidability Results			
Class	$w \in L$	$L = \emptyset$	$L_1 = L_2$
Reg	+ (A.37)	+ (A.39)	+ (A.41)
CFL	+ (B.14)	+ (B.16)	–

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Class	$w \in L$	$L = \emptyset$	$L_1 = L_2$
Reg	+ (A.37)	+ (A.39)	+ (A.41)
CFL	+ (B.14)	+ (B.16)	–

Closure Results					
Class	$L_1 \cdot L_2$	$L_1 \cup L_2$	$L_1 \cap L_2$	$\overline{L}$	L^*
Reg	+ (A.28)	+ (A.18)	+ (A.16)	+ (A.14)	+ (A.29)
CFL	+ (B.18)	+ (B.18)	– (B.19)	– (B.19)	+ (B.18)

Closure Properties

Seen:

- Closure under concatenation, union and iteration
- Non-closure under intersection and complement

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Next:

- Automata model for CFLs

Outline of Part B

Pushdown Automata I

- **Goal:** introduce an automata model which exactly accepts CFLs
- **Clear:** DFA not sufficient
  (missing "counting capability", e.g. for $\{a^n b^n \mid n \geq 1\}$)
- DFA will be extended to pushdown automata by
  - adding a pushdown store which stores symbols from a pushdown alphabet and uses a special bottom symbol
  - adding push and pop operations to transitions

Definition B.20

A pushdown automaton (PDA) is of the form $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ where

- Q is a finite set of states
- Σ is the (finite) input alphabet
- Γ is the (finite) pushdown alphabet
- $\Delta \subseteq (Q \times \Gamma \times \Sigma_\varepsilon) \times (Q \times \Gamma^*)$ is a finite set of transitions
- $q_0 \in Q$ is the initial state
- Z_0 is the (pushdown) bottom symbol
- $F \subseteq Q$ is a set of final states

Interpretation of $((q, Z, x), (q', \delta)) \in \Delta$: if the PDA $\mathfrak{A}$ is in state q where Z is on top of the stack and x is the next input symbol (or empty), then $\mathfrak{A}$ reads x, replaces Z by δ, and changes into the state q'.

Configurations, Runs, Acceptance

Definition B.21

Let $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ be a PDA.

- An element of $Q \times \Gamma^* \times \Sigma^*$ is called a configuration of $\mathfrak{A}$.
- The initial configuration for input $w \in \Sigma^*$ is given by (q_0, Z_0, w).
- The set of final configurations is given by $F \times \{\varepsilon\} \times \{\varepsilon\}$.
- If $((q, Z, x), (q', \delta)) \in \Delta$, then $(q, Z\gamma, xw) \vdash (q', \delta\gamma, w)$ for every $\gamma \in \Gamma^*$, $w \in \Sigma^*$.

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- If $((q, Z, x), (q', \delta)) \in \Delta$, then $(q, Z\gamma, xw) \vdash (q', \delta\gamma, w)$ for every $\gamma \in \Gamma^*$, $w \in \Sigma^*$.
- $\mathfrak{A}$ accepts $w \in \Sigma^*$ if $(q_0, Z_0, w) \vdash^* (q, \varepsilon, \varepsilon)$ for some $q \in F$.
- The language accepted by $\mathfrak{A}$ is $L(\mathfrak{A}) := \{w \in \Sigma^* \mid \mathfrak{A} \text{ accepts } w\}$.
- A language L is called PDA-recognisable if $L = L(\mathfrak{A})$ for some PDA $\mathfrak{A}$.
- Two PDA $\mathfrak{A}_1, \mathfrak{A}_2$ are called equivalent if $L(\mathfrak{A}_1) = L(\mathfrak{A}_2)$.

Examples

Example B.22

1. PDA which recognises $L = \{a^n b^n \mid n \geq 1\}$
   (on the board)

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2. PDA which recognises $L = \{ww^R \mid w \in \{a, b\}^*\}$
   (palindromes of even length; on the board)

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1. PDA which recognises $L = \{a^n b^n \mid n \geq 1\}$
   (on the board)
2. PDA which recognises $L = \{ww^R \mid w \in \{a, b\}^*\}$
   (palindromes of even length; on the board)

Observation: $\mathfrak{A}_2$ is nondeterministic: whenever a construction transition is applicable, the pushdown could also be deconstructed

Deterministic PDA

Definition B.23

A PDA $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is called deterministic (DPDA) if for every $q \in Q, Z \in \Gamma$,

1. for every $x \in \Sigma_\varepsilon$, there is at most one (q, Z, x)-transition in Δ and
2. if there is a (q, Z, a)-transition in Δ for some $a \in \Sigma$, then there is no (q, Z, ε)-transition in Δ.

Remark: this excludes two types of nondeterminism:

1. if $((q, Z, x), (q'_1, \delta_1)), ((q, Z, x), (q'_2, \delta_2)) \in \Delta$:
$$(q'_1, \delta_1\gamma, w) \dashv (q, Z\gamma, xw) \vdash (q'_2, \delta_2\gamma, w)$$
2. if $((q, Z, a), (q'_1, \delta_1)), ((q, Z, \varepsilon), (q'_2, \delta_2)) \in \Delta$:
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Corollary B.24

In a DPDA, every configuration has at most one $\vdash$-successor.

Expressiveness of DPDA

One can show: determinism restricts the set of acceptable languages (DPDA-recognisable languages are closed under complement, which is generally not true for PDA-recognisable languages)

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Expressiveness of DPDA

One can show: determinism restricts the set of acceptable languages (DPDA-recognisable languages are closed under complement, which is generally not true for PDA-recognisable languages)

Example B.25

The set of palindromes of even length is PDA-recognisable, but not DPDA-recognisable (without proof).

PDA and Context-Free Languages I

Theorem B.26

A language is context-free iff it is PDA-recognisable.

PDA and Context-Free Languages I

Theorem B.26

A language is context-free iff it is PDA-recognisable.

Proof.

$\Leftarrow$: omitted

$\Rightarrow$: let $G = \langle N, \Sigma, P, S \rangle$ be a CFG. Construction of PDA $\mathfrak{A}_G$ recognising $L(G)$:

- $\mathfrak{A}_G$ simulates a derivation of G where always the leftmost nonterminal of a sentence is replaced ("leftmost derivation")
- begin with S on pushdown
- if nonterminal on top: apply a corresponding production rule
- if terminal on top: match with next input symbol

(cf. formal construction on following slide)

□

PDA and Context-Free Languages II

Proof of Theorem B.26 (continued).

$\Rightarrow$: Formally: $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

- $Q := \{q_0\}$
- $\Gamma := N \cup \Sigma$
- for each $A \to \alpha \in P$: $((q_0, A, \varepsilon), (q_0, \alpha)) \in \Delta$
- for each $a \in \Sigma$: $((q_0, a, a), (q_0, \varepsilon)) \in \Delta$
- $Z_0 := S$
- $F := Q$

□

PDA and Context-Free Languages II

Proof of Theorem B.26 (continued).

$\Rightarrow$: Formally: $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F\rangle$ is given by

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- $Z_0 := S$
- $F := Q$

$\square$

Example B.27

"Bracket language", given by G:

$$S \to \langle\rangle \mid \langle S\rangle \mid SS$$

(on the board)

Outline of Part B

Outlook

- Equivalence problem for CFG and PDA ("$L(X_1) = L(X_2)$?")
  (generally undecidable, decidable for DPDA)
- Pumping Lemma for CFL
- Greibach Normal Form for CFG
- Construction of parsers for compilers
- Non-context-free grammars and languages (context-sensitive and recursively enumerable languages, Turing machines—see Week 4)