Model Checking

Lecture #18: Reduced Ordered Binary Decision Diagrams

[Baier & Katoen, Chapter 6.7]

Joost-Pieter Katoen

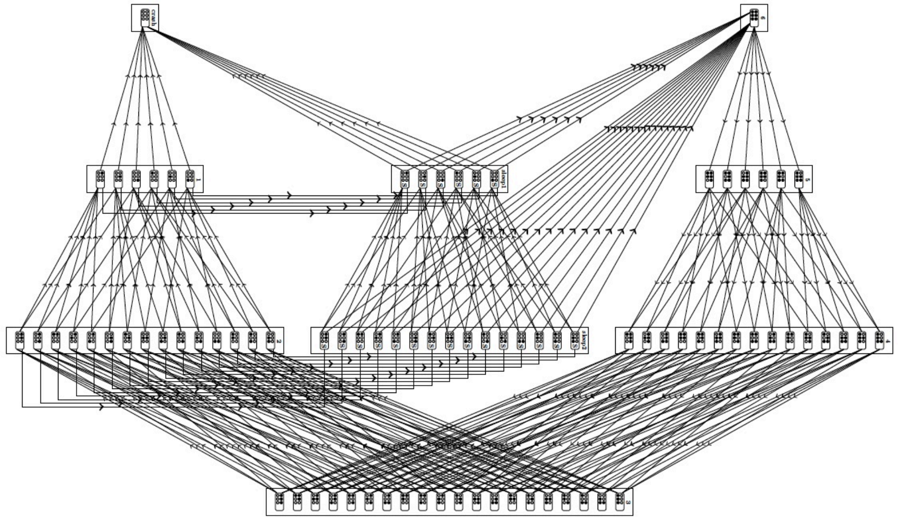
Software Modeling and Verification Group

Model Checking Course, RWTH Aachen, WiSe 2019/2020

Overview

Overview

State Spaces Can Be Gigantic



A model of the Hubble telescope

Treating Gigantic Models?

- Use compact data structures
- Make models smaller prior to (or: during) model checking
- Try to make them even smaller
- If possible, try to obtain the smallest possible model
- While preserving the properties of interest
- Do this all algorithmically and possibly fast

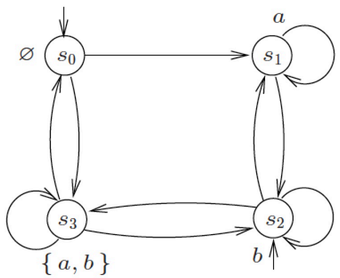
Symbolic CTL Model Checking

- Explicit representation of transition system: state explosion problem
- Idea: reformulate model-checking in a symbolic way
- Concept: represent sets of states and transitions symbolically
- Approach: binary encoding of states + switching functions for sets
- Compactly represent switching functions by binary decision diagrams
- Alternative: conjunctive normal form (used in SAT-based model checking)

Basic Approach

- let $TS = (S, \rightarrow, I, AP, L)$ be a "large" finite transition system
  - the set of actions is irrelevant here and is omitted, i.e., $\rightarrow \subseteq S \times S$
- For $n \geq \lceil \log |S| \rceil$, let injective function $enc : S \rightarrow \{0,1\}^n$
  - the encoding of the states by bit vectors of length n
  - elements in $\{0,1\}^n \setminus enc(S)$ encode unreachable pseudo states
- Identify the states $s \in S = enc^{-1}(\{0,1\}^n)$ with $enc(s) \in \{0,1\}^n$
- And $T \subseteq S$ by its characteristic function $\chi_T : \{0,1\}^n \rightarrow \{0,1\}$
  $\chi_T(enc(s)) = 1$ if and only if $s \in T$
- And $\rightarrow \subseteq S \times S$ by the Boolean function $\Delta : \{0,1\}^{2n} \rightarrow \{0,1\}$
  such that $\Delta\left(enc(s), enc(s')\right) = 1$ if and only if $s \rightarrow s'$

Symbolic Representation of Transition System



Function: $\Delta(\underbrace{x_1, x_2}_{s}, \underbrace{x_1', x_2'}_{s'}) = 1$ if and only if $s \rightarrow s'$

$$\begin{aligned}
\Delta(x_1, x_2, x_1', x_2') = \quad & (\neg x_1 \wedge \neg x_2 \wedge \neg x_1' \wedge x_2') \\
\vee \quad & (\neg x_1 \wedge \neg x_2 \wedge x_1' \wedge x_2') \\
\vee \quad & (\neg x_1 \wedge x_2 \wedge x_1' \wedge \neg x_2') \\
\vee \quad & \ldots \\
\vee \quad & (x_1 \wedge x_2 \wedge x_1' \wedge x_2')
\end{aligned}$$

Overview

Switching Functions

- Let $Var = \{z_1, \ldots, z_m\}$ be a finite set of Boolean variables, $m \geq 0$
- An evaluation is a function $\eta : Var \to \{0, 1\}$
  - shorthand $[z_1 = b_1, \ldots, z_m = b_m]$ for $\eta(z_1) = b_1, \ldots, \eta(z_m) = b_m$
- Let $Eval(Var)$ denote the set of all evaluations for $Var = z_1, \ldots, z_m$
- $f : Eval(Var) \to \{0, 1\}$ is a switching function for $Var = \{z_1, \ldots, z_m\}$
- Logical operations and quantification are defined as:
  - $f_1(\cdot) \wedge f_2(\cdot) \;=\; \min\{f_1(\cdot), f_2(\cdot)\}$
  - $f_1(\cdot) \vee f_2(\cdot) \;=\; \max\{f_1(\cdot), f_2(\cdot)\}$
  - $\exists z.\, f(\cdot) \;=\; f(\cdot)|_{z=0} \vee f(\cdot)|_{z=1}$, and
  - $\forall z.\, f(\cdot) \;=\; f(\cdot)|_{z=0} \wedge f(\cdot)|_{z=1}$

Impossible Polynomial Data Structure

There is no polynomial-size data structure for all switching functions with $|Eval(z_1, \ldots, z_m)| \;=\; 2^m$; i.e., the number of switching functions is 2^{2^m}.

Proof.

- Suppose there is a data structure that can represent K_m switching functions by at most 2^{m-1} bits
- Then $K_m \;\leq\; \sum_{i=0}^{2^{m-1}} 2^i \;=\; 2^{2^{m-1}+1} - 1 \;<\; 2^{2^{m-1}+1}$
- But then there are at least

$$2^{2^m} \;-\; 2^{2^{m-1}+1} \;=\; 2^{2^{m-1}+1} \cdot \left(2^{2^m - 2^{m-1} - 1} \;-\; 1\right) \;=\; 2^{2^{m-1}+1} \cdot \left(2^{2^{m-1}-1} - 1\right)$$

switching functions whose representation needs more than 2^{m-1} bits. □

Representing Switching Functions

- Truth tables
  - very space inefficient: 2^n entries for n variables
  - satisfiability and equivalence check: easy; boolean operations also easy
  - ... but have to consider exponentially many lines (so are hard)
- Disjunctive Normal Form (DNF)
  - satisfiability is easy: find a disjunct with complementary literals
  - negation and conjunction complicated
  - equivalence checking ($f = g$?) is coNP-complete
- Conjunctive Normal Form (CNF)
  - satisfiability problem is NP-complete (Cook's theorem)
  - negation and disjunction complicated

Representing Switching Functions

representation	*compact?*	*sat*	*equiv*	$\wedge$	$\vee$	$\neg$
propositional formula	often	hard	hard	easy	easy	easy
DNF	sometimes	easy	hard	hard	easy	hard
CNF	sometimes	hard	hard	easy	hard	hard
(ordered) truth table	never	hard	hard	hard	hard	hard

Perhaps There is Some Hope

Nevertheless there are data structures which yield compact representations for many switching functions that appear in practical applications

for hardware circuits, ordered binary decision diagrams (OBDDs) are successful

Representing Switching Functions

representation	*compact?*	*sat*	*equiv*	$\wedge$	$\vee$	$\neg$
propositional formula	often	hard	hard	easy	easy	easy
DNF	sometimes	easy	hard	hard	easy	hard
CNF	sometimes	hard	hard	easy	hard	hard
(ordered) truth table	never	hard	hard	hard	hard	hard
reduced ordered binary decision diagram	often	easy	easy*	medium	medium	easy

* provided appropriate implementation techniques are used

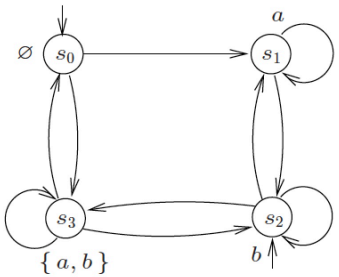
Binary Decision Tree

- The BDT for function f on $Var = \{ z_1, \ldots, z_m \}$ has depth m
  - outgoing edges for node at level i stand for $z_i = 0$ (dashed) and $z_i = 1$ (solid)
- For evaluation $s = [z_1 = b_1, \ldots, z_m = b_m]$, $f(s)$ is the value of the leaf
  - reached by traversing the BDT from the root using branch $z_i = b_i$ at level i
- The sub-tree of node v at level i for variable ordering $z_1 < \ldots < z_m$ represents

$$f_v = f|_{z_1=b_1,\ldots,z_{i-1}=b_{i-1}}$$

  - which is a switching function over $\{ z_i, \ldots, z_m \}$ and
  - where $z_1 = b_1, \ldots, z_{i-1} = b_{i-1}$ is the sequence of decisions made along the path from the root to node v

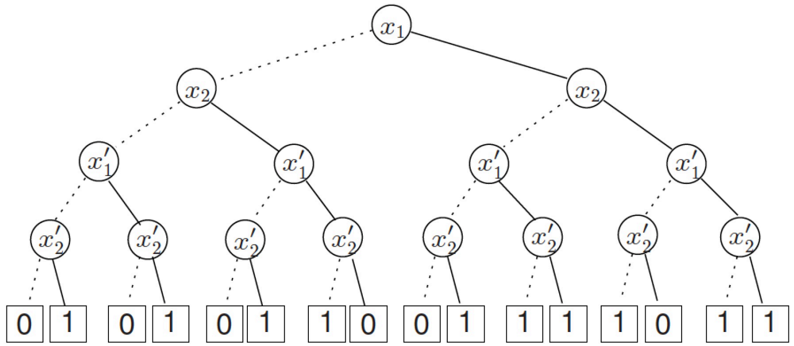
Symbolic Representation of Transition System



Switching function: $\Delta(\underbrace{x_1, x_2}_{s}, \underbrace{x_1', x_2'}_{s'}) = 1$ if and only if $s \rightarrow s'$

$$\begin{array}{rcl} \Delta(x_1, x_2, x_1', x_2') = & & (\neg x_1 \wedge \neg x_2 \wedge \neg x_1' \wedge x_2') \\ & \vee & (\neg x_1 \wedge \neg x_2 \wedge x_1' \wedge x_2') \\ & \vee & (\neg x_1 \wedge x_2 \wedge x_1' \wedge \neg x_2') \\ & \vee & \ldots \\ & \vee & (x_1 \wedge x_2 \wedge x_1' \wedge x_2') \end{array}$$

Transition Relation as a BDT



A BDT representing Δ for our example using ordering $x_1 < x_2 < x_1' < x_2'$

Facts About BDTs

- BDTs are not compact
  - a BDT for switching function f on n variables has 2^n leafs
  - $\Rightarrow$ they are as space inefficient as truth tables!
- $\Rightarrow$ BDTs contain quite some redundancy
  - all leafs with value one (zero) could be collapsed into a single leaf
  - a similar scheme could be adopted for isomorphic subtrees
- The size of a BDT does not change if the variable order changes

Overview

Ordered Binary Decision Diagram

- OBDDs rely on compactifying BDT representations
- Idea: skip redundant fragments of BDT representations
- Collapse sub-trees with all terminals having same value
- Identify nodes with isomorphic sub-trees
- This yields directed acyclic graphs with out-degree two
- Inner nodes are labeled with variables
- Leafs are labeled with function values (zero and one)

Ordered BDDs

Definition: Ordered BDDs

Let $\wp = (z_1, \ldots, z_m)$ be a (total) variable ordering for $Var = \{ z_1, \ldots, z_m \}$ where, i.e., $z_1 <_\wp \ldots <_\wp z_m$.

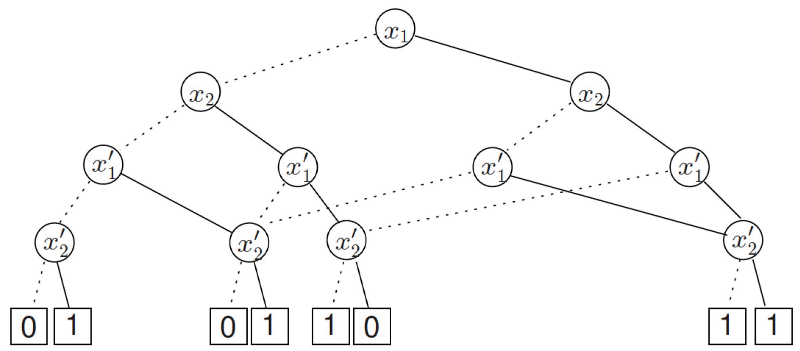
An $\wp$-OBDD is a tuple $\mathfrak{B} = (V, V_I, V_T, succ_0, succ_1, var, val, v_0)$ with:

- a finite set V of nodes, partitioned into V_I (inner) and V_T (terminals)
  - and a distinguished root (node) $v_0 \in V_I$
- successor functions $succ_0$, $succ_1 : V_I \to V$
  - such that each node $v \in V \setminus \{v_0\}$ has at least one predecessor
  - i.e., all nodes of the OBDD $\mathfrak{B}$ are reachable from the root
- labeling functions $var : V_I \to Var$ and $val : V_T \to \{ 0, 1 \}$ satisfying for $v \in V_I$ and $w \in \{ succ_0(v), succ_1(v) \}$:

$$\big(var(v) = z_i \wedge w \in V_I\big) \;\Rightarrow\; var(w) = z_j \text{ with } z_i <_\wp z_j$$

Examples

Example: Transition Relation as OBDD



Example OBDD representing $f_\to$ for our example with $x_1 <_\wp x_2 <_\wp x_1' <_\wp x_2'$

OBDD Semantics

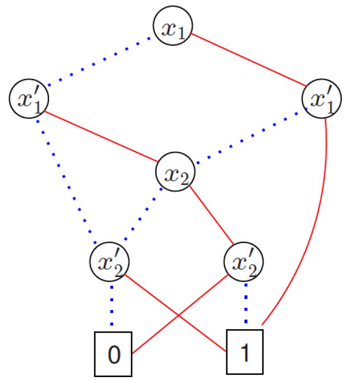
Definition: OBDD semantics

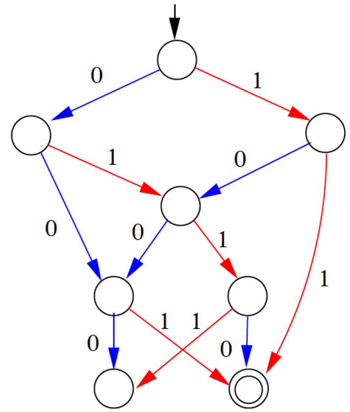
The semantics of $\wp$-OBDD $\mathfrak{B}$ is the switching function $f_{\mathfrak{B}}$ where

$$f_{\mathfrak{B}}([z_1 = b_1, \ldots, z_m = b_m])$$

is the value of the resulting leaf when traversing $\mathfrak{B}$ starting in v_0 and branching according to the evaluation $[z_1 = b_1, \ldots, z_m = b_m]$.

Intermezzo: OBDDs and DFA





each OBDD $\mathfrak{B}$ is a deterministic finite-state automaton $\mathfrak{A}_{\mathfrak{B}}$ with $f_{\mathfrak{B}}^{-1}(1) = L(\mathfrak{A}_{\mathfrak{B}})$.

Bottom-up Characterisation of $f_{\mathfrak{B}}$

Let $\mathfrak{B}$ be a $\wp$-OBDD.

The switching function f_v for node $v \in V$ can be obtained as follows:

- If $v \in V_T$, then f_v is the constant switching function with value $val(v)$
- If $v \in V_I$ with $var(v) = z$, then $f_v = \underbrace{\left(\neg z \wedge f_{succ_0(v)}\right) \vee \left(z \wedge f_{succ_1(v)}\right)}_{\text{Shannon expansion}}$

Furthermore, $f_{\mathfrak{B}} = f_{v_0}$ for the root v_0 of $\mathfrak{B}$.

The function $f_{\mathfrak{B}}$ is the switching function represented by OBDD $\wp - \mathfrak{B}$.

Consistent Co-factors in OBDDs

Definition: consistent co-factors

Let f be a switching function for Var and let $\wp = (z_1, \ldots, z_m)$ be a variable ordering for Var, i.e., $z_1 <_{\wp} \ldots <_{\wp} z_m$.

The switching function g is a $\wp$-consistent cofactor of f if

$$g = f|_{z_1=b_1,\ldots,z_i=b_i} \quad \text{for some } i \in \{0, 1, \ldots, m\}.$$

It holds that:

- for each node v of an $\wp$-OBDD $\mathfrak{B}$, f_v is a $\wp$-consistent cofactor of $f_{\mathfrak{B}}$
- for each $\wp$-consistent cofactor g of $f_{\mathfrak{B}}$, $f_v = g$ for some node $v \in \mathfrak{B}$

Overview

Reduced OBDDs

Definition: reduced OBDD

A $\wp$-OBDD $\mathfrak{B}$ is reduced if for every pair (v, w) of nodes in $\mathfrak{B}$ it holds:

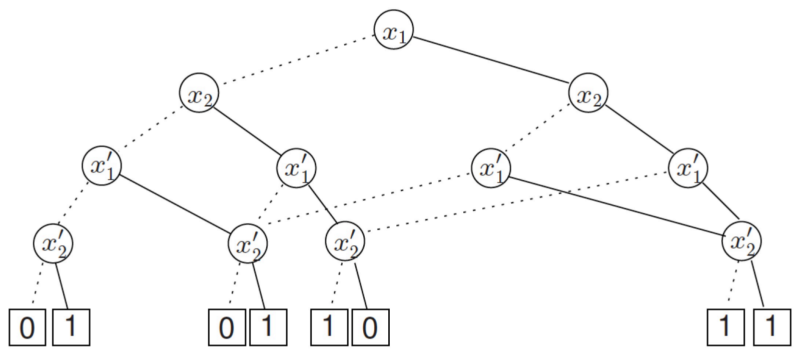
$$v \neq w \quad \text{implies} \quad f_v \neq f_w.$$

A reduced $\wp$-OBDD is abbreviated as $\wp$-ROBDD.

In $\wp$-ROBDDs every $\wp$-consistent cofactor is represented by exactly one node.

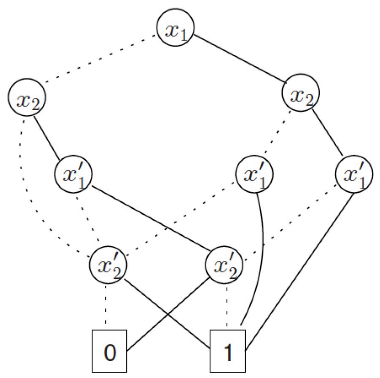
Example Reduced OBDDs

Example: Transition Relation as OBDD

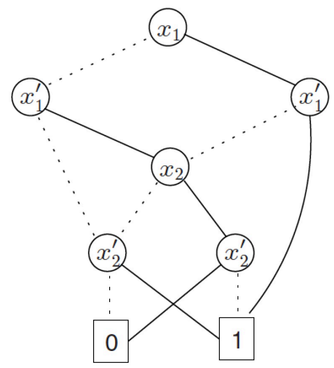


An example OBDD representing $f_{\rightarrow}$ for our example using $x_1 < x_2 < x_1' < x_2'$

Example: Transition Relation as ROBDD



(a) ordering $x_1 < x_2 < x_1' < x_2'$



(b) ordering $x_1 <' x_1' <' x_2 <' x_2'$

Universality and Canonicity Theorem

[Fortune, Hopcroft & Schmidt, 1978]

For finite set *Var* of Boolean variables and variable ordering $\wp$ for *Var*:

(a) For each switching function f on *Var*, $f = f_{\mathfrak{B}}$ for some $\wp$-ROBDD $\mathfrak{B}$.

(b) For any $\wp$-ROBDDs $\mathfrak{B}$ and $\mathfrak{C}$ with $f_{\mathfrak{B}} = f_{\mathfrak{C}}$, $\mathfrak{B}$ and $\mathfrak{C}$ are isomorphic[1].

[1] agree up to renaming of nodes

Proof

The Importance of Canonicity

- Absence of redundant vertices
  - if $f_{\mathfrak{B}}$ does not depend on x_i, ROBDD $\mathfrak{B}$ does not contain an x_i node
- Test for equivalence: $f(x_1, \ldots, x_n) \equiv g(x_1, \ldots, x_n)$?
  - generate ROBDDs $\mathfrak{B}_f$ and $\mathfrak{B}_g$, and check isomorphism
- Test for validity: for all $x_1, \ldots, x_n$, is $f(x_1, \ldots, x_n) = 1$?
  - generate ROBDD $\mathfrak{B}_f$ and check whether it only consists of a 1-leaf
- Test for implication: $f(x_1, \ldots, x_n) \rightarrow g(x_1, \ldots, x_n)$?
  - generate ROBDD $\mathfrak{B}_{f \wedge \neg g}$ and check if it just consists of a 0-leaf
- Test for satisfiability
  - f is satisfiable if and only if $\mathfrak{B}_f$ has a reachable 1-leaf

Minimality of ROBDDs

Let $\mathfrak{B}$ be an $\wp$-OBDD for f.

Then: $\mathfrak{B}$ is reduced iff $size(\mathfrak{B}) \leq size(\mathfrak{C})$ for each $\wp$-OBDD $\mathfrak{C}$ for f.

Proof.

This follows from the fact that:

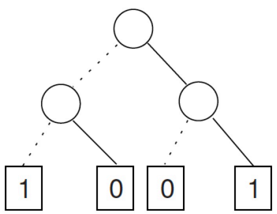
1. Each $\wp$-consistent cofactor of f is represented in any $\wp$-OBDD for f by at least one node, and
2. A $\wp$-OBDD $\mathfrak{B}$ for f is reduced iff there is a 1-to-1 correspondence between the nodes in $\mathfrak{B}$ and the $\wp$-consistent cofactors of $\mathfrak{B}$.

□

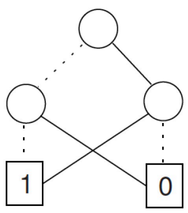
Reducing OBDDs

- Generate an OBDD (or BDT) for a boolean expression, then reduce
  - by means of a recursive descent over the OBDD
- Elimination of duplicate leafs
  - for a duplicate 0-leaf (or 1-leaf), redirect all in-edges to just one of them
- Elimination of "don't care" (non-leaf) vertices
  - if $succ_0(v) = succ_1(v) = w$, delete v and redirect all its in-edges to w
- Elimination of isomorphic subtrees
  - if $v \neq w$ are roots of isomorphic subtrees, remove w
  - and redirect all incoming edges to w to v

How to Reduce an OBDD?

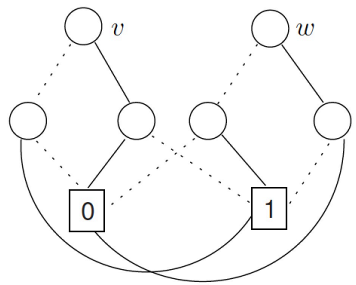


becomes

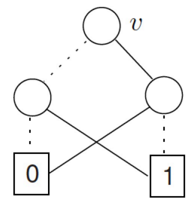


(special case of) isomorphism rule

How to Reduce an OBDD?

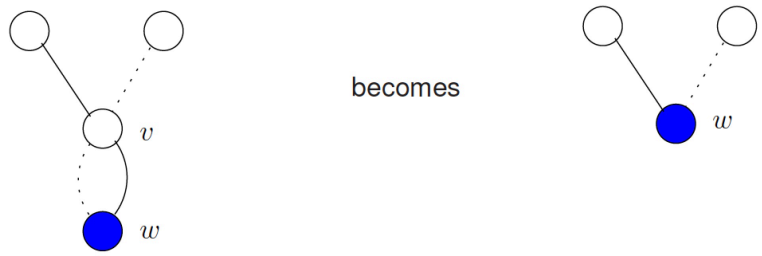


becomes



isomorphism rule

How to Reduce an OBDD?



elimination rule

Example

Soundness

if $\mathfrak{C}$ arises from a $\wp$-OBDD $\mathfrak{B}$ by the elimination or isomorphism rule, then $\mathfrak{C}$ is a $\wp$-OBDD with $f_{\mathfrak{B}} = f_{\mathfrak{C}}$.

Proof.

Elimination rule for v with $var(v) = z$, and $w = succ_0(v) = succ_1(v)$:

$$f_v = \left(\neg z \wedge f_{succ_0(v)}\right) \vee \left(z \wedge f_{succ_1(v)}\right) = (\neg z \wedge f_w) \vee (z \wedge f_w) = f_w$$

Isomorphism rule for v, w with $var(v) = var(w) = z$ yields:

$$\begin{aligned} f_v &= \left(\neg z \wedge f_{succ_0(v)}\right) \vee \left(z \wedge f_{succ_1(v)}\right) \\ &= \left(\neg z \wedge f_{succ_0(w)}\right) \vee \left(z \wedge f_{succ_1(w)}\right) \\ &= f_w \end{aligned}$$

$\square$

as each reduction rule decreases the # nodes, repeatedly applying them terminates

Completeness of Reduction Rules

$\wp$-OBDD $\mathfrak{B}$ is reduced iff no reduction rule is applicable to $\mathfrak{B}$.

Overview

Summary

- ROBDDs are directed acyclic graphs aimed at succinctly representing switching functions
- They provide a compact representation for many switching functions
- In an ROBDD, each co-factor is represented by exactly one node
- ROBDDs are canonical for a given variable ordering
- Any OBDD can be reduced by two reduction rules (applied in any order)
-

Next Lecture

Thursday January 16, 10:30