# Model Checking

Lecture #2: Transition Systems

[Baier & Katoen, Chapter 2]

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## Overview







## Overview







## Transition systems

- Model to describe the behaviour of systems
- Digraphs where nodes represent states, and edges model transitions
- State:
  - the current colour of a traffic light
  - the current values of all program variables + the program counter
  - the current value of the registers plus the values of the input bits
- Transition: ("state change")
  - a switch from one colour to another
  - the execution of a program statement
  - the change of the registers and output bits for a new input

## Transition system

**Definition: Transition system**

A transition system $TS$ is a tuple $(S, Act, \rightarrow, I, AP, L)$ where

- $S$ is a set of states
- $Act$ is a set of actions
- $\longrightarrow \subseteq S \times Act \times S$ is a transition relation
- $I \subseteq S$ is a set of initial states
- $AP$ is a set of atomic propositions
- $L : S \rightarrow 2^{AP}$ is a labelling function

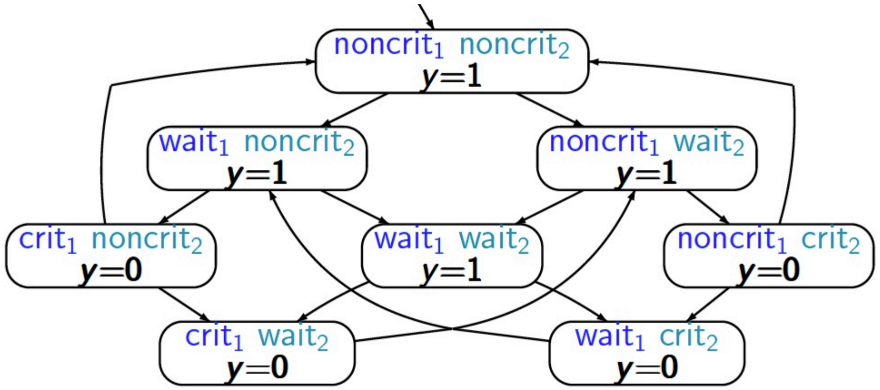
$S$ and $Act$ are either finite or countably infinite

Notation: $s \xrightarrow{\alpha} s'$ as abbreviation of $(s, \alpha, s') \in \longrightarrow$





## A Mutual Exclusion Algorithm




For simplicity, actions are omitted in this example.





## Direct Successors and Predecessors

$$Post(s, \alpha) = \left\{ s' \in S \mid s \xrightarrow{\alpha} s' \right\}, \quad Post(s) = \bigcup_{\alpha \in Act} Post(s, \alpha)$$

$$Pre(s, \alpha) = \left\{ s' \in S \mid s' \xrightarrow{\alpha} s \right\}, \quad Pre(s) = \bigcup_{\alpha \in Act} Pre(s, \alpha).$$

$$Post(C, \alpha) = \bigcup_{s \in C} Post(s, \alpha), \quad Post(C) = \bigcup_{s \in C} Post(s) \text{ for } C \subseteq S.$$

$$Pre(C, \alpha) = \bigcup_{s \in C} Pre(s, \alpha), \quad Pre(C) = \bigcup_{s \in C} Pre(s) \text{ for } C \subseteq S.$$

State $s$ is called terminal if and only if $Post(s) = \varnothing$





## Transition System "Behaviour"

The possible behaviours of a TS result from:

select non-deterministically an initial state $s \in I$
`while` $s$ is not a terminal
do
  select non-deterministically a transition $s \xrightarrow{\alpha} s'$
  perform the action $\alpha$ and set $s = s'$
od

## Executions

### Definition: Executions

- An execution fragment $\rho \in (S \times Act)^{\omega}$ of transition systems $TS$ is an infinite, alternating sequence of states and actions:

$$\rho \;=\; s_0\, \alpha_1\, s_1\, \alpha_2\, s_2\, \alpha_3 \ldots \text{ such that } s_i \xrightarrow{\alpha_{i+1}} s_{i+1} \text{ for all } 0 \leq i.$$

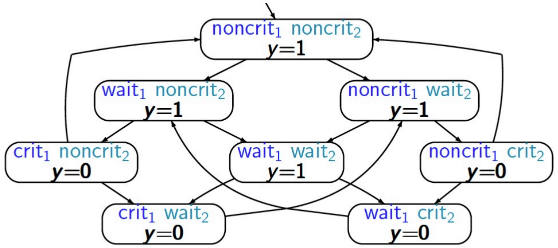
  We also denote $\rho$ by: $s_0 \xrightarrow{\alpha_1} s_1 \xrightarrow{\alpha_2} s_2 \xrightarrow{\alpha_3} \ldots$.
- $\rho$ is maximal if $\rho$ is infinite or finite and ending in a terminal state.
- $\rho$ is initial if it starts in an initial state, i.e., $s_0 \in I$.
- An execution is an initial, maximal execution fragment.
- Omitting the actions from an execution yields a path.

A state $s$ is reachable in $TS$ if $s$ occurs in some execution of $TS$.





## Example Executions








## Transition Systems versus Finite Automata

As opposed to finite automata, a transition system:

- has no accept/final states
- is not “accepting” a (regular) language
- may have countably infinite set of states and actions
- may be infinitely branching
- actions are used to “glue” small transition systems

Transition systems are used to model reactive systems, i.e., systems that continuously interact with their environment.





## Overview

## Traces

- Actions are mainly used to model the (possibility of) interaction
  synchronous or asynchronous communication
- Here, focus on the states that are visited during executions
  the states themselves are not "observable", but just their atomic propositions
- Traces are sequences of the form $L(s_0)\, L(s_1)\, L(s_2) \ldots$
  record the (sets of) atomic propositions along an execution
- For transition systems without terminal states[1]:
  traces are infinite words over the alphabet $2^{AP}$, i.e., they are in $\left(2^{AP}\right)^{\omega}$

[1] This is an assumption commonly used throughout this lecture.





## Traces

**Definition: Traces**

Let $TS = (S, Act, \rightarrow, I, AP, L)$ be transition system without terminal states.

- The trace of execution

$$\rho \;=\; s_0 \xrightarrow{\alpha_1} s_1 \xrightarrow{\alpha_2} s_2 \xrightarrow{\alpha_3} \ldots$$

  is the infinite word $trace(\rho) \;=\; L(s_0)\, L(s_1)\, L(s_2) \ldots$ over $\left(2^{AP}\right)^{\omega}$.
  Prefixes of traces are finite traces.
- The traces of a set $\Pi$ of executions (or paths) is defined by:

$$trace(\Pi) \;=\; \{\, trace(\pi) \mid \pi \in \Pi \,\}.$$

- The traces of state $s$ are $Traces(s) = trace(Paths(s))$.
- The traces of transition system $TS$: $Traces(TS) = \bigcup_{s \in I} Traces(s)$.





## Example Traces

Consider the mutex transition system. Let $AP = \{\, crit_1, crit_2 \,\}$.
The trace of the path:

$$\begin{aligned}\pi \;=\; & \langle n_1, n_2, y=1\rangle \rightarrow \langle w_1, n_2, y=1\rangle \rightarrow \langle c_1, n_2, y=0\rangle \rightarrow \\ & \langle n_1, n_2, y=1\rangle \rightarrow \langle n_1, w_2, y=1\rangle \rightarrow \langle n_1, c_2, y=0\rangle \rightarrow \ldots\end{aligned}$$

is:

$$trace(\pi) \;=\; \varnothing\, \varnothing\, \{\, crit_1 \,\}\, \varnothing\, \varnothing\, \{\, crit_2 \,\}\, \varnothing\, \varnothing\, \{\, crit_1 \,\}\, \varnothing\, \varnothing\, \{\, crit_2 \,\} \ldots$$

The finite trace of the finite path fragment:

$$\begin{aligned}\widehat{\pi} \;=\; & \langle n_1, n_2, y=1\rangle \rightarrow \langle w_1, n_2, y=1\rangle \rightarrow \langle w_1, w_2, y=1\rangle \rightarrow \\ & \langle w_1, c_2, y=0\rangle \rightarrow \langle w_1, n_2, y=1\rangle \rightarrow \langle c_1, n_2, y=0\rangle\end{aligned}$$

is:

$$trace(\widehat{\pi}) \;=\; \varnothing\, \varnothing\, \varnothing\, \{\, crit_2 \,\}\, \varnothing\, \{\, crit_1 \,\}$$





## Overview

## Transition Systems are Universal

Transition systems can model the behaviour of:

- Sequential programs
- Multi-threaded programs
- Communicating sequential programs
- Sequential hardware circuits
- Petri nets
- State Charts
- ... and many more





## Program Graphs

Let *Var* be a collection of typed variables.

A program graph is a finite, rooted directed graph with:

- a finite set *Loc* of vertices, called locations
- a set of initial vertices (roots), called initial locations
- a set of labelled edges that connect locations with:
  - a Boolean condition over variables, e.g., $x < 10$
  - an action $\alpha \in Act$, e.g., $x := x{+}1$

  Intuition: `if` $x < 10$ `then` $x := x{+}1$
- an effect function describing the effect of an action on a variable valuation $Var \rightarrow \mathbb{R}$, e.g.,

$$Effect(x := x{+}1, \underbrace{[x \mapsto 5, y \mapsto 0]}_{\eta(x)=6, \eta(y)=0}) = \underbrace{[x \mapsto 6, y \mapsto 0]}_{\eta'(x)=6, \eta'(y)=0}$$

- an initial Boolean condition, e.g., $x{=}10 \wedge y < 3$





## Program Graphs

**Definition: Program graph**

A program graph *PG* over set *Var* of typed variables is a tuple

$$(Loc, Act, Effect, \longrightarrow, Loc_0, g_0) \quad \text{where}$$

- *Loc* is a set of locations with initial locations $Loc_0 \subseteq Loc$
- *Act* is a set of actions
- $Effect : Act \times Eval(Var) \rightarrow Eval(Var)$ is the effect function
- $\longrightarrow \subseteq Loc \times \underbrace{Cond(Var)}_{\text{Boolean conditions over } Var} \times Act \times Loc$ is the edge relation
- $g_0 \in Cond(Var)$ is the initial condition.

Notation: $\ell \xrightarrow{g:\alpha} \ell'$ denotes $(\ell, g, \alpha, \ell') \in \longrightarrow$





## Example

Pseudo-code thread *i*:

$$
\begin{array}{ll}
 & \text{int k := 0;} \\
 & \text{b := [true, true];} \\
\ell_0 \left\{ \begin{array}{l} \text{while (true) do} \\ \quad b[i] := false; \end{array} \right. & \\
\ell_1 \left\{ \begin{array}{l} \quad \text{while (k != i) do} \end{array} \right. & \\
\ell_2 \left\{ \begin{array}{l} \qquad \text{while (not b[1-i]) do} \\ \qquad\quad \text{k := i;} \\ \qquad \text{end} \end{array} \right. & \\
 & \quad \text{end} \\
\ell_3 \left\{ \begin{array}{l} \quad \text{critical\_section;} \end{array} \right. & \\
\ell_4 \left\{ \begin{array}{l} \quad \text{b[i] := true;} \end{array} \right. & \\
 & \text{end}
\end{array}
$$

initially b = [true, true]
and k = 0

## Program Graphs $\mapsto$ Transition Systems

- Basic strategy: unfolding
  - state = location (current control) $\ell$ + valuation $\eta$
  - initial state = initial location satisfying the initial condition
- Propositions and labelling
  - propositions: "at $\ell$" and "$x \in D$" for $D \subseteq dom(x)$
  - $\langle \ell, \eta \rangle$ is labelled with "at $\ell$" and all conditions that hold in $\eta$
- If $\ell \xrightarrow{g:\alpha} \ell'$ and $g$ holds for the current valuation $\eta$, then

$$\underbrace{\langle \ell, \eta \rangle}_{\text{current state}} \xrightarrow{\alpha} \underbrace{\langle \ell', Effect(\alpha, \eta) \rangle}_{\text{next state}}$$





## Program Graphs $\mapsto$ Transition Systems

**Definition: Transition system of a program graph**

The transition system $TS(PG)$ of program graph

$$PG = (Loc, Act, Effect, \longrightarrow, Loc_0, g_0)$$

over set $Var$ of variables is the tuple $(S, Act, \longrightarrow, I, AP, L)$ where

- $S = Loc \times Eval(Var)$
- $\longrightarrow \subseteq S \times Act \times S$ is defined by the rule:

$$\frac{\ell \xrightarrow{g:\alpha} \ell' \quad \wedge \quad \eta \models g}{\langle \ell, \eta \rangle \xrightarrow{\alpha} \langle \ell', Effect(\alpha, \eta) \rangle}$$

- $I = \{\langle \ell, \eta \rangle \mid \ell \in Loc_0, \eta \models g_0\}$
- $L(\langle \ell, \eta \rangle) = \{\ell\} \cup \{g \in Cond(Var) \mid \eta \models g\}$.





## Example

Pseudo-code thread $i$:

```
    int k := 0;
    b := [true, true];
ℓ0 { while (true) do
       b[i] := false;
ℓ1 {   while (k != i) do
ℓ2 {     while (not b[1-i]) do
           k := i;
         end
       end
ℓ3 {   critical_section;
ℓ4 {   b[i] := true;
    end
```

initially b = [true, true]
and k = 0





## Overview

# Modelling Multi-Threading

- Transition systems
  - suited for modelling sequential data-dependent systems
  - and for modelling sequential hardware circuits
- How about concurrent systems?
  - multi-threading
  - distributed algorithms and communication protocols
- Can we model:
  - multi-threaded programs with shared variables?





# Interleaving

- Abstract from decomposition of system in components
- Actions of independent components are merged or "interleaved"
  - a single processor is available
  - on which the actions of the processes are interlocked
- No assumptions are made on the order of processes
  - possible orders for non-terminating independent processes $P$ and $Q$:

$$\begin{array}{cccccccccc} P & Q & P & Q & P & Q & Q & Q & P & \ldots \\ P & P & Q & P & P & Q & P & P & Q & \ldots \\ P & Q & P & P & Q & P & P & P & Q & \ldots \end{array}$$

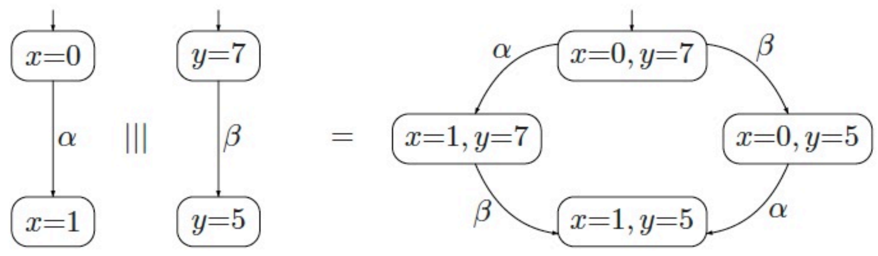
  - assumption: there is a scheduler with an a priori unknown strategy





# Justification

$$\underbrace{x := x+1}_{=\alpha} \;|||\; \underbrace{y := y-2}_{=\beta}$$




the effect of concurrently executed, independent actions $\alpha$ and $\beta$ is equal regardless of their execution order





# Interleaving of transition systems

**Definition: Interleaving of transition systems**

Let $TS_i = (S_i, Act_i, \rightarrow_i, I_i, AP_i, L_i)$ $i$=1, 2, be two transition systems.

Transition system

$$TS_1 \,|||\, TS_2 = (S_1 \times S_2, Act_1 \uplus Act_2, \rightarrow, I_1 \times I_2, AP_1 \uplus AP_2, L)$$

where $L(\langle s_1, s_2 \rangle) = L_1(s_1) \cup L_2(s_2)$ and the transition relation $\rightarrow$ is defined by the inference rules:

$$\frac{s_1 \xrightarrow{\alpha}_1 s_1'}{\langle s_1, s_2 \rangle \xrightarrow{\alpha} \langle s_1', s_2 \rangle} \quad \text{and} \quad \frac{s_2 \xrightarrow{\alpha}_2 s_2'}{\langle s_1, s_2 \rangle \xrightarrow{\alpha} \langle s_1, s_2' \rangle}$$

## Interleaving of Program Graphs

For program graphs $PG_1$ (on $Var_1$) and $PG_2$ (on $Var_2$) without shared variables, i.e., $Var_1 \cap Var_2 = \varnothing$,

$$TS(PG_1) \mid\mid\mid TS(PG_2)$$

faithfully describes the concurrent behaviour of $PG_1$ and $PG_2$

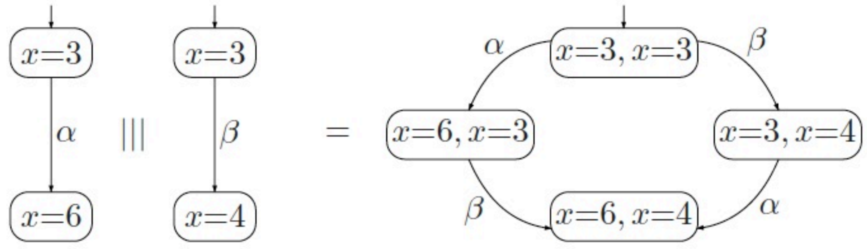
what if they have variables in common?





## Shared Variables

$$\underbrace{x := 2 \cdot x}_{\text{action } \alpha} \mid\mid\mid \underbrace{x := x+1}_{\text{action } \beta} \quad \text{with initially } x = 3$$




$\langle x{=}6, x{=}4 \rangle$ is an inconsistent state!

$\Rightarrow$ this is not a faithful mode of the concurrent execution of $\alpha$ and $\beta$.





## Modelling Multi-threaded Program Graphs

- If $PG_1$ and $PG_2$ share no variables:

$$TS(PG_1) \mid\mid\mid TS(PG_2)$$

  interleaving of transition systems

- If $PG_1$ and $PG_2$ share some variables:

$$TS(PG_1 \mid\mid\mid PG_2)$$

  interleaving of program graphs (defined next)

- In general: $TS(PG_1) \mid\mid\mid TS(PG_2) \neq TS(PG_1 \mid\mid\mid PG_2)$





## Interleaving of Program Graphs

**Definition: Interleaving of program graphs**

Let $PG_i = \left(Loc_i, Act_i, Effect_i, \longrightarrow_i, Loc_{0,i}, g_{0,i}\right)$ over variables $Var_i$, for $i{=}1, 2$.
Program graph $PG_1 \mid\mid\mid PG_2$ over $Var_1 \cup Var_2$ is defined by:

$$\left(Loc_1 \times Loc_2, Act_1 \uplus Act_2, Effect, \longrightarrow, Loc_{0,1} \times Loc_{0,2}, g_{0,1} \wedge g_{0,2}\right)$$

where $\longrightarrow$ is defined by the inference rules:

$$\frac{\ell_1 \xrightarrow{g:\alpha}_1 \ell_1'}{\langle \ell_1, \ell_2 \rangle \xrightarrow{g:\alpha} \langle \ell_1', \ell_2 \rangle} \quad \text{and} \quad \frac{\ell_2 \xrightarrow{g:\alpha}_2 \ell_2'}{\langle \ell_1, \ell_2 \rangle \xrightarrow{g:\alpha} \langle \ell_1, \ell_2' \rangle}$$

and $Effect(\alpha, \eta) \;=\; Effect_i(\alpha, \eta)$ if $\alpha \in Act_i$.

# A Toy Example

$\underbrace{x := 2{\cdot}x}_{\text{action } \alpha} \;|||\; \underbrace{x := x+1}_{\text{action } \beta}$ with initially $x = 3$





# An Example with Two Threads

Pseudo-code thread $i$=0:

```
    int k := 0;
    b := [true, true];
ℓ0 { while (true) do
       b[i] := false;
ℓ1 {   while (k != i) do
ℓ2 {     while (not b[1-i]) do
           k := i;
         end
       end
ℓ3 {   critical_section;
ℓ4 {   b[i] := true;
    end
```

|||

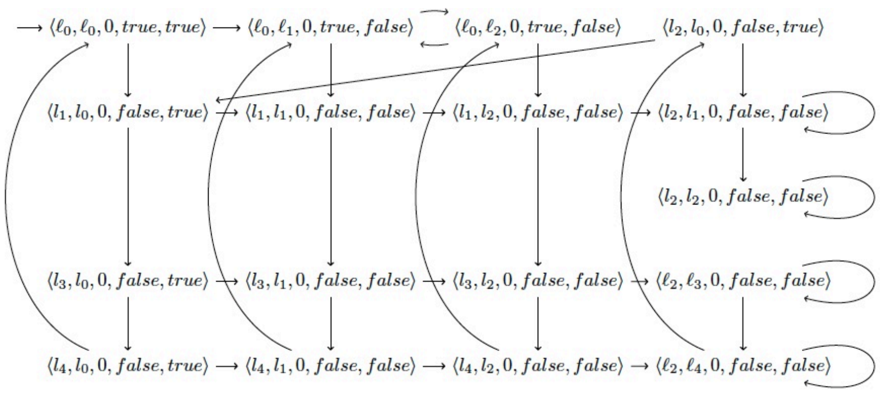
Pseudo-code thread $i$=1:

```
    int k := 0;
    b := [true, true];
ℓ0 { while (true) do
       b[i] := false;
ℓ1 {   while (k != i) do
ℓ2 {     while (not b[1-i]) do
           k := i;
         end
       end
ℓ3 {   critical_section;
ℓ4 {   b[i] := true;
    end
```





# The Transition System




We treated the states $\langle \ell_i, \ell_j, 0, b[0], b[1] \rangle$ and $\langle \ell_j, \ell_i, 1, b[1], b[0] \rangle$ as equivalent so as to reduce the size of the transition system.





# On Atomicity

$$\underbrace{x := x+1;\, y := 2x+1;\, z := y \textbf{ div } x}_{\text{non-atomic}} \;|||\; x := 0$$

Possible execution fragment:

$$\langle x = 11 \rangle \xrightarrow{x:=x+1} \langle x = 12 \rangle \xrightarrow{y:=2x+1} \langle x = 12 \rangle \xrightarrow{x:=0} \langle x = 0 \rangle \xrightarrow{z:=y/x} \dagger \ldots$$

$$\underbrace{\langle x := x+1;\, y := 2x+1;\, z := y \textbf{ div } x \rangle}_{\text{atomic}} \;|||\; x := 0$$

Either the left process or the right process is completed first:

$$\langle x = 11 \rangle \xrightarrow{x:=x+1} \langle x = 12 \rangle \xrightarrow{y:=2x+1} \langle x = 12 \rangle \xrightarrow{z:=y/x} \langle x = 12 \rangle \xrightarrow{x:=0} \langle x = 0 \rangle$$

# Peterson's Algorithm

$P_1$ **loop forever**

$\vdots$ (* non-critical actions *)

$\langle b_1 := \text{true};\ x := 2 \rangle$; (* request *)

**wait until** $(x = 1 \lor \neg b_2)$

**do** critical section **od**

$b_1 :=$ false (* release *)

$\vdots$ (* non-critical actions *)

**end loop**

$b_i$ is true if and only if process $P_i$ is waiting or in critical section

if both processes want to enter their critical section, $x$ decides who gets access





# Accessing a Bank Account

Person Left behaves as follows:

**while** true {
......
*nc* : $\langle b_1, x = \text{true}, 2; \rangle$
*wt* : **wait until**$(x == 1 \,||\, \neg\, b_2)\,\{$
*cs* : ...@account...}
$b_1$ = false;
......
}

Person Right behaves as follows:

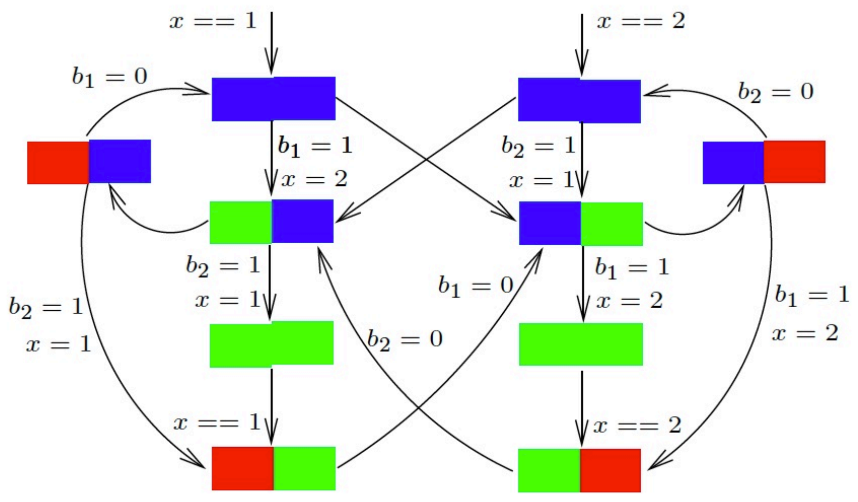
**while** true {
......
*nc* : $\langle b_2, x = \text{true}, 1; \rangle$
*wt* : **wait until**$(x == 2 \,||\, \neg\, b_1)\,\{$
*cs* : ...@account...}
$b_2$ = false;
......
}

Can we guarantee that only one person at a time has access to the bank account?





# The Transition System




Manual inspection reveals that mutual exclusion is guaranteed





# A Non-atomic Version

Person Left behaves as follows:

**while** true {
......
*nc* : $x = 2$;
*rq* : $b_1$ = true;
*wt* : **wait until**$(x == 1 \,||\, \neg\, b_2)\,\{$
*cs* : ...@account...}
$b_1$ = false;
......
}

Person Right behaves as follows:

**while** true {
......
*nc* : $x = 1$;
*rq* : $b_2$ = true;
*wt* : **wait until**$(x == 2 \,||\, \neg\, b_1)\,\{$
*cs* : ...@account...}
$b_2$ = false;
......
}

## On Atomicity Again

Assume that the location inbetween the assignments $x := \ldots$ and $b_i :=$ true in program graph $PG_i$ is called $rq_i$. Possible state sequence:

$$\langle nc_1, \quad nc_2, \quad x = 1, \quad b_1 = \text{false}, \quad b_2 = \text{false}\rangle$$
$$\langle nc_1, \quad rq_2, \quad x = 1, \quad b_1 = \text{false}, \quad b_2 = \text{false}\rangle$$
$$\langle rq_1, \quad rq_2, \quad x = 2, \quad b_1 = \text{false}, \quad b_2 = \text{false}\rangle$$
$$\langle wt_1, \quad rq_2, \quad x = 2, \quad b_1 = \text{true}, \quad b_2 = \text{false}\rangle$$
$$\langle cs_1, \quad rq_2, \quad x = 2, \quad b_1 = \text{true}, \quad b_2 = \text{false}\rangle$$
$$\langle cs_1, \quad wt_2, \quad x = 2, \quad b_1 = \text{true}, \quad b_2 = \text{true}\rangle$$
$$\langle cs_1, \quad cs_2, \quad x = 2, \quad b_1 = \text{true}, \quad b_2 = \text{true}\rangle$$

This is a counterexample to the mutual exclusion property.





## Overview







## This Can be Modelled by Transition Systems Too

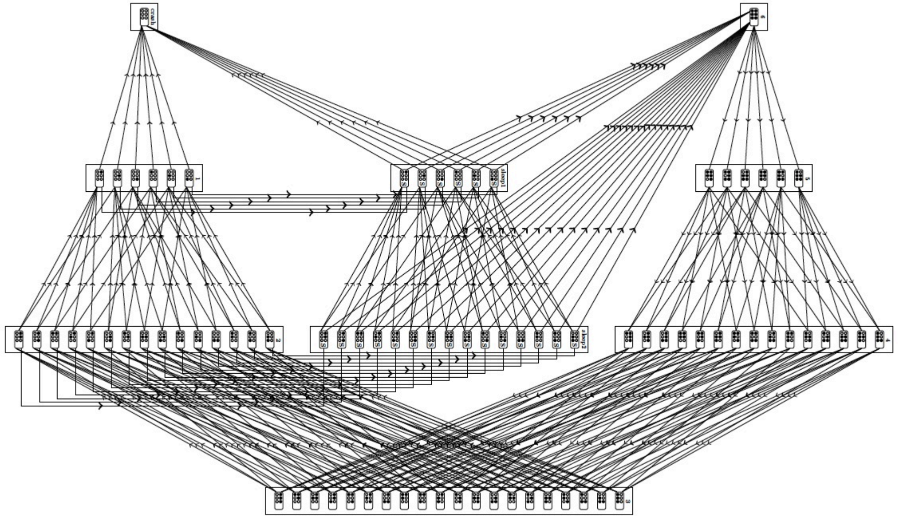
- Programs manipulating ddynamic data structures
- Multi-threaded programs communicating via handshaking
- Multi-threaded programs communicating via unbounded buffers
- Multi-threaded programs using weak memory models
- Multi-threaded programs with fences (or: memory barriers)
- And many others





## Overview

## State Spaces Can Be Gigantic



A model of the Hubble telescope





## Sequential Programs

- The # states of a program graph is worst case:

$$\left|\#\text{program locations}\right| \cdot \prod_{\text{variable } x} \left| dom(x) \right|$$

$\Rightarrow$ # states grows exponentially in the # program variables
- $N$ variables with $k$ possible values each yields $k^N$ states

- A program with 10 locations, 3 bools, 5 integers (in range $0 \ldots 9$):

$$10 \cdot 2^3 \cdot 10^5 \; = \; 800,000 \text{ states}$$

- Adding a single 50-positions bit-array yields $800,000 \cdot 2^{50}$ states





## Multi-Threaded Programs

- The # states of $P_1 \mid\mid\mid \ldots \mid\mid\mid P_n$ is maximally:

$$\#\text{states of } P_1 \times \ldots \times \#\text{states of } P_n$$

$\Rightarrow$ # states grows exponentially in # components

- The composition of $N$ components of size $k$ each yields $k^N$ states





## State Explosion Problem

The exponential growth of the state space in terms of the number of variables (as for program graphs) and number of threads (as for multi-threaded systems) gives rise to the state explosion problem.

In their basic form, model checking consists of enumerating and analysing the set of reachable states. Unfortunately, the number of states of even a relatively small system is often far greater than can be handled in a realistic computer.

# Next Lecture

Thursday October 10, 10:30