Model Checking

Lecture #8: LTL Model Checking By Automata

[Baier & Katoen, Chapter 5.2]

Joost-Pieter Katoen

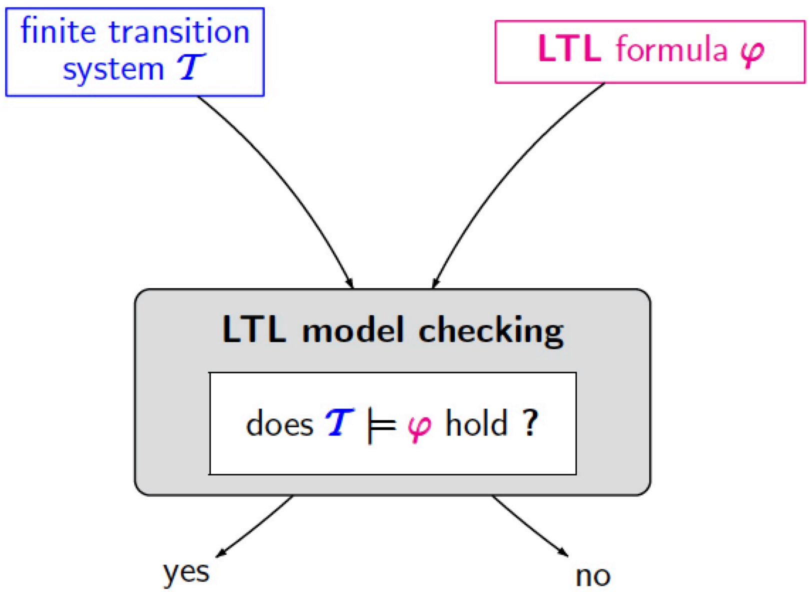
Software Modeling and Verification Group

Model Checking Course, RWTH Aachen, WiSe 2019/2020

Overview

1. Linear Temporal Logic
2. LTL Model Checking
3. From LTL to GNBA
4. Complexity
5. Summary

Topic



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LTL Syntax

Definition: LTL syntax

BNF grammar for LTL formulas with proposition $a \in AP$:

$$\varphi \ ::= \ \mathrm{true} \ \big| \ a \ \big| \ \varphi_1 \wedge \varphi_2 \ \big| \ \neg\varphi \ \big| \ \bigcirc\varphi \ \big| \ \varphi_1 \,\mathsf{U}\, \varphi_2$$

- Propositional logic
  - $a \in AP$ — atomic proposition
  - $\neg\varphi$ and $\varphi \wedge \psi$ — negation and conjunction
- Temporal modalities
  - $\bigcirc\varphi$ — neXt state fulfills φ
  - $\varphi \,\mathsf{U}\, \psi$ — φ holds Until a ψ-state is reached

Linear Temporal Logic (LTL) is a logic to describe LT properties

Derived Operators

$\Diamond\varphi \equiv \mathrm{true}\,\mathsf{U}\,\varphi$ "some time in the future"

$\Box\varphi \equiv \neg\Diamond\neg\varphi$ "from now on forever"

Semantics Over Words

Definition: LTL semantics over infinite words

The LT-property induced by LTL formula φ over AP is:

$$Words(\varphi) = \left\{ \sigma \in \left(2^{AP}\right)^{\omega} \mid \sigma \vDash \varphi \right\}, \text{where } \vDash \text{ is the smallest relation with:}$$

$$\begin{array}{lllll}
\sigma & \vDash & \mathrm{true} & & \\
\sigma & \vDash & a & \text{iff} & a \in A_0 \quad (\text{i.e., } A_0 \vDash a) \\
\sigma & \vDash & \varphi_1 \wedge \varphi_2 & \text{iff} & \sigma \vDash \varphi_1 \text{ and } \sigma \vDash \varphi_2 \\
\sigma & \vDash & \neg\varphi & \text{iff} & \sigma \nvDash \varphi \\
\sigma & \vDash & \bigcirc\varphi & \text{iff} & \sigma[1..] = A_1A_2A_3\ldots \vDash \varphi \\
\sigma & \vDash & \varphi_1 \,\mathsf{U}\, \varphi_2 & \text{iff} & \exists j \geq 0.\ \sigma[j..] \vDash \varphi_2 \text{ and } \sigma[i..] \vDash \varphi_1,\ 0 \leq i < j
\end{array}$$

for $\sigma = A_0A_1A_2\ldots$, let $\sigma[i..] = A_iA_{i+1}A_{i+2}\ldots$ be the suffix of σ from index i on.

Semantics of $\Box$, $\Diamond$, $\Box\Diamond$ and $\Diamond\Box$

$$\begin{array}{lllll}
\sigma & \vDash & \Diamond\varphi & \text{iff} & \exists j \geq 0.\ \sigma[j..] \vDash \varphi \\
\sigma & \vDash & \Box\varphi & \text{iff} & \forall j \geq 0.\ \sigma[j..] \vDash \varphi \\
\sigma & \vDash & \Box\Diamond\varphi & \text{iff} & \underbrace{\forall j \geq 0.\ \exists i \geq j.\ \sigma[i\ldots] \vDash \varphi}_{\text{infinitely often } \varphi} \\
\sigma & \vDash & \Diamond\Box\varphi & \text{iff} & \underbrace{\exists j \geq 0. \forall i \geq j.\ \sigma[i\ldots] \vDash \varphi}_{\text{persistence of } \varphi}
\end{array}$$

Semantics over Transition Systems

Let $TS = (S, Act, \rightarrow, I, AP, L)$ be a transition system and φ be an LTL-formula over AP.

- For infinite path fragment π of TS:

$$\pi \models \varphi \quad \text{iff} \quad trace(\pi) \models \varphi$$

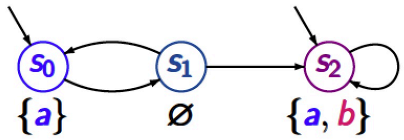
- For state $s \in S$:

$$s \models \varphi \quad \text{iff} \quad \forall \pi \in Paths(s).\ \pi \models \varphi$$

- For transition system TS:

$$TS \models \varphi \quad \text{iff} \quad Traces(TS) \subseteq Words(\varphi) \quad \text{iff} \quad \forall s \in I.\ s \models \varphi$$

Example



$AP = \{a, b\}$

$\mathcal{T} \models a$ as $s_0 \models a$ and $s_2 \models a$

$\mathcal{T} \not\models \Diamond\Box a$ as $s_0\, s_1\, s_0\, s_1 \ldots \not\models \Diamond\Box a$

$\mathcal{T} \models \Diamond\Box b \vee \Box\Diamond(\neg a \wedge \neg b)$ as $s_2 \models b$, $s_1 \not\models a, b$

$\mathcal{T} \models \Box(a \rightarrow (\bigcirc\neg a \vee b))$ as $s_2 \models b$, $s_0 \models \bigcirc\neg a$

Overview

The LTL Model Checking Problem

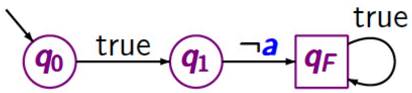
Given:

1. finite transition system TS, and
2. LTL-formula φ

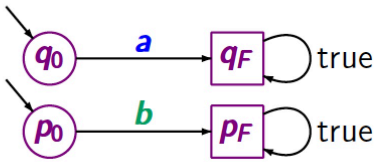
:

decide whether $TS \models \varphi$, and if $TS \not\models \varphi$, provide a counterexample.

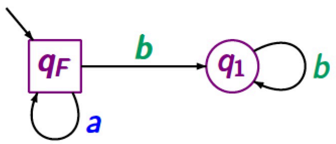
NBA for LTL Formulae



$\mathcal{L}_\omega(\mathcal{A}) = Words(\bigcirc \neg a)$

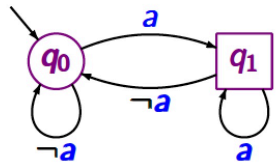


$\mathcal{L}_\omega(\mathcal{A}) = Words(a \vee b)$

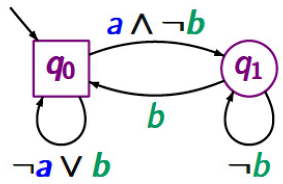


$\mathcal{L}_\omega(\mathcal{A}) = Words(\Box a)$

NBA for LTL Formulae



$\mathcal{L}_\omega(\mathcal{A}) = Words(\Box\Diamond a)$



$\mathcal{L}_\omega(\mathcal{A}) = Words(\Box(a \rightarrow \Diamond b))$

e.g., $\left.\begin{array}{l} \varnothing\,\varnothing\,\varnothing\,\varnothing \ldots = \varnothing^\omega \\ (\{a\}\,\{b\})^\omega \end{array}\right\}$ are accepted by $\mathcal{A}$

A Naive Attempt

$TS \models \varphi$ if and only if $Traces(TS) \subseteq Words(\varphi)$

if and only if $Traces(TS) \subseteq \mathfrak{L}_\omega(\mathfrak{A}_\varphi)$

if and only if $Traces(TS) \cap \overline{\mathfrak{L}_\omega(\mathfrak{A}_\varphi)} = \varnothing$

if and only if $Traces(TS) \cap \mathfrak{L}_\omega(\overline{\mathfrak{A}_\varphi}) = \varnothing$.

Naive idea: check whether TS has no behaviour accepted by NBA $\overline{\mathfrak{A}_\varphi}$

But complementation of NBA yields a blow-up:
if $\mathfrak{A}$ has n states, $\overline{\mathfrak{A}}$ has c^{n^2} states in worst case

$\Rightarrow$ use the fact that: $\mathfrak{L}_\omega(\overline{\mathfrak{A}_\varphi}) = \mathfrak{L}_\omega(\mathfrak{A}_{\neg\varphi})$

Approach

$TS \models \varphi$ if and only if $Traces(TS) \subseteq Words(\varphi)$

if and only if $Traces(TS) \subseteq \mathfrak{L}_\omega(\mathfrak{A}_\varphi)$

if and only if $Traces(TS) \cap \overline{\mathfrak{L}_\omega(\mathfrak{A}_\varphi)} = \varnothing$

if and only if $Traces(TS) \cap \mathfrak{L}_\omega(\overline{\mathfrak{A}_\varphi}) = \varnothing$

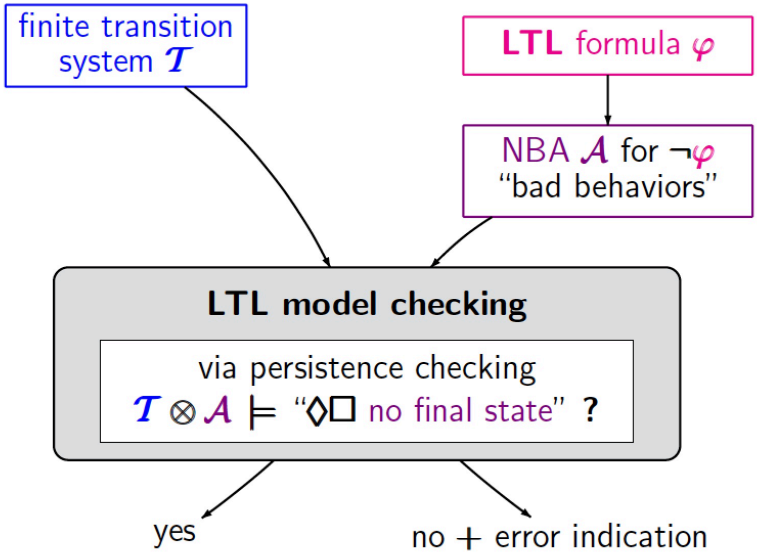
if and only if $Traces(TS) \cap \mathfrak{L}_\omega(\mathfrak{A}_{\neg\varphi}) = \varnothing$

if and only if $TS \otimes \mathfrak{A}_{\neg\varphi} \models \Diamond\Box\,\neg F$

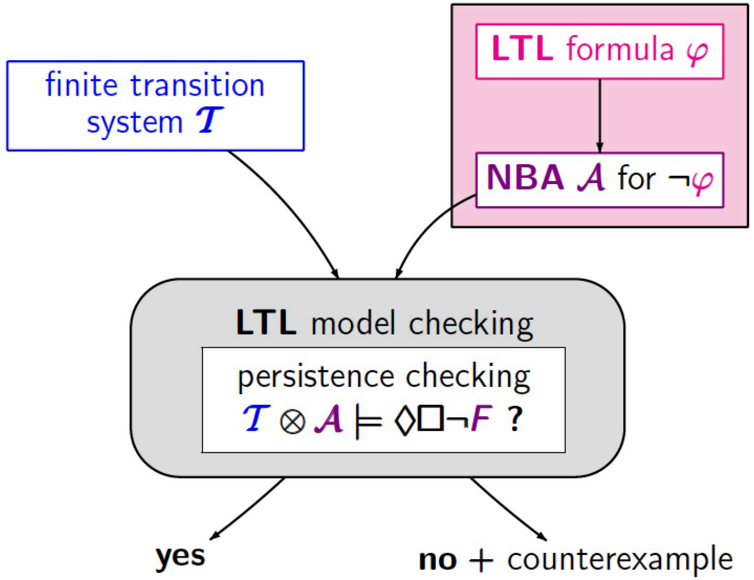
where F is the set of accept states of NBA $\mathfrak{A}_{\neg\varphi}$.

LTL model checking is thus reduced to persistence checking

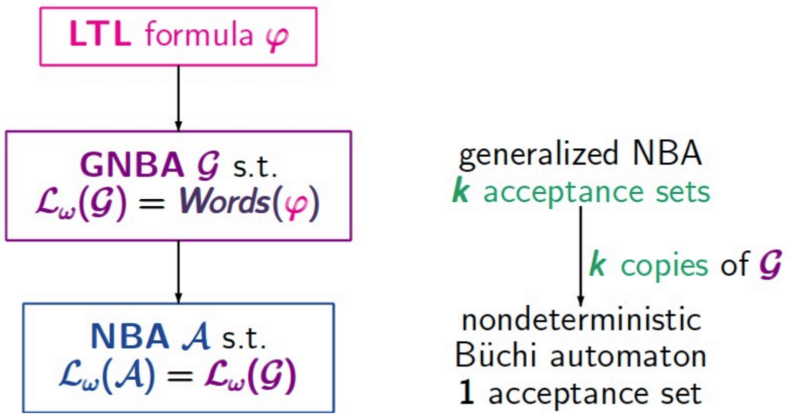
Automata-Based LTL Model Checking



Overview



Overview



Recap: Generalized Büchi Automata

Definition: Generalized Büchi automata

A generalized NBA (GNBA) $\mathfrak{G}$ is a tuple $(Q, \Sigma, \delta, Q_0, \mathfrak{F})$ where Q, Σ, δ, Q_0 are as before and

$$\mathfrak{F} = \{ F_1, \ldots, F_k \} \quad \text{with} \quad F_i \subseteq Q$$

for some natural $k \in \mathbb{N}$.

Run $q_0 \, q_1 \ldots \in Q^\omega$ is accepting if $\forall F_j \in \mathfrak{F}$: $q_i \in F_j$ for infinitely many i

The size of $\mathfrak{G}$, denoted $|\mathfrak{G}|$, is the number of states and transitions in $\mathfrak{G}$

GNBA and NBA are Equally Expressive

For every GNBA $\mathfrak{G}$ there exists an NBA $\mathfrak{A}$ with

$$\mathfrak{L}_\omega(\mathfrak{G}) \;=\; \mathfrak{L}_\omega(\mathfrak{A}) \quad \text{with} \quad |\mathfrak{A}| = O(|\mathfrak{G}| \cdot |\mathfrak{F}|)$$

where $\mathfrak{F} = \{ F_1, \ldots, F_k \}$ denotes the set of acceptance sets in $\mathfrak{G}$.

Proof.

For k=0, 1, this result follows directly. For $k > 1$, make k copies of $\mathfrak{G}$:

- initial states of NBA := the initial states in the first copy
- final states of NBA := accept set F_1 in the first copy
- on visiting in i-th copy a state in F_i, then move to the $(i+1)$-st copy

□

Overview

How to Obtain a GNBA?

GNBA $\mathfrak{G}_\varphi$ over 2^{AP} for LTL-formula φ with $\mathfrak{L}_\omega(\mathfrak{G}_\varphi) = Words(\varphi)$:

- Assume φ only contains the operators $\wedge$, $\neg$, $\bigcirc$ and U

  $\vee$, $\rightarrow$, $\Diamond$, $\Box$, W, and so on, are derived from these base operators

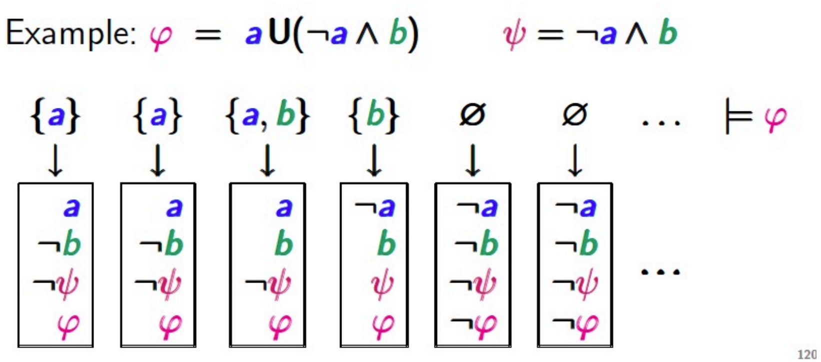
- States are elementary sets of sub-formulas in φ
  - for $\sigma = A_0 A_1 \ldots \in Words(\varphi)$, expand $A_i \subseteq AP$ with sub-formulas of φ
  - ... to obtain the infinite word $\bar{\sigma} \;=\; B_0 B_1 \ldots$ with B_i a set of sub-formulas of φ such that

    $$\psi \in B_i \qquad \text{if and only if} \qquad \sigma^i = A_i A_{i+1} \ldots \models \psi$$

  - $\bar{\sigma}$ is intended to be a run of GNBA $\mathfrak{G}_\varphi$ for σ

- Transitions are derived from semantics $\bigcirc$ and expansion law for U
- Accept sets guarantee that: $\bar{\sigma}$ is an accepting run for σ iff $\sigma \models \varphi$

States by Example



Closure

Definition: Closure

The closure of LTL-formula φ is the set $cl(\varphi)$ consisting of all sub-formulas ψ of φ and their negation $\neg\psi$ where ψ and $\neg\neg\psi$ are identified.

Example

For $\varphi = a \,\mathsf{U}\, (\neg a \wedge b)$ we have

$$cl(\varphi) \;=\; \{\, a, b, \neg a, \neg b, \neg a \wedge b, \neg(\neg a \wedge b), \varphi, \neg\varphi \,\}.$$

We cannot take B_i as arbitrary subset of $cl(\varphi)$.
They must be elementary.

Elementary Sets

Definition: Elementary sets

$B \subseteq cl(\varphi)$ is elementary if all following conditions hold:

1. B is maximally consistent, i.e., for all $\varphi_1 \wedge \varphi_2, \psi \in cl(\varphi)$:
   - $\varphi_1 \wedge \varphi_2 \in B \;\Leftrightarrow\; \varphi_1 \in B$ and $\varphi_2 \in B$
   - $\psi \notin B \;\Leftrightarrow\; \neg\psi \in B$
   - $\mathrm{true} \in cl(\varphi) \;\Rightarrow\; \mathrm{true} \in B$
2. B is locally consistent, i.e., for all $\varphi_1 \,\mathsf{U}\, \varphi_2 \in cl(\varphi)$:
   - $\varphi_2 \in B \;\Rightarrow\; \varphi_1 \,\mathsf{U}\, \varphi_2 \in B$
   - $\varphi_1 \,\mathsf{U}\, \varphi_2 \in B$ and $\varphi_2 \notin B \;\Rightarrow\; \varphi_1 \in B$

Examples

Elementary or not? LTLMC3.2-49

Let $\varphi = a\,\mathsf{U}(\neg a \wedge b)$.

$B_1 = \{a, b, \neg a \wedge b, \varphi\}$	not elementary propositional inconsistent
$B_2 = \{\neg a, b, \varphi\}$	not elementary, not maximal as $\neg a \wedge b \notin B_2$ $\neg(\neg a \wedge b) \notin B_2$
$B_3 = \{\neg a, b, \neg a \wedge b, \neg\varphi\}$	not elementary not locally consistent for U
$B_4 = \{\neg a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}$	elementary

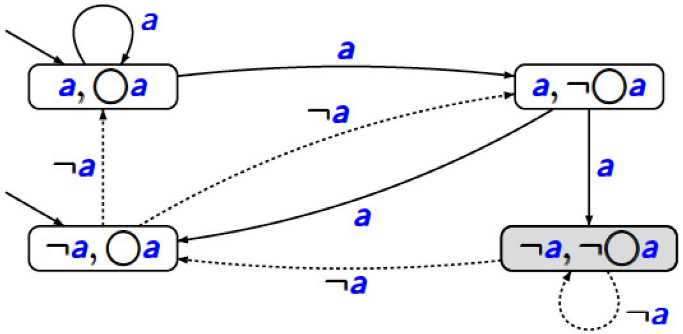
Automaton Construction

Definition: The GNBA for and LTL Formula

For LTL-formula φ, let $\mathfrak{G}_\varphi = (Q, 2^{AP}, \delta, Q_0, \mathfrak{F})$ where

- Q is the set of all elementary sets of formulas $B \subseteq cl(\varphi)$ with
  $Q_0 \;=\; \{\, B \in Q \mid \varphi \in B \,\}$
- If $A \neq B \cap AP$, then $\delta(B, A) \;=\; \varnothing$.
- $\delta(B, B \cap AP)$ is the set $B' \subseteq Q$ satisfying:
  (i) For every $\bigcirc\psi \in cl(\varphi)$: $\bigcirc\psi \in B \;\Leftrightarrow\; \psi \in B'$, and
  (ii) For every $\varphi_1 \,\mathsf{U}\, \varphi_2 \in cl(\varphi)$:
  $$\varphi_1 \,\mathsf{U}\, \varphi_2 \in B \;\Leftrightarrow\; (\varphi_2 \in B \vee (\varphi_1 \in B \wedge \varphi_1 \,\mathsf{U}\, \varphi_2 \in B'))$$
- $\mathfrak{F} \;=\; \{\, \mathfrak{F}_{\varphi_1 \mathsf{U} \varphi_2} \mid \varphi_1 \,\mathsf{U}\, \varphi_2 \in cl(\varphi) \,\}$ where
  $\mathfrak{F}_{\varphi_1 \mathsf{U} \varphi_2} \;=\; \{\, B \in Q \mid \varphi_1 \,\mathsf{U}\, \varphi_2 \notin B \text{ or } \varphi_2 \in B \,\}$

Example (1)



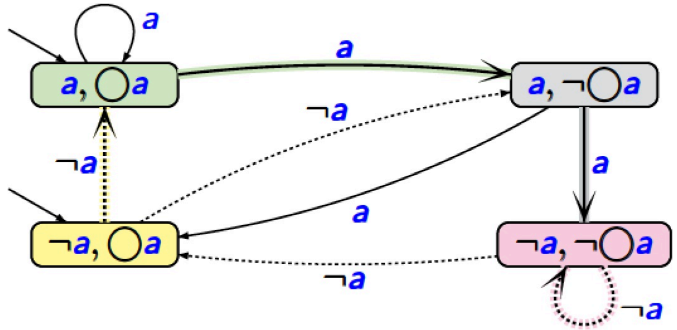
initial states: formula-sets B with $\bigcirc a \in B$

transition relation:

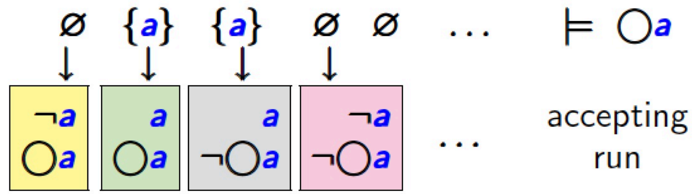
if $\bigcirc a \in B$ then $\delta(B, B \cap \{a\}) = \{B' : a \in B'\}$

if $\bigcirc a \notin B$ then $\delta(B, B \cap \{a\}) = \{B' : a \notin B'\}$

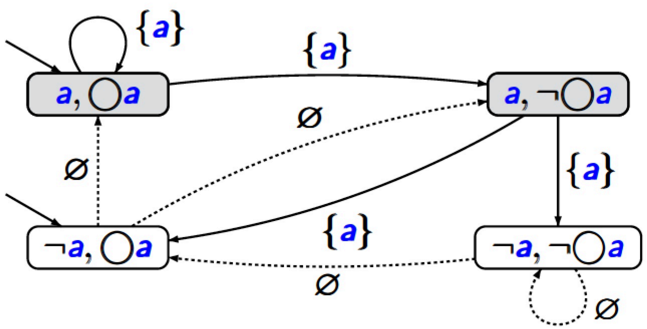
Example (2)



set of acceptance sets: $\mathcal{F} = \varnothing$



Example (3)



for all words $\sigma = A_0\, A_1\, A_2\, A_3 \ldots \in \mathcal{L}_\omega(\mathcal{G})$: $\quad A_1 = \{a\}$

proof: Let $B_0\, B_1\, B_2 \ldots$ be an accepting run for σ.

$\Longrightarrow$ $\bigcirc a \in B_0$ and therefore $a \in B_1$

$\Longrightarrow$ the outgoing edges of B_1 have label $\{a\}$

$\Longrightarrow$ $\{a\} = B_1 \cap AP = A_1$

Main Theorem

[Vardi, Wolper & Sistla 1986]

For any LTL-formula φ (over AP) there exists a GNBA $\mathfrak{G}_\varphi$ over 2^{AP} with:

(a) $Words(\varphi) = \mathcal{L}_\omega(\mathfrak{G}_\varphi)$

(b) $\mathfrak{G}_\varphi$ can be constructed in time and space $O\left(2^{|\varphi|}\right)$

(c) #accepting sets of $\mathfrak{G}_\varphi$ is bounded above by $O(|\varphi|)$.

Corollary

For every LTL-formula φ, $Words(\varphi)$ is ω-regular.

NBA More Expressive Than LTL

There is no LTL formula φ with $Words(\varphi) = E$ for the LT-property:

$$E = \left\{ A_0 A_1 A_2 \ldots \in \left(2^{\{a\}}\right)^{\omega} \mid a \in A_{2i} \text{ for } i \geq 0 \right\}$$

But there exists an NBA $\mathfrak{A}$ with $\mathfrak{L}_{\omega}(\mathfrak{A}) = E$.

Proof.

Omitted. □

Overview

Lower Bound

There exists a family of LTL formulas φ_n with $|\varphi_n| = O(poly(n))$ such that every NBA $\mathfrak{A}_{\varphi_n}$ for φ_n has at least 2^n states.

Proof.

On the black board. □

Complexity

The time and space complexity of automata-based LTL model checking is

$$O(\,|TS| \cdot 2^{|\varphi|}\,)$$

Proof.

1. the closure of LTL formula φ has size in $O(|\varphi|)$
2. the number of elementary sets is in $O(2^{|\varphi|})$
3. the number of states in the GNBA $\mathfrak{G}_{\varphi}$ is in $O(2^{|\varphi|})$
4. the number of acceptance sets in GNBA $\mathfrak{G}_{\varphi}$ is in $O(|\varphi|)$
5. the size of the NBA $\mathfrak{A}_{\varphi}$ is in $O(|\varphi| \cdot 2^{|\varphi|})$
6. the size of $TS \otimes \mathfrak{A}_{\varphi}$ is in $O(\,|TS| \cdot 2^{|\varphi|}\,)$
7. determining $TS \otimes \mathfrak{A}_{\varphi} \models \Diamond \Box \neg F$ is in $O(|TS \otimes \mathfrak{A}_{\varphi}|)$. □

Overview

Summary

- LTL model checking exploits a GNBA $\mathfrak{A}_{\neg\varphi}$ for the negation of φ
- States of the GNBA are subsets of certain sub-formulas of φ
- Taking these subsets give rises to an exponential blow-up. This cannot be avoided
- For each until-sub-formula of φ, the GNBA has one acceptance set
- Each LTL-formula describes an ω-regular LT property
- LTL is strictly less expressive than ω-regular expressions
- LTL model checking by automata is linear in the size of the transition system and exponential in the size of φ

Next Lecture

Friday November 15, 14:30