Probabilistic Programming

Lecture #8+#9: Loop Invariants

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Overview

Overview

Motivation

- Reasoning about loops is the hardest task in program verification
- Why?
  - Weakest preconditions of loops are defined as fixed points
  - They can be approximated iteratively
  - But: Recognise a pattern to yield a closed-form formula for taking a loop n times
  - Taking the limit yields the required fixed point

  These last two steps are the source of undecidability
- "Practical" approach: capture the effect of a loop by a loop invariant

A loop invariant is a property of a program loop that is true before (and after) each iteration.

Loop invariants

Recall that for while-loops we have:

$$wlp(\text{while}(G)\{P\}, F) \;=\; \text{gfp } X.\ (G \wedge wlp(P, X)) \vee (\neg G \wedge F)$$

To determine the effect a the while-loop, one exploits an "invariant" $I \in \mathbb{P}$

Loop invariant

Predicate $I \in \mathbb{P}$ is a loop invariant if it satisfies:

1. $G \;\Rightarrow\; I$
2. $\neg G \;\wedge\; I \;\Rightarrow\; F$, and
3. $G \wedge I \Rightarrow wlp(P, I)$.

Satisfaction of I is invariant under (guarded) iteration of the loop body P.

Characteristic functions for loops

Recall:

$$wp(\text{while } (G)\{\,P\,\}, f) \;=\; \text{lfp } X.\ \underbrace{([G] \cdot wp(P, X) + [\neg G] \cdot f)}_{\text{characteristic function } \Phi_f(X) \text{ for wp}}$$

and

$$wlp(\text{while } (G)\{\,P\,\}, f) \;=\; \text{gfp } X.\ \underbrace{([G] \cdot wlp(P, X) + [\neg G] \cdot f)}_{\text{characteristic function } \Psi_f(X) \text{ for wlp}}$$

Loop invariants

For $I, F \in \mathbb{P}$ and probabilistic loop $\text{while}(G)\{P\}$ it holds:

$$(\neg G \;\wedge\; I \;\Rightarrow\; F \text{ and } G \;\wedge\; I \Rightarrow wlp(P, I)) \quad \text{iff} \quad [I] \sqsubseteq \Psi_{[F]}([I])$$

where $\Psi_{[F]}$ is the wlp-characteristic function of the probabilistic loop for postcondition $[F]$.

Proof.

On the black board. □

Example of a loop invariant.

Overview

Probabilistic invariants

Probabilistic invariants

Let Φ_f be the wp-characteristic function of $P' = \text{while}(G)\{P\}$ with respect to post-expectation $f \in \mathbb{E}$ and let $I \in \mathbb{E}$. Then:

1. I is a wp-superinvariant of P' w.r.t. f iff $\Phi_f(I) \leq I$.
2. I is a wp-subinvariant of P' w.r.t. f iff $I \leq \Phi_f(I)$.

Sub- and superinvariants for wlp are defined analogously (but are bounded, i.e., then $I \in \mathbb{E}_{\leq 1}$, and Φ_f is replaced by Ψ_f.)

Lemma

$[G] \cdot I \leq wp(P, I)$ iff $I \leq \Phi_{[\neg G] \cdot I}(I)$

Proof.

Left as an exercise. □

Induction for upper bounds on wp

Induction on complete lattices (Park's lemma)

Let $(D, \sqsubseteq)$ be a complete lattice and $\Phi : D \to D$ continuous. Then:

$$\forall d \in D. \quad \Phi(d) \sqsubseteq d \quad \text{implies} \quad \text{lfp } \Phi \sqsubseteq d.$$

Applying this induction principle yields upper bounds on weakest pre-expectations.

Upper bounds on weakest pre-expectations

For $\text{while}(G)\{P\}$ and $f, I \in \mathbb{E}$ we have:

$$\underbrace{\Phi_f(I) \leq I}_{\text{wp-superinvariant}} \quad \text{implies} \quad wp(\text{while}(G)\{P\}, f) \leq I$$

Every wp-superinvariant of a loop for f
is an upper bound to the wp of the loop (and f).

Example

Co-induction for lower bounds on wlp

Co-induction on complete lattices (Park's lemma)

Let $(D, \sqsubseteq)$ be a complete lattice and $\Phi : D \to D$ continuous. Then:

$$\forall d \in D. \quad d \sqsubseteq \Phi(d) \quad \text{implies} \quad d \sqsubseteq \text{gfp } \Phi.$$

Applying this induction principle yields lower bounds on weakest liberal pre-expectations.

Lower bounds on weakest liberal pre-expectations

For $\text{while}(G)\{P\}$ and $f, I \in \mathbb{E}_{\leq 1}$ we have:

$$\underbrace{I \leq \Psi_f(I)}_{\text{wlp-subinvariant}} \quad \text{implies} \quad I \leq wlp(\text{while}(G)\{P\}, f)$$

Every wlp-subinvariant of a loop for f
is a lower bound to the wp of the loop (and f).

Example

Verification of loops (1)

The following procedure can be followed for induction and co-induction:

1. Find an appropriate loop invariant I
2. Push I through the characteristic function of the loop once, i.e., compute $\Phi(I)$
3. Check whether this took us down or up in the partial order $\leq$:
   - 3.1 $\Phi(I) \leq I$, for induction, or
   - 3.2 $I \leq \Phi(I)$, for co-induction.

The key difficulty is to find an appropriate invariant I. This is undecidable.

Remarks about unsound versions of (co-)induction

Let $(D, \sqsubseteq)$ be a complete lattice and $\Phi : D \to D$ continuous. The following statements are not valid:

$$\forall d \in D.\quad d \sqsubseteq \Phi(d) \quad \text{implies} \quad d \sqsubseteq \text{lfp } \Phi.$$

and

$$\forall d \in D.\quad \Phi(d) \sqsubseteq d \quad \text{implies} \quad \text{gfp } \Phi \sqsubseteq d.$$

As a consequence, co-induction for lower bounds on wp of loops and induction on upper bounds on wlp of loops is unsound.

Counterexample

On the black board.

Overview

ω-invariants

ω-invariants

Let $n \in \mathbb{N}$, $f \in \mathbb{E}$ and Φ_f be the wp-characteristic function of the loop while$(G)\{P\}$.

The monotonically increasing[1] sequence $(I)_{n\in\mathbb{N}}$ is a wp-ω-subinvariant of the loop w.r.t. f iff

$$I_0 \leq \Phi_f(\mathbf{0}) \quad \text{and} \quad I_{n+1} \leq \Phi_f(I_n) \quad \text{for all } n.$$

In a similar way, wlp-ω-superinvariants are defined, where $f, I_n \in \mathbb{E}_{\leq 1}$, $(I)_{n\in\mathbb{N}}$ is monotonically decreasing, and Φ_f is replaced by Ψ_f.

[1] But not necessarily strictly increasing.

Bounds on loops using ω-invariants

Theorem

1. Let $(I)_{n\in\mathbb{N}}$ be a wp-ω-subinvariant of while$(G)\{P\}$ w.r.t. $f \in \mathbb{E}$. Then:

$$\sup_{n\in\mathbb{N}} I_n \leq wp(\text{while}(G)\{P\}, f) .$$

2. Let $(I)_{n\in\mathbb{N}}$ be a wlp-ω-superinvariant of while$(G)\{P\}$ w.r.t. f. Then:

$$wlp(\text{while}(G)\{P\}, f) \leq \inf_{n\in\mathbb{N}} I_n .$$

Proof.

□

Verification of loops (2)

The following procedure can be followed using ω sub-/superinvariants:

1. Find an appropriate ω-invariant, i.e., a sequence $(I)_{n\in\mathbb{N}}$
2. Check that $(I)_{n\in\mathbb{N}}$ is indeed an ω-invariant:
   - 2.1 Push I_n through the characteristic function
   - 2.2 Check whether this took us above I_{n+1} (for wp) or below I_{n+1} (for wlp) in the partial order $\leq$
3. Find the supremum (for wp) or the infimum (for wlp) of $(I)_{n\in\mathbb{N}}$

In addition to finding appropriate invariants I_n, we have to reason about the limits of the ω-invariants. This is undecidable too.

Overview

Total correctness proof rules for loops

Total correctness proof rules for loops

Let $f \in \mathbb{E}$ with $f \leq \mathbf{k}$, for some $k \in \mathbb{N}$, and let $J \in \mathbb{E}$ be $\mathbf{k}$-bounded too such that $I \in \mathbb{E}$ defined by: $I = [\neg G] \cdot f + [G] \cdot J$. Then it holds:

1. If $I = [F]$ for some predicate $F \in \mathbb{P}$, then:

$$T \cdot I \leq wp(\text{while}(G)\{P\}, f)$$

where $T = wp(\text{while}(G)\{P\}, \mathbf{1})$ is the loop's termination probability.

2. If $[F] \leq T$ for some predicate $F \in \mathbb{P}$, then:

$$[F] \cdot I \leq wp(\text{while}(G)\{P\}, f)$$

3. If $\varepsilon \cdot I \leq T$ for some $\varepsilon > 0$, then:

$$I \leq wp(\text{while}(G)\{P\}, f).$$

Almost-surely terminating loops

Proof rules for a.s.-terminating loops

Let $\text{while}(G)\{P\}$ be almost-surely terminating, i.e., $wp(\text{while}(G)\{P\}, \mathbf{1}) = 1$, and let $I \in \mathbb{E}_{\leq 1}$. Then:

1. $$I \leq \Phi_f(I) \quad \text{implies} \quad I \leq wp(\text{while}(G)\{P\}, f) \ .$$

2. $$\Psi_f(I) \leq I \quad \text{implies} \quad wlp(\text{while}(G)\{P\}, f) \leq I \ .$$

Bound refinement

Bound refinement

Let loop $\text{while}(G)\{P\}$, $f, I \in \mathbb{E}$. Then:

1. If $\Phi_f(I) \leq I$:

$$wp(\text{while}(G)\{P\}, f) \leq I \quad \text{implies} \quad wp(\text{while}(G)\{P\}, f) \leq \Phi_f(I) \ .$$

2. If $I \leq \Phi_f(I)$:

$$I \leq wp(\text{while}(G)\{P\}, f) \quad \text{implies} \quad \Phi_f(I) \leq wp(\text{while}(G)\{P\}, f) \ .$$

If $\Phi_f(I) \neq I$, we thus obtain tighter upper and lower bounds, respectively.

The sequence $\Phi(I), \Phi^2(I), \Phi^3(I), \ldots$ converges.
For upper bounds to the greatest fixed point that is below (or equal to) I.
(This fixed point itself is an upper bound too.)
This is the Tarski-Kantorovich fixpoint principle.

Overview

Invariant synthesis for linear programs

1. Speculate that a loop invariant can be expressed as linear expression:

$$[\alpha_1 \cdot x_1 + \ldots + \alpha_n \cdot x_n + \alpha_{n+1} \ \ll \ 0] \cdot (\beta_1 \cdot x_1 + \ldots + \beta_n \cdot x_n + \beta_{n+1})$$

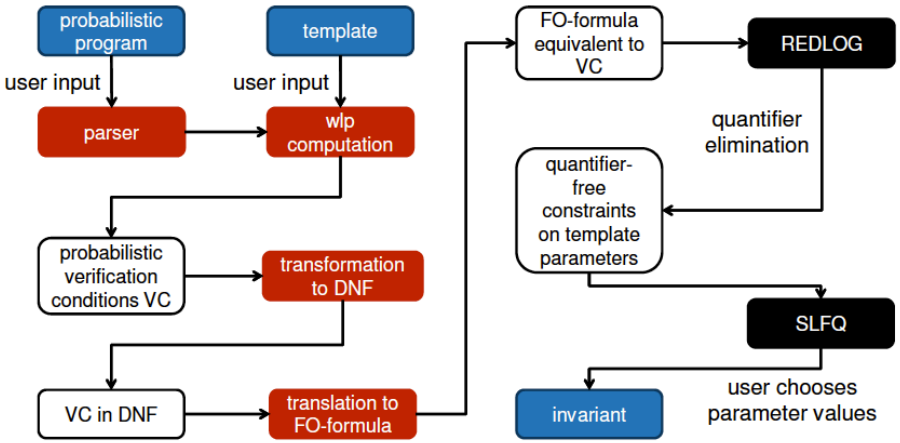
   - where n is at most the number of program variables in the program
   - x_i are program variables
   - α_i, β_i are real-valued invariant-template variables
   - $\ll \in \{<, \leq\}$ is a binary comparison operator

2. Transform these numerical constraints into non-linear FO formulas.
3. Use constraint solving techniques to obtain constraints on α_i, β_i.

Soundness and completeness

For any linear probabilistic program and linear loop invariant
this method will find
all parameter solutions that make the template valid, and no others.

PRINSYS tool: Probabilistic invariant synthesis



moves.rwth-aachen.de/prinsys

Duelling cowboys: when does A win?

```
int cbDuel(float a, b) {
  int t := A;
  int c := 1;
  while (c = 1) {
    if (t = A) {
      (c := 0 [a] t := B);
    } else {
      (c := 0 [b] t := A);
    }
  }
  return t;
}
```

Aim: find expectation I

Satisfying $I \leq [t = A]$ upon termination.

Observation

On entering the loop, c = 1 and either t = A or t = B.

Template suggestion

$$I = \underbrace{[t = A \wedge c = 0]}_{A \text{ wins duel}} \cdot 1 + \underbrace{[t = A \wedge c = 1]}_{A\text{'s turn}} \cdot \alpha + \underbrace{[t = B \wedge c = 1]}_{B\text{'s turn}} \cdot \beta$$

Duelling cowboys: when does A win?

Invariant template: $I = [\mathrm{t} = A \wedge \mathrm{c} = 0] \cdot 1 + [\mathrm{t} = A \wedge \mathrm{c} = 1] \cdot \alpha + [\mathrm{t} = B \wedge \mathrm{c} = 1] \cdot \beta$

Initially, $\mathrm{t} = A \wedge \mathrm{c} = 1$ and thus $\alpha = Pr\{A \text{ wins duel}\}$.

Running PrinSys yields: $a \cdot \beta - a + \alpha - \beta \leq 0 \quad \wedge \quad b \cdot \alpha - \alpha + \beta \leq 0$

Simplification yields: $\beta \leq (1-b) \cdot \alpha$ and $\alpha \leq \dfrac{a}{a+b-a \cdot b}$

Maximisation: $\beta = (1-b) \cdot \alpha$ and $\alpha = \dfrac{a}{a+b-a \cdot b}$

Probabilistic loop invariant

$$I = [\mathrm{t} = A \wedge \mathrm{c} = 0] \cdot 1 + [\mathrm{t} = A \wedge \mathrm{c} = 1] \cdot \frac{a}{a+b-ab} + [\mathrm{t} = B \wedge \mathrm{c} = 1] \cdot \frac{(1-b)a}{a+b-a \cdot b}$$

Annotated program for post-expectation $[t = A]$

```
1  int cowboyDuel(a, b) {
2    ⟨(1−b)a / a+b−ab⟩
3    ⟨min{a/a+b−ab, (1−b)a/a+b−ab}⟩
4    (t := A [] t := B);
5    ⟨[t = A] · a/a+b−ab + [t = B] · (1−b)a/a+b−ab⟩
6    c := 1;
7    ⟨[t = A ∧ c = 0] · 1 + [t = A ∧ c = 1] · a/a+b−ab  +[t = B ∧ c = 1] · (1−b)a/a+b−ab⟩
8    while (c = 1) {
9      ⟨[t = A ∧ c = 1] · a/a+b−ab + [t = B ∧ c = 1] · (1−b)a/a+b−ab⟩
10     ⟨[t = A ∧ c ≠ 1] · a + [t = A ∧ c = 1] · a/a+b−ab
           +[t = B ∧ c = 0] · (1 − b) + [t = B ∧ c = 1] · (1−b)a/a+b−ab⟩
11     if (t = A) {
12       (c := 0 [a] t := B);
13     } else {
14       (c := 0 [b] t := A);
15     }
16     ⟨[t = A ∧ c = 0] · 1 + [t = A ∧ c = 1] · a/a+b−ab  +[t = B ∧ c = 1] · (1−b)a/a+b−ab⟩
17   }
18   ⟨[c ≠ 1] · ( [t = A ∧ c = 0] · 1 + [t = A ∧ c = 1] · a/a+b−ab
         +[t = B ∧ c = 1] · (1−b)a/a+b−ab )⟩
19   ⟨[t = A]⟩
20   return t; // the survivor
21 }
```


Playing with geometric distributions

- X is a random variable, geometrically distributed with parameter p
- Y is a random variable, geometrically distributed with parameter q

Q: generate a sample x, say, according to the random variable $X - Y$

```
int XminY1(float p, q){ // 0 <= p, q <= 1
  int x := 0;
  bool flip := false;
  while (not flip) { // take a sample of X to increase x
    (x +:= 1 [p] flip := true);
  }
  flip := false;
  while (not flip) { // take a sample of Y to decrease x
    (x -:= 1 [q] flip := true);
  }
  return x; // a sample of X-Y
}
```


An alternative program

```
int XminY2(float p, q){
  int x := 0;
  bool flip := false;
  (flip := false [0.5] flip := true); // flip a fair coin
  if (not flip) {
    while (not flip) { // sample X to increase x
      (x +:= 1 [p] flip := true);
    }
  } else {
    flip := false; // reset flip
    while (not flip) { // sample Y to decrease x
      x -:= 1;
      (skip [q] flip := true);
    }
  }
return x; // a sample of X-Y
}
```

Program equivalence: $X - Y$

```
int XminY1(float p, q){
  int x, c := 0, 1;
  while (c) {
    (x +:= 1 [p] c := 0);
  }
  c := 1;
  while (c) {
    (x -:= 1 [q] c := 0);
  }
  return x;
}
```

```
int XminY2(float p, q){
  int x := 0;
  (c := 0 [0.5] c := 1);
  if (c) {
    while (c) {
      (x +:= 1 [p] c := 0);
    }
  } else {
    c := 1;
    while (c) {
      x -:= 1;
      (skip [q] c := 0);
    }
  }
return x;
}
```

Using wp, one can prove that the expectations of $f = x$ coincide if and only if $q = \frac{1}{2-p}$.

Program equivalence: $X - Y$

```
int XminY1(float p, q){
  int x, f := 0, 0;
  while (f = 0) {
    (x++ [p] f := 1);
  }
  f := 0;
  while (f = 0) {
    (x-- [q] f := 1);
  }
  return x;
}
```

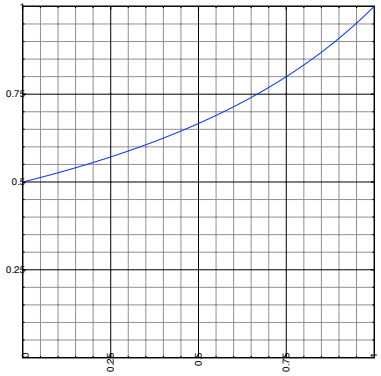
```
int XminY2(float p, q){
  int x, f := 0, 0;
  (f := 0 [0.5] f := 1);
  if (f = 0) {
    while (f = 0) {
      (x++ [p] f := 1);
    }
  } else {
    f := 0;
    while (f = 0) {
      x--;
      (skip [q] f := 1);
    }
  }
  return x;
}
```

Using template $I = x + [f = 0] \cdot \alpha$ we find:

$\alpha_{11} = \frac{p}{1-p}$, $\alpha_{12} = -\frac{q}{1-q}$, $\alpha_{21} = \alpha_{11}$ and $\alpha_{22} = -\frac{1}{1-q}$.

Expected value of x is $\frac{p}{1-p} - \frac{q}{1-q}$ and $\frac{p}{2(1-p)} - \frac{1}{2(1-q)}$.

Graphically this means ...



Both programs yield the same expected outcome for x all points on the curve $q = \frac{1}{2-p}$.