Probabilistic Programming

Lecture #4: Probabilistic GCL

Joost-Pieter Katoen

Software Modeling and Verification Chair | RWTH AACHEN UNIVERSITY

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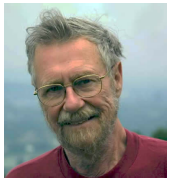
Overview

Dijkstra's guarded command language: Syntax



`skip`	empty statement
`diverge`	divergence
`x := E`	assignment
`prog1 ; prog2`	sequential composition
`if (G) prog1 else prog2`	choice
`prog1 [] prog2`	non-deterministic choice
`while (G) prog`	iteration

Elementary pGCL ingredients

- Program variables $x \in \mathit{Vars}$ whose values are fractional numbers
- Arithmetic expressions E over the program variables
- Boolean expressions G (guarding a choice or loop) over the program variables
- A distribution expression $\mu : \Sigma \to \mathit{Dist}(\mathbb{Q})$
- A probability expression $p : \Sigma \to [0,1] \cap \mathbb{Q}$

Probabilistic GCL: Syntax

Kozen | McIver | Morgan



- `skip` — empty statement
- `diverge` — divergence
- `x := E` — assignment
- `x :r= mu` — **random assignment** $(x :\approx \mu)$
- `prog1 ; prog2` — sequential composition
- `if (G) prog1 else prog2` — choice
- `prog1 [p] prog2` — **probabilistic choice**
- `while (G) prog` — iteration

Conditioning in the form of observe-statements omitted for now.

Let's start simple

```
x := 0 [0.5] x := 1;
y := -1 [0.5] y := 0
```

This program admits four runs and yields the outcome:

$$Pr[x=0, y=0] = Pr[x=0, y=-1] = Pr[x=1, y=0] = Pr[x=1, y=-1] = 1/4$$

A loopy program

For $0 < p < 1$ an arbitrary probability:

```
bool c := true;
int i := 0;
while (c) {
    i++;
    (c := false [p] c := true)
}
```

The loopy program models a geometric distribution with parameter p.

$$Pr[i = N] = (1-p)^{N-1} \cdot p \quad \text{for } N > 0$$

On termination

```
bool c := true;
int i := 0;
while (c) {
    i++;
    (c := false [p] c := true)
}
```

This program does not always terminate. It almost surely terminates.

The good, the bad, and the ugly

Duelling cowboys

```
int cowboyDuel(float a, b) {
  int t := A [0.5] t := B;
  bool c := true;
  while (c) {
    if (t = A) {
      (c := false [a] t := B);
    } else {
      (c := false [b] t := A);
    }
  }
  return t;
}
```


Random assignments

The random assignment $x :\approx \mu$ works as follows:

1. evaluate distribution expression μ in the current program state s
2. sample from the resulting probability distribution $\mu(s)$ yielding value v with probability $\mu(s)(v)$
3. assign the value v to the variable x.

For denoting distribution expressions, we use the bra–ket notation.

$$\frac{1}{2} \cdot [a\rangle + \frac{1}{3} \cdot [b\rangle + \frac{1}{6} \cdot [c\rangle$$

denotes the distribution μ with $\mu(a) = 1/2$, $\mu(b) = 1/3$, and $\mu(c) = 1/6$. The support set of μ equals $\{ a, b, c \}$

Examples on the black board.

Overview

Why formal semantics matters

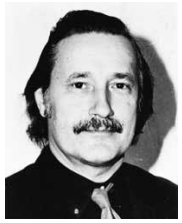
- Unambiguous meaning to all programs
- Basis for proving correctness
  - of programs
  - of program transformations
  - of program equivalence
  - of static analysis
  - of compilers
  - .....

The inventors of semantics

Tony Hoare

Robert W. Floyd

Gordon Plotkin



Christopher Strachey

Dana Scott

Approaches to semantics

- Operational semantics: (developed by Plotkin)
  - The meaning of a program in terms of how it executes on an abstract machine.
  - Useful for modelling the execution behaviour of a program.
- Axiomatic semantics: (developed by Floyd and Hoare)
  - Provides correctness assertions for each program construct.
  - Useful for verifying that a program's computed results are correct with respect to the specification.
- Denotational semantics: (developed by Strachey and Scott)
  - Provides a mapping of language constructs onto mathematical objects.
  - Useful for obtaining an abstract insight into the working of a program.

Today: operational semantics of pGCL in terms of Markov chains.

Later: denotational semantics of pGCL in terms of weakest preconditions.

Structural operational semantics: ingredients

- Variable valuation $s : \mathit{Vars} \to \mathbb{Q}$ maps each program variable onto a value (here: rational numbers)
- Expression valuation[1], let $[\![E]\!]$ denote the valuation of expression E
- Configuration (aka: state) $\langle P, s \rangle$ denotes that
  - program P is about to be executed (aka: program counter)
  - and the current variable valuation equals s.
- Transition rules for the execution of commands: $\langle P, s \rangle \longrightarrow \langle P', s' \rangle$

  transition rules are written as $\dfrac{\text{premise}}{\text{conclusion}}$

  where the premise is omitted if it is vacuously true.

[1] Here, we will not go into the details of this (simple) part.

Recall: Markov chains

A Markov chain (MC) is a triple $(\Sigma, \sigma_I, \mathbf{P})$ with:

- Σ being a countable set of states
- $\sigma_I \in \Sigma$ the initial state, and
- $\mathbf{P} : \Sigma \to \mathit{Dist}(\Sigma)$ the transition probability function

where $\mathit{Dist}(\Sigma)$ is a discrete probability measure on Σ.

Operational semantics

Aim: Model the behaviour of a pGCL program P by the MC $[\![P]\!]$.

Approach:

- Take states of the form
  - $\langle Q, s \rangle$ with program Q or $\downarrow$, and variable valuation $s : \mathit{Vars} \to \mathbb{Q}$
  - $\langle \mathit{sink} \rangle$ models program termination (successful or violated observation)
- Take initial state $\sigma_I = \langle P, s \rangle$ where s fulfils the initial conditions
- Transition relation $\to$ is the smallest relation satisfying the SOS rules on the next slides
- Where transition probabilities equal to one are omitted

Transition rules (1)

$$\langle \mathtt{skip}, s \rangle \to \langle \downarrow, s \rangle \qquad \langle \mathtt{diverge}, s \rangle \to \langle \mathtt{diverge}, s \rangle$$

$$\langle \downarrow, s \rangle \to \langle \mathit{sink} \rangle \qquad \langle \mathit{sink} \rangle \to \langle \mathit{sink} \rangle$$

$$\langle x := E, s \rangle \to \langle \downarrow, s[x := s([\![E]\!])] \rangle$$

$$\frac{\mu(s)(v) = a > 0}{\langle x :\approx \mu, s \rangle \xrightarrow{a} \langle \downarrow, s[x := v] \rangle}$$

$$\langle P [\, p\,] \, Q, s \rangle \to \mu \text{ with } \mu(\langle P, s \rangle) = p \text{ and } \mu(\langle Q, s \rangle) = 1 - p$$

Transition rules (2)

$$\frac{\langle P, s \rangle \to \mu}{\langle P; Q, s \rangle \to \nu} \text{ with } \nu(\langle P'; Q', s' \rangle) = \mu(\langle P', s' \rangle) \text{ where } \downarrow; Q = Q$$

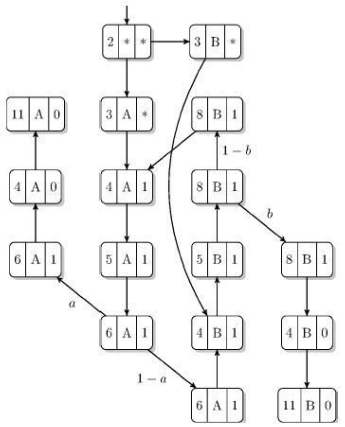
$$\frac{s \models G}{\langle \text{if } (G)\{P\} \text{ else } \{Q\}, s \rangle \to \langle P, s \rangle} \qquad \frac{s \not\models G}{\langle \text{if } (G)\{P\} \text{ else } \{Q\}, s \rangle \to \langle Q, s \rangle}$$

$$\frac{s \models G}{\langle \text{while}(G)\{P\}, s \rangle \to \langle P; \text{while } (G)\{P\}, s \rangle} \qquad \frac{s \not\models G}{\langle \text{while}(G)\{P\}, s \rangle \to \langle \downarrow, s \rangle}$$

Example

Duelling cowboys

```
int cowboyDuel(float a, b) {
  int t := A [0.5] t := B;
  bool c := true;
  while (c) {
    if (t = A) {
      (c := false [a] t := B);
    } else {
      (c := false [b] t := A);
    }
  }
  return t;
}
```



This (parametric) MC is finite. Once we count the number of shots before one of the cowboys dies, the MC becomes countably infinite.

Playing with geometric distributions

- X is a random variable, geometrically distributed with parameter p
- Y is a random variable, geometrically distributed with parameter q

Q: generate a sample x, say, according to the random variable $X - Y$

```
int XminY1(float p, q){ // 0 <= p, q <= 1
  int x := 0;
  bool flip := false;
  while (not flip) { // take a sample of X to increase x
    (x +:= 1 [p] flip := true);
  }
  flip := false;
  while (not flip) { // take a sample of Y to decrease x
    (x -:= 1 [q] flip := true);
  }
  return x; // a sample of X-Y
}
```


An alternative program

```
int XminY2(float p, q){
  int x := 0;
  bool flip := false;
  (flip := false [0.5] flip := true); // flip a fair coin
  if (not flip) {
    while (not flip) { // sample X to increase x
      (x +:= 1 [p] flip := true);
    }
  } else {
    flip := false; // reset flip
    while (not flip) { // sample Y to decrease x
      x -:= 1;
      (skip [q] flip := true);
    }
  }
return x; // a sample of X-Y
}
```

Program equivalence: $X - Y$

```
int XminY1(float p, q){
  int  x, c := 0, 1;
  while (c) {
    (x +:= 1 [p] c := 0);
  }
  c := 1;
  while (c) {
    (x -:= 1 [q] c := 0);
  }
  return x;
}
```

```
int XminY2(float p, q){
  int x := 0;
  (c := 0 [0.5] c := 1);
  if (c) {
    while (c) {
      (x +:= 1 [p] c := 0);
    }
  } else {
    c := 1;
    while (c) {
      x -:= 1;
      (skip [q] c := 0);
    }
  }
return x;
}
```

The probability that $x = k$ for some $k \in \mathbb{Z}$ coincides for both programs if and only if $q = \frac{1}{2-p}$.

The outcome of a pGCL program

Unlike a deterministic program, a pGCL program P has not an output for a given input. Instead, it yields a unique probability distribution over its final states.

In fact, this is a sub-distribution (probability mass at most one), as with a certain probability P may diverge.

Let P be a pGCL program and s an input state. Then the distribution over final states obtained by running P starting in s is given by $Pr(s \models \Diamond \langle \downarrow, \cdot \rangle)$.

Reachability probabilities

If the MC $[\![P]\!]$ of pGCL program P has finitely many states, reachability probabilities can be obtained in an automated manner. This applies to the cowboy example for given probabilities a and b.

The same holds for expected rewards, e.g., the expected number of steps until termination of a finite-state program P.

Overview

Probabilistic GCL with recursion: Syntax

- `skip` — empty statement
- `diverge` — divergence
- `x := E` — assignment
- `x :r= mu` — random assignment ($x :\approx \mu$)
- `prog1 ; prog2` — sequential composition
- `if (G) prog1 else prog2` — choice
- `prog1 [p] prog2` — probabilistic choice
- `while (G) prog` — iteration
- `P = prog` — **process definition**
- `call P` — **process invocation**

Recursion does not increase the expressive power, but is often convenient.

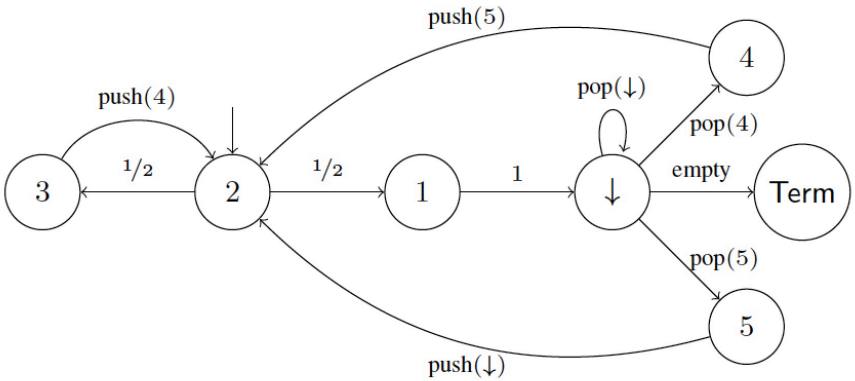
Pushdown Markov chains

Pushdown Markov chain

A pushdown Markov chain D is a tuple $(\Sigma, \sigma_I, \Gamma, \gamma_0, \Delta)$ where:

- Σ is a countable set of (control) states
- $\sigma_I \in \Sigma$ is the initial (control) state
- Γ is a finite stack alphabet
- $\gamma_0 \in \Gamma$ is the bottom-of-the-stack symbol
- $\Delta : \Sigma \times \Gamma \to \mathit{Dist}(\Sigma) \times (\Gamma \setminus \{\gamma_0\})^*$ is the probability transition relation

Recursion: pushdown Markov chains



$\{\mathsf{skip}^1\}\ [1/2]^2\ \{\mathsf{call}\ P^3;\ \mathsf{call}\ P^4;\ \mathsf{call}\ P^5\}$