Concurrency Theory

Winter Semester 2019/20

Lecture 5: Fixed-Point Theory

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https://moves.rwth-aachen.de/teaching/ws-19-20/ct/

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Exam in Concurrency Theory

- Written exam
- Date: Fri 14 Feb 08:30–10:30
- Registration via RWTHonline by 15 Jan
- No specific requirements for admission

Outline of Lecture 5

Recap: Hennessy-Milner Logic with Recursion

The Fixed-Point Theorem

The Fixed-Point Theorem for Finite Lattices

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Recap: Hennessy-Milner Logic with Recursion

Introducing Recursion

Solution: employ recursion!

- $Inv(\langle a \rangle \mathrm{tt}) \equiv \langle a \rangle \mathrm{tt} \wedge [a]\, Inv(\langle a \rangle \mathrm{tt})$
- $Pos([a]\mathrm{ff}) \equiv [a]\mathrm{ff} \vee \langle a \rangle\, Pos([a]\mathrm{ff})$

Interpretation: the sets of states $X, Y \subseteq S$ satisfying the respective formula should solve the corresponding equation, i.e.,

- $X = \langle \cdot a \cdot \rangle(S) \cap [\cdot a \cdot](X)$
- $Y = [\cdot a \cdot](\emptyset) \cup \langle \cdot a \cdot \rangle(Y)$

Open questions

- Do such recursive equations (always) have solutions?
- If so, are they unique?
- How can we decide whether a process satisfies a recursive formula (“model checking”)?

Syntax of HML with One Recursive Variable

Initially: only one variable (for simplicity)
Later: mutual recursion

Definition (Syntax of HML with one variable)

The set HMF_X of Hennessy-Milner formulae with one variable X over a set of actions Act is defined by the following syntax:

$$
\begin{array}{rcll}
F & ::= & X & \text{(variable)} \\
& | & \mathrm{tt} & \text{(true)} \\
& | & \mathrm{ff} & \text{(false)} \\
& | & F_1 \wedge F_2 & \text{(conjunction)} \\
& | & F_1 \vee F_2 & \text{(disjunction)} \\
& | & \langle \alpha \rangle F & \text{(diamond)} \\
& | & [\alpha] F & \text{(box)}
\end{array}
$$

where $\alpha \in Act$.

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Semantics of HML with One Recursive Variable I

So far: $[\![F]\!] \subseteq S$ for $F \in \mathit{HMF}$ and LTS $(S, \mathit{Act}, \longrightarrow)$

Now: semantics of formula depends on states that (are assumed to) satisfy X

Definition (Semantics of HML with one variable)

Let $(S, \mathit{Act}, \longrightarrow)$ be an LTS and $F \in \mathit{HMF}_X$. The semantics of F,

$$[\![F]\!] : 2^S \to 2^S,$$

is defined by

$$\begin{aligned}
[\![X]\!](T) &:= T \\
[\![\mathrm{tt}]\!](T) &:= S \\
[\![\mathrm{ff}]\!](T) &:= \emptyset \\
[\![F_1 \wedge F_2]\!](T) &:= [\![F_1]\!](T) \cap [\![F_2]\!](T) \\
[\![F_1 \vee F_2]\!](T) &:= [\![F_1]\!](T) \cup [\![F_2]\!](T) \\
[\![\langle\alpha\rangle F]\!](T) &:= \langle\cdot\alpha\cdot\rangle([\![F]\!](T)) \\
[\![[\alpha]F]\!](T) &:= [\cdot\alpha\cdot]([\![F]\!](T))
\end{aligned}$$

Semantics of HML with One Recursive Variable II

- Idea underlying the definition of

$$[\![.]\!] : HMF_X \to (2^S \to 2^S) :$$

  if $T \subseteq S$ gives the set of states that satisfy X, then $[\![F]\!](T)$ will be the set of states that satisfy F
- How to determine this T?
- According to previous discussion: as solution of recursive equation of the form $X = F_X$ where $F_X \in HMF_X$
- But: solution not unique; therefore write:

$$X \stackrel{min}{=} F_X \qquad \text{or} \qquad X \stackrel{max}{=} F_X$$

- In the following we will see:
  1. Equation $X = F_X$ always solvable
  2. Least and greatest solutions are unique and can be obtained by fixed-point iteration

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Complete Lattices

Definition (Complete lattice)

A complete lattice is a partial order $(D, \sqsubseteq)$ such that all subsets of D have LUBs and GLBs. In this case,

$$\bot := \bigsqcup \emptyset \; (= \bigsqcap D) \qquad \text{and} \qquad \top := \bigsqcap \emptyset \; (= \bigsqcup D)$$

respectively denote the least and greatest element of D.

Application to HML with Recursion

Lemma

Let $(S, Act, \longrightarrow)$ *be an LTS. Then* $(2^S, \subseteq)$ *is a complete lattice with*

- $\bigsqcup \mathcal{T} = \bigcup \mathcal{T} = \bigcup_{T \in \mathcal{T}} T$ *for all* $\mathcal{T} \subseteq 2^S$
- $\bigsqcap \mathcal{T} = \bigcap \mathcal{T} = \bigcap_{T \in \mathcal{T}} T$ *for all* $\mathcal{T} \subseteq 2^S$
- $\bot = \bigsqcup \emptyset = \bigsqcap 2^S = \emptyset$
- $\top = \bigsqcap \emptyset = \bigsqcup 2^S = S$

Proof.

omitted $\square$

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The Fixed-Point Theorem

The Fixed-Point Theorem for Finite Lattices

Fixed Points

Definition 5.1 (Fixed point)

Let D be some domain, $d \in D$, and $f : D \to D$. If

$$f(d) = d$$

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1. The (only) fixed points of $f_1 : \mathbb{N} \to \mathbb{N} : n \mapsto n^2$ are 0 and 1
2. A subset $T \subseteq \mathbb{N}$ is a fixed point of $f_2 : 2^{\mathbb{N}} \to 2^{\mathbb{N}} : T \mapsto T \cup \{1, 2\}$ iff $\{1, 2\} \subseteq T$

Monotonicity of Functions

Definition 5.3 (Monotonicity)

Let $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$ be partial orders. A function $f : D \to D'$ is called monotonic (w.r.t. $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$) if, for every $d_1, d_2 \in D$,

$$d_1 \sqsubseteq d_2 \implies f(d_1) \sqsubseteq' f(d_2).$$

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3. Let $\mathcal{T} := \{T \subseteq \mathbb{N} \mid T \text{ finite}\}$.
   Then $f_3 : \mathcal{T} \to \mathbb{N} : T \mapsto \sum_{n \in T} n$ is monotonic w.r.t. $(2^{\mathbb{N}}, \subseteq)$ and $(\mathbb{N}, \leq)$.

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   Then $f_3 : \mathcal{T} \to \mathbb{N} : T \mapsto \sum_{n \in T} n$ is monotonic w.r.t. $(2^{\mathbb{N}}, \subseteq)$ and $(\mathbb{N}, \leq)$.
4. $f_4 : 2^{\mathbb{N}} \to 2^{\mathbb{N}} : T \mapsto \mathbb{N} \setminus T$ is not monotonic w.r.t. $(2^{\mathbb{N}}, \subseteq)$
   (since, e.g., $\emptyset \subseteq \mathbb{N}$ but $f_4(\emptyset) = \mathbb{N} \not\subseteq f_4(\mathbb{N}) = \emptyset$).

The Fixed-Point Theorem I

Alfred Tarski (1901–1983)

Theorem 5.5 (Tarski's fixed-point theorem)

Let $(D, \sqsubseteq)$ be a complete lattice and $f : D \to D$ monotonic. Then f has a least fixed point $\mathrm{fix}(f)$ and a greatest fixed point $\mathrm{FIX}(f)$ given by

$$\mathrm{fix}(f) = \bigsqcap \{d \in D \mid f(d) \sqsubseteq d\} \qquad \textit{(GLB of all pre-fixed points of } f\textit{)}$$

$$\mathrm{FIX}(f) = \bigsqcup \{d \in D \mid d \sqsubseteq f(d)\} \qquad \textit{(LUB of all post-fixed points of } f\textit{)}$$

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Proof.

on the board □

The Fixed-Point Theorem II

Example 5.6 (cf. Example 5.2)

- Let $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}} : T \mapsto T \cup \{1, 2\}$
- As seen in Example 5.2: $f(T) = T$ iff $\{1, 2\} \subseteq T$

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- As seen in Example 5.2: $f(T) = T$ iff $\{1, 2\} \subseteq T$
- Theorem 5.5 for fix:

$$\begin{aligned}
\mathrm{fix}(f) &= \bigsqcap\{d \in D \mid f(d) \sqsubseteq d\} & \\
&= \bigcap\{T \subseteq \mathbb{N} \mid f(T) \subseteq T\} & \text{(Lemma 4.15)} \\
&= \bigcap\{T \subseteq \mathbb{N} \mid T \cup \{1, 2\} \subseteq T\} & \text{(Def. } f\text{)} \\
&= \bigcap\{T \subseteq \mathbb{N} \mid \{1, 2\} \subseteq T\} & \\
&= \{1, 2\} &
\end{aligned}$$

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The Fixed-Point Theorem for Finite Lattices

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Theorem 5.7 (Fixed-point theorem for finite lattices)

Let $(D, \sqsubseteq)$ be a finite complete lattice and $f : D \to D$ monotonic. Then

$$\mathrm{fix}(f) = f^m(\bot) \quad \textit{and} \quad \mathrm{FIX}(f) = f^M(\top)$$

for some $m, M \in \mathbb{N}$ where $f^0(d) := d \quad \textit{and} \quad f^{k+1}(d) := f(f^k(d))$.

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on the board □

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on the board □

Example 5.8

- Let $f : 2^{\{0,1,2\}} \to 2^{\{0,1,2\}} : T \mapsto T \cup \{1\} \setminus \{2\}$ (monotonic on $(2^{\{0,1,2\}}, \subseteq)$)

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- $f^0(\bot) = \emptyset$, $f^1(\bot) = \{1\}$, $f^2(\bot) = \{1\} = f^1(\bot)$
  $\implies$ $\mathrm{fix}(f) = \{1\}$ after $m = 1$ iterations

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- $f^0(\top) = \{0, 1, 2\}$, $f^1(\top) = \{0, 1\}$, $f^2(\top) = \{0, 1\} = f^1(\top)$
  $\implies \mathrm{FIX}(f) = \{0, 1\}$ after $M = 1$ iterations

Application to HML with Recursion

Lemma 5.9

Let $(S, Act, \longrightarrow)$ *be an LTS and* $F \in HMF_X$. *Then*

1. $[\![F]\!] : 2^S \to 2^S$ *is monotonic w.r.t.* $(2^S, \subseteq)$

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If, in addition, S *is finite, then*

4. $\mathrm{fix}([\![F]\!]) = [\![F]\!]^m(\emptyset)$ *for some* $m \in \mathbb{N}$
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Proof.

1. by induction on the structure of F (details omitted)
2. by Lemma 4.15 and Theorem 5.5
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4. by Lemma 4.15 and Theorem 5.7
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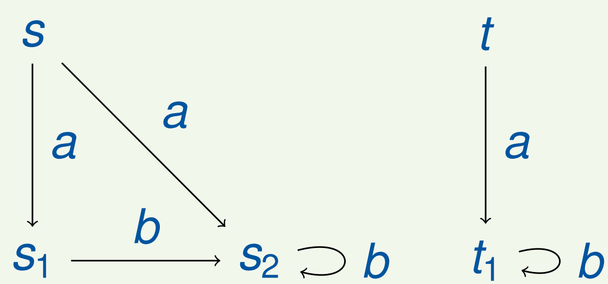
The Fixed-Point Theorem for Finite Lattices

An Example

Example 5.10



Let $S := \{s, s_1, s_2, t, t_1\}$.

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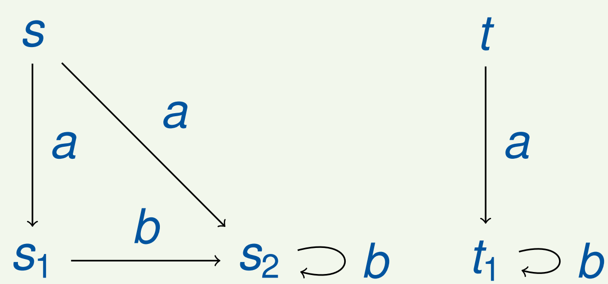

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1. Solution of

$$X \overset{max}{=} \langle b \rangle \mathrm{tt} \wedge [b]X$$

   ("all b^*-successors have a b-successor"): on the board

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The Fixed-Point Theorem for Finite Lattices

An Example

Example 5.10

$$\begin{tikzcd} s \arrow[d, "a"] \arrow[dr, "a"] & & t \arrow[d, "a"] \\ s_1 \arrow[r, "b"] & s_2 \arrow[loop right, "b"] & t_1 \arrow[loop right, "b"] \end{tikzcd}$$

Let $S := \{s, s_1, s_2, t, t_1\}$.

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("all b^*-successors have a b-successor"): on the board
2. Solution of
$$Y \overset{min}{=} \langle b \rangle \mathrm{tt} \vee \langle \{a, b\} \rangle Y$$
("a b-transition is reachable"): on the board