Concurrency Theory

Winter Semester 2019/20

Lecture 4: Hennessy-Milner Logic with Recursion

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https://moves.rwth-aachen.de/teaching/ws-19-20/ct/

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Outline of Lecture 4

Recap: Hennessy-Milner Logic and Process Traces

Adding Recursion to HML

HML with One Recursive Variable

Algebraic Foundations

Recap: Hennessy-Milner Logic and Process Traces

Syntax of HML

Definition (Syntax of HML)

The set *HMF* of Hennessy-Milner formulae over a set of actions *Act* is defined by the following syntax:

$$
\begin{aligned}
F ::= {} & \mathrm{tt} && \text{(true)} \\
\mid {} & \mathrm{ff} && \text{(false)} \\
\mid {} & F_1 \wedge F_2 && \text{(conjunction)} \\
\mid {} & F_1 \vee F_2 && \text{(disjunction)} \\
\mid {} & \langle \alpha \rangle F && \text{(diamond)} \\
\mid {} & [\alpha] F && \text{(box)}
\end{aligned}
$$

where $\alpha \in Act$.

Abbreviations for $L = \{\alpha_1, \ldots, \alpha_n\}$ ($n \in \mathbb{N}$):

- $\langle L \rangle F := \langle \alpha_1 \rangle F \vee \ldots \vee \langle \alpha_n \rangle F$
- $[L]F := [\alpha_1]F \wedge \ldots \wedge [\alpha_n]F$
- In particular, $\langle \emptyset \rangle F :=$ ff and $[\emptyset]F :=$ tt

Semantics of HML

Definition (Semantics of HML)

Let $(S, Act, \longrightarrow)$ be an LTS and $F \in HMF$. The set of processes in S that satisfy F, $[\![F]\!] \subseteq S$, is defined by:

$$[\![\mathrm{tt}]\!] := S \qquad [\![\mathrm{ff}]\!] := \emptyset$$
$$[\![F_1 \wedge F_2]\!] := [\![F_1]\!] \cap [\![F_2]\!] \qquad [\![F_1 \vee F_2]\!] := [\![F_1]\!] \cup [\![F_2]\!]$$
$$[\![\langle\alpha\rangle F]\!] := \langle\cdot\alpha\cdot\rangle([\![F]\!]) \qquad [\![[\alpha]F]\!] := [\cdot\alpha\cdot]([\![F]\!])$$

where $\langle\cdot\alpha\cdot\rangle, [\cdot\alpha\cdot] : 2^S \to 2^S$ are given by

$$\langle\cdot\alpha\cdot\rangle(T) := \{s \in S \mid \exists s' \in T : s \xrightarrow{\alpha} s'\}$$
$$[\cdot\alpha\cdot](T) := \{s \in S \mid \forall s' \in S : s \xrightarrow{\alpha} s' \implies s' \in T\}$$

We write $s \models F$ iff $s \in [\![F]\!]$. Two HML formulae are equivalent (written $F \equiv G$) iff they are satisfied by the same processes in every LTS.

Closure under Negation

Observation: negation is not one of the HML constructs
Reason: HML is closed under negation

Lemma

For every $F \in HMF$ there exists $F^c \in HMF$ such that $[\![F^c]\!] = S \setminus [\![F]\!]$ for every LTS $(S, Act, \longrightarrow)$.

Proof.

Definition of F^c:

$$\begin{aligned} tt^c &:= ff & ff^c &:= tt \\ (F_1 \wedge F_2)^c &:= F_1^c \vee F_2^c & (F_1 \vee F_2)^c &:= F_1^c \wedge F_2^c \\ (\langle\alpha\rangle F)^c &:= [\alpha]F^c & ([\alpha]F)^c &:= \langle\alpha\rangle F^c \end{aligned}$$

Process Traces

Goal: reduce processes to the action sequences they can perform

Definition (Trace language)

For every $P \in Prc$, let

$$Tr(P) := \{w \in Act^* \mid \text{ex. } P' \in Prc \text{ such that } P \xrightarrow{w} P'\}$$

be the trace language of P (where $\xrightarrow{w} := \xrightarrow{a_1} \circ \ldots \circ \xrightarrow{a_n}$ for $w = a_1 \ldots a_n$).

$P, Q \in Prc$ are called trace equivalent if $Tr(P) = Tr(Q)$.

Example (One-place buffer)

$B = in.\overline{out}.B$

$\implies Tr(B) = (in \cdot \overline{out})^* \cdot (in + \varepsilon)$

Recap: Hennessy-Milner Logic and Process Traces

HML and Process Traces

Lemma

Let $(Prc, Act, \longrightarrow)$ be an LTS, and let $P, Q \in Prc$ satisfy the same HMF (i.e., $\forall F \in HMF : P \models F \iff Q \models F$). Then $Tr(P) = Tr(Q)$.

Proof.

on the board □

Remark: the converse does not hold.

Example

- Let $P := a.(b.\text{nil} + c.\text{nil}) \in Prc$, $Q := a.b.\text{nil} + a.c.\text{nil} \in Prc$
- Then $Tr(P) = Tr(Q) = \{\varepsilon, a, ab, ac\}$
- Let $F := [a](\langle b\rangle \text{tt} \wedge \langle c\rangle \text{tt}) \in HMF$
- Then $P \models F$ but $Q \not\models F$
- [Later: $P, Q \in Prc$ HML-equivalent iff bismilar]

Outline of Lecture 4

Finiteness of HML

Observation: HML formulae only describe finite part of process behaviour

- each modal operator ($[.]$, $\langle . \rangle$) talks about one step
- only finite nesting of operators (modal depth)

Adding Recursion to HML

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- $F := (\langle a \rangle [a] \mathrm{ff}) \vee \langle b \rangle \mathrm{tt} \in HMF$ has modal depth 2
- Checking F involves analysis of all behaviours of length ≤ 2

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- $F := (\langle a \rangle [a] \mathrm{ff}) \vee \langle b \rangle \mathrm{tt} \in HMF$ has modal depth 2
- Checking F involves analysis of all behaviours of length ≤ 2

But: sometimes necessary to refer to arbitrarily long computations (e.g., “no deadlock state reachable”)

- possible solution: support infinite conjunctions and disjunctions

Infinite Conjunctions

Example 4.2

- Let $C = a.C$, $D = a.D + a.\mathrm{nil}$
- Then $C \models [a]\langle a\rangle \mathrm{tt}$ but $D \not\models [a]\langle a\rangle \mathrm{tt}$ (i.e., C and D distinguishable by formula of depth 2)

Adding Recursion to HML

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- Then $C \models [a]\langle a\rangle\text{tt}$ but $D \not\models [a]\langle a\rangle\text{tt}$ (i.e., C and D distinguishable by formula of depth 2)
- Now redefine D as $D_n = a.D_n + a.E_n$ where $n \in \mathbb{N}$, $E_k = a.E_{k-1}$ $(1 \leq k \leq n)$, $E_0 = \text{nil}$
- Then (for $[\alpha]^k F := \underbrace{[\alpha]\ldots[\alpha]}_{k \text{ times}} F$ where $F \in HMF$):
  - $C \models [a]^k\langle a\rangle\text{tt}$ for all $k \in \mathbb{N}$
  - $D_n \models [a]^k\langle a\rangle\text{tt}$ for all $0 \leq k \leq n$
  - $D_n \not\models [a]^k\langle a\rangle\text{tt}$ for all $k > n$

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  - $D_n \models [a]^k \langle a\rangle\mathrm{tt}$ for all $0 \leq k \leq n$
  - $D_n \not\models [a]^k \langle a\rangle\mathrm{tt}$ for all $k > n$
- Conclusion: no single HML formula can distinguish C and all D_n
  - unsatisfactory as behaviour clearly different
- Generally: invariant property "always $\langle a\rangle\mathrm{tt}$" not expressible
- Requires infinite conjunction:

$$Inv(\langle a\rangle\mathrm{tt}) = \langle a\rangle\mathrm{tt} \wedge [a]\langle a\rangle\mathrm{tt} \wedge [a][a]\langle a\rangle\mathrm{tt} \wedge \dots = \bigwedge_{k\in\mathbb{N}} [a]^k \langle a\rangle\mathrm{tt}$$

Infinite Disjunctions

Dually: possibility properties expressible by infinite disjunctions

Example 4.3

- Let $C = a.C$, $D = a.D + a.\text{nil}$ as before
- C has no possibility to terminate
- D has the option to terminate (i.e., to eventually satisfy $[a]\text{ff}$) at any time by choosing the $a.\text{nil}$ branch

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- D has the option to terminate (i.e., to eventually satisfy $[a]\text{ff}$) at any time by choosing the $a.\text{nil}$ branch
- Representable by infinite disjunction:

$$Pos([a]\text{ff}) = [a]\text{ff} \lor \langle a\rangle[a]\text{ff} \lor \langle a\rangle\langle a\rangle[a]\text{ff} \lor \ldots = \bigvee_{k\in\mathbb{N}} \langle a\rangle^k[a]\text{ff}$$

Adding Recursion to HML

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Problem: infinite formulae not easy to handle

Adding Recursion to HML

Introducing Recursion

Solution: employ recursion!

- $Inv(\langle a\rangle tt) \equiv \langle a\rangle tt \wedge [a]\, Inv(\langle a\rangle tt)$
- $Pos([a]ff) \equiv [a]ff \vee \langle a\rangle\, Pos([a]ff)$

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- $Pos([a]\mathrm{ff}) \equiv [a]\mathrm{ff} \vee \langle a \rangle\, Pos([a]\mathrm{ff})$

Interpretation: the sets of states $X, Y \subseteq S$ satisfying the respective formula should solve the corresponding equation, i.e.,

- $X = \langle \cdot a \cdot \rangle(S) \cap [\cdot a \cdot](X)$
- $Y = [\cdot a \cdot](\emptyset) \cup \langle \cdot a \cdot \rangle(Y)$

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Open questions

- Do such recursive equations (always) have solutions?
- If so, are they unique?
- How can we decide whether a process satisfies a recursive formula (“model checking”)?

Existence of Solutions

Example 4.4

- Consider again $C = a.C$, $D = a.D + a.\mathrm{nil}$

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- Invariant: $X \equiv \langle a\rangle \mathrm{tt} \land [a]X$
  - $X = \emptyset$ is a solution (as no process can satisfy both $\langle a\rangle \mathrm{tt}$ and $[a]\mathrm{ff}$)
  - but we expect $C \in X$ (as C can perform a invariantly)
  - in fact, $X = \{C\}$ also solves the equation (and is the greatest solution w.r.t. $\subseteq$)

$\Longrightarrow$ write $X \overset{max}{=} \langle a\rangle \mathrm{tt} \land [a]X$

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$\Longrightarrow$ write $X \stackrel{max}{=} \langle a \rangle \mathrm{tt} \land [a]X$

- Possibility: $Y \equiv [a]\mathrm{ff} \lor \langle a \rangle Y$
  - greatest solution: $Y = \{C, D, \mathrm{nil}\}$
  - but we expect $C \notin Y$ (as C cannot terminate at all)
  - here: least solution w.r.t. $\subseteq$: $Y = \{D, \mathrm{nil}\}$

$\Longrightarrow$ write $Y \stackrel{min}{=} [a]\mathrm{ff} \lor \langle a \rangle Y$

Uniqueness of Solutions

Uniqueness of solutions

- Use greatest solutions for properties that hold unless the process has a finite computation that disproves it.
- Use least solutions for properties that hold if the process has a finite computation that proves it.

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Example 4.5

Let $(S, Act, \longrightarrow)$ be an LTS, $s \in S$, and $F \in HMF$.

- Invariant: $Inv(F) \equiv X$ for $X \stackrel{max}{=} F \wedge [Act]X$
  - $s \models Inv(F)$ if all states reachable from s satisfy F

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  - $s \models Inv(F)$ if all states reachable from s satisfy F
- Possibility: $Pos(F) \equiv Y$ for $Y \overset{min}{=} F \vee \langle Act \rangle Y$
  - $s \models Pos(F)$ if a state satisfying F is reachable from s

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- Invariant: $Inv(F) \equiv X$ for $X \overset{max}{=} F \wedge [Act]X$
  - $s \models Inv(F)$ if all states reachable from s satisfy F
- Possibility: $Pos(F) \equiv Y$ for $Y \overset{min}{=} F \vee \langle Act \rangle Y$
  - $s \models Pos(F)$ if a state satisfying F is reachable from s
- Safety: $Safe(F) \equiv X$ for $X \overset{max}{=} F \wedge ([Act]\text{ff} \vee \langle Act \rangle X)$
  - $s \models Safe(F)$ if s has a complete (i.e., infinite or terminating) transition sequence where each state satisfies F

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- Use greatest solutions for properties that hold unless the process has a finite computation that disproves it.
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Let $(S, Act, \longrightarrow)$ be an LTS, $s \in S$, and $F \in HMF$.

- Invariant: $Inv(F) \equiv X$ for $X \stackrel{max}{=} F \wedge [Act]X$
  - $s \models Inv(F)$ if all states reachable from s satisfy F
- Possibility: $Pos(F) \equiv Y$ for $Y \stackrel{min}{=} F \vee \langle Act \rangle Y$
  - $s \models Pos(F)$ if a state satisfying F is reachable from s
- Safety: $Safe(F) \equiv X$ for $X \stackrel{max}{=} F \wedge ([Act]\mathrm{ff} \vee \langle Act \rangle X)$
  - $s \models Safe(F)$ if s has a complete (i.e., infinite or terminating) transition sequence where each state satisfies F
- Eventuality: $Evt(F) \equiv Y$ for $Y \stackrel{min}{=} F \vee (\langle Act \rangle \mathrm{tt} \wedge [Act]Y)$
  - $s \models Evt(F)$ if each complete transition sequence starting in s contains a state satisfying F

Outline of Lecture 4

HML with One Recursive Variable

Syntax of HML with One Recursive Variable

Initially: only one variable (for simplicity)
Later: mutual recursion

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Initially: only one variable (for simplicity)
Later: mutual recursion

Definition 4.6 (Syntax of HML with one variable)

The set HMF_X of Hennessy-Milner formulae with one variable X over a set of actions Act is defined by the following syntax:

$$
\begin{array}{rlll}
F ::= & X & & \text{(variable)} \\
| & \mathrm{tt} & & \text{(true)} \\
| & \mathrm{ff} & & \text{(false)} \\
| & F_1 \wedge F_2 & & \text{(conjunction)} \\
| & F_1 \vee F_2 & & \text{(disjunction)} \\
| & \langle \alpha \rangle F & & \text{(diamond)} \\
| & [\alpha] F & & \text{(box)}
\end{array}
$$

where $\alpha \in Act$.

Semantics of HML with One Recursive Variable I

So far: $[\![F]\!] \subseteq S$ for $F \in \mathit{HMF}$ and LTS $(S, \mathit{Act}, \longrightarrow)$

Now: semantics of formula depends on states that (are assumed to) satisfy X

Semantics of HML with One Recursive Variable I

So far: $[\![F]\!] \subseteq S$ for $F \in HMF$ and LTS $(S, Act, \longrightarrow)$

Now: semantics of formula depends on states that (are assumed to) satisfy X

Definition 4.7 (Semantics of HML with one variable)

Let $(S, Act, \longrightarrow)$ be an LTS and $F \in HMF_X$. The semantics of F,

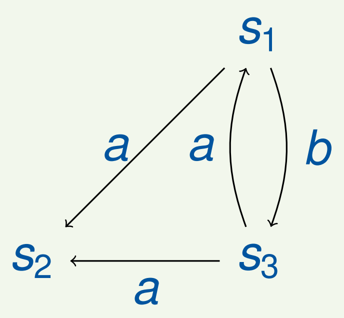
$$[\![F]\!] : 2^S \rightarrow 2^S,$$

is defined by

$$\begin{aligned}
[\![X]\!](T) &:= T \\
[\![\mathrm{tt}]\!](T) &:= S \\
[\![\mathrm{ff}]\!](T) &:= \emptyset \\
[\![F_1 \wedge F_2]\!](T) &:= [\![F_1]\!](T) \cap [\![F_2]\!](T) \\
[\![F_1 \vee F_2]\!](T) &:= [\![F_1]\!](T) \cup [\![F_2]\!](T) \\
[\![\langle\alpha\rangle F]\!](T) &:= \langle\cdot\alpha\cdot\rangle([\![F]\!](T)) \\
[\![[\alpha]F]\!](T) &:= [\cdot\alpha\cdot]([\![F]\!](T))
\end{aligned}$$

Semantics of HML with One Recursive Variable II

Example 4.8



Let $S := \{s_1, s_2, s_3\}$.

Semantics of HML with One Recursive Variable II

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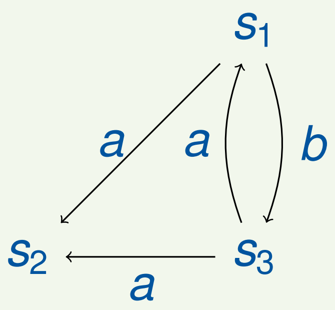

Let $S := \{s_1, s_2, s_3\}$.

- $[\![\langle a \rangle X]\!](\{s_1\}) = \{s_3\}$

Semantics of HML with One Recursive Variable II

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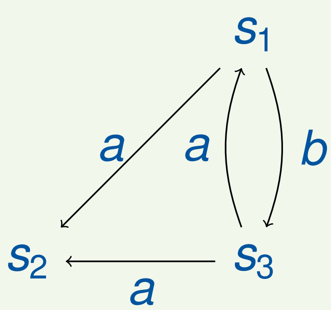

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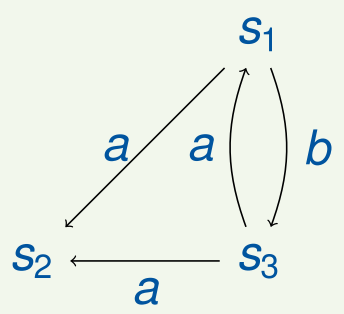

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- $[\![\langle a \rangle X]\!](\{s_1, s_2\}) = \{s_1, s_3\}$
- $[\![[b]X]\!](\{s_2\}) = \{s_2, s_3\}$

Semantics of HML with One Recursive Variable III

- Idea underlying the definition of

$$[\![.]\!] : HMF_X \to (2^S \to 2^S) :$$

  if $T \subseteq S$ gives the set of states that satisfy X, then $[\![F]\!](T)$ will be the set of states that satisfy F

Semantics of HML with One Recursive Variable III

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  if $T \subseteq S$ gives the set of states that satisfy X, then $[\![F]\!](T)$ will be the set of states that satisfy F
- How to determine this T?
- According to previous discussion: as solution of recursive equation of the form $X = F_X$ where $F_X \in HMF_X$

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$$X \stackrel{min}{=} F_X \qquad \text{or} \qquad X \stackrel{max}{=} F_X$$

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$$X \stackrel{min}{=} F_X \qquad \text{or} \qquad X \stackrel{max}{=} F_X$$

- In the following we will see:
  1. Equation $X = F_X$ always solvable
  2. Least and greatest solutions are unique and can be obtained by fixed-point iteration

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Adding Recursion to HML

HML with One Recursive Variable

Algebraic Foundations

Partial Orders

Definition 4.9 (Partial order)

A partial order (PO) $(D, \sqsubseteq)$ consists of a set D, called domain, and of a relation $\sqsubseteq \subseteq D \times D$ such that, for every $d_1, d_2, d_3 \in D$,

reflexivity: $d_1 \sqsubseteq d_1$

transitivity: $d_1 \sqsubseteq d_2$ and $d_2 \sqsubseteq d_3 \implies d_1 \sqsubseteq d_3$

antisymmetry: $d_1 \sqsubseteq d_2$ and $d_2 \sqsubseteq d_1 \implies d_1 = d_2$

It is called total if, in addition, always $d_1 \sqsubseteq d_2$ or $d_2 \sqsubseteq d_1$.

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It is called total if, in addition, always $d_1 \sqsubseteq d_2$ or $d_2 \sqsubseteq d_1$.

Example 4.10

1. $(\mathbb{N}, \leq)$ is a total partial order
2. $(\mathbb{N}, <)$ is not a partial order (since not reflexive)

Partial Orders

Definition 4.9 (Partial order)

A partial order (PO) $(D, \sqsubseteq)$ consists of a set D, called domain, and of a relation $\sqsubseteq \subseteq D \times D$ such that, for every $d_1, d_2, d_3 \in D$,

reflexivity: $d_1 \sqsubseteq d_1$

transitivity: $d_1 \sqsubseteq d_2$ and $d_2 \sqsubseteq d_3 \implies d_1 \sqsubseteq d_3$

antisymmetry: $d_1 \sqsubseteq d_2$ and $d_2 \sqsubseteq d_1 \implies d_1 = d_2$

It is called total if, in addition, always $d_1 \sqsubseteq d_2$ or $d_2 \sqsubseteq d_1$.

Example 4.10

1. $(\mathbb{N}, \leq)$ is a total partial order
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4. $(\Sigma^*, \sqsubseteq)$ is a (non-total) partial order, where Σ is some alphabet and $\sqsubseteq$ denotes prefix ordering ($u \sqsubseteq v \iff \exists w \in \Sigma^* : uw = v$)

Upper and Lower Bounds

Definition 4.11 ((Least) upper bounds and (greatest) lower bounds)

Let $(D, \sqsubseteq)$ be a partial order and $T \subseteq D$.

1. An element $d \in D$ is called an upper bound of T if $t \sqsubseteq d$ for every $t \in T$ (notation: $T \sqsubseteq d$). It is called least upper bound (LUB) (or supremum) of T if additionally $d \sqsubseteq d'$ for every upper bound d' of T (notation: $d = \bigsqcup T$).

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2. In $(2^{\mathbb{N}}, \subseteq)$, every subset $T \subseteq 2^{\mathbb{N}}$ has an LUB and GLB:

$$\bigsqcup T = \bigcup T \qquad \text{and} \qquad \bigsqcap T = \bigcap T$$

Complete Lattices

Definition 4.13 (Complete lattice)

A complete lattice is a partial order $(D, \sqsubseteq)$ such that all subsets of D have LUBs and GLBs. In this case,

$$\bot := \bigsqcup \emptyset \; (= \bigsqcap D) \qquad \text{and} \qquad \top := \bigsqcap \emptyset \; (= \bigsqcup D)$$

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Application to HML with Recursion

Lemma 4.15

Let $(S, Act, \longrightarrow)$ *be an LTS. Then* $(2^S, \subseteq)$ *is a complete lattice with*

- $\bigsqcup \mathcal{T} = \bigcup \mathcal{T} = \bigcup_{T \in \mathcal{T}} T$ *for all* $\mathcal{T} \subseteq 2^S$
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Proof.

omitted □