Probabilistic Programming

Lecture #5: Weakest Preconditions

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Overview

Overview

Approaches to semantics

- Operational semantics: (developed by Plotkin)
  - The meaning of a program in terms of how it executes on an abstract machine.
  - Useful for modelling the execution behaviour of a program.
- Axiomatic semantics: (developed by Floyd and Hoare)
  - Provides correctness assertions for each program construct.
  - Useful for verifying that a program's computed results are correct with respect to the specification.
- Denotational semantics: (developed by Strachey and Scott)
  - Provides a mapping of language constructs onto mathematical objects.
  - Useful for obtaining an abstract insight into the working of a program.

Today: denotational semantics of Dijkstra's GCL in terms of weakest preconditions. No probabilities yet.

Next lecture: how to extent preconditions to the probabilistic setting.

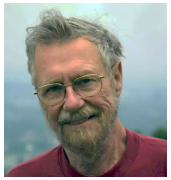
Code-level reasoning

Proving properties of programs: not by executing them,
but by reasoning at the syntax level of programs.

Compositionality: determine the correctness of composed program P
by reasoning about its parts in isolation and
then obtain P's correctness result by combining those parts' analyses.

Overview

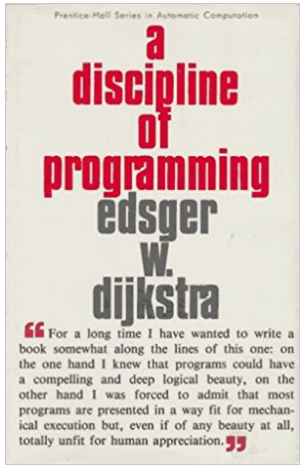
Dijkstra's guarded command language

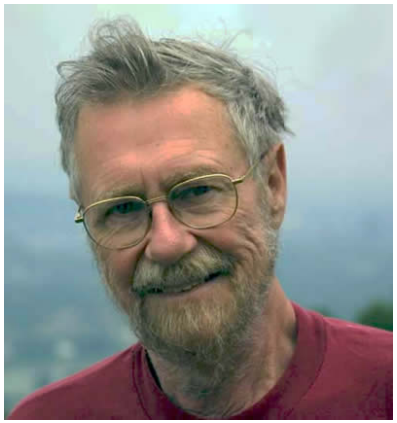


- `skip` — empty statement
- `diverge` — divergence
- `x := E` — assignment
- `prog1 ; prog2` — sequential composition
- `if (G) prog1 else prog2` — choice
- `prog1 [] prog2` — non-deterministic choice
- `while (G) prog` — iteration

For simplicity: we omit non-deterministic choice.

A discipline of programming





Some preliminaries

- Variable valuation $s : \mathit{Vars} \to \mathbb{Q}$ maps each program variable onto a value (here: rational numbers)
- Let $\mathbb{S}$ denote the set of variable valuations.
- Let $[\![E]\!]$ denote the valuation of expression E
- The indicator function of guard G is denoted by $[G]$:

$$[G](s) = \begin{cases} 1 & \text{if } s \vDash G \\ 0 & \text{if } s \nvDash G \end{cases}$$

These are also known as Iverson brackets.

Predicate transformers

Predicates

A predicate F maps program states onto Booleans, i.e., $F : \mathbb{S} \to \mathbb{B}$.

Let $\mathbb{P}$ denote the set of all predicates and $F \sqsubseteq G$ if and only if $F \Rightarrow G$.

$(\mathbb{P}, \sqsubseteq)$ is a complete lattice.

Proof.

Predicate F equals $\{ s \in \mathbb{S} \mid s \vDash F \}$. Thus $P = 2^{\mathbb{S}}$. Partial order $\sqsubseteq$ equals $\subseteq$. □

Predicate transformer

A predicate transformer is a total function between predicates.

Examples

- Predicates are sets of variable valuations.
- Program statements can be viewed as predicate transformers
- One is interested in preconditions that are least restrictive

Overview

Weakest preconditions

Weakest precondition

For program P and $E, F \in \mathbb{P}$, the predicate transformer $wp(P, \cdot) : \mathbb{P} \to \mathbb{P}$ is defined by $wp(P, F) = E$ if and only if when P starts in an initial state satisfying E it holds:

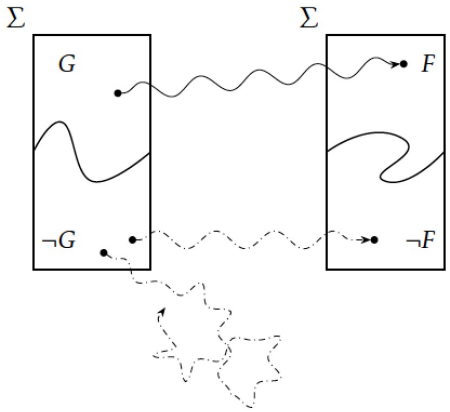
1. the execution of P terminates in a state satisfying F, and
2. for any $H \in \mathbb{P}$ such that P terminates in a state satisfying F, $H \Rightarrow E$.

$wp(P, F)$ is called the weakest precondition on the initial state of P such that P terminates in a final state satisfying the postcondition F.

Weakest preconditions correspond to so-called total correctness.

Examples.

Weakest precondition G w.r.t. postcondition F



This holds for every deterministic program.

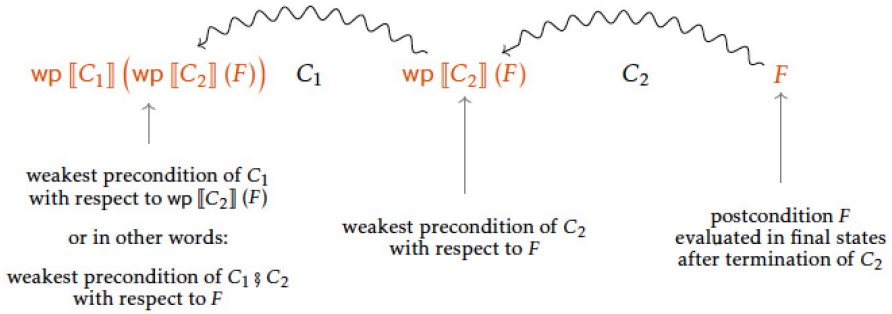
Weakest preconditions

Consider the program P and postcondition $F \in \mathbb{P}$.

Then $wp(P, F) = E$ means:

1. From any state $s \vDash E$, the program P terminates in some state $t \vDash F$.
2. From any state $s \nvDash E$, it holds:
   - 2.1 either P terminates in a state $t \nvDash F$
   - 2.2 or P does not terminate at all.

Backward reasoning



Weakest preconditions reason in a backward manner about programs.

Predicate transformer semantics of Dijkstra's GCL

Syntax	Semantics $wp(P, F)$
`skip`	F
`diverge`	$false$
`x := E`	$F[x := E]$
`P ; Q`	$wp(P, wp(Q, F))$
`if (G) P else Q`	$(G \wedge wp(P, F)) \vee (\neg G \wedge wp(Q, F))$
`while (G) P`	$\mathrm{lfp}\, X.\ ((G \wedge wp(P, X)) \vee (\neg G \wedge F))$

lfp is the least fixed point wrt. the ordering $\sqsubseteq = \Rightarrow$ on the set $\mathbb{P}$ of predicates.

Loop-free examples

Loops

$$wp(\text{while } (G)\{\, P \,\}, F) \;=\; \mathrm{lfp}\, X.\ \underbrace{((G \wedge wp(P, X)) \vee (\neg G \wedge F))}_{\Phi(X)}$$

Scott continuity of Φ

The function $\Phi : \mathbb{P} \to \mathbb{P}$ (defined as above) is continuous on $(\mathbb{P}, \sqsubseteq)$.

Proof.

By structural induction on the program P. □

Corollary

By Kleene's fixpoint theorem, it follows $\mathrm{lfp}\ \Phi = \sup_{n \in \mathbb{N}} \Phi^n(\mathrm{false})$.

$\Phi^n(\mathrm{false})$ denotes the wp of running while $(G)\{\, P \,\}$ exactly n times starting from the empty set of states.

A loopy program example

```
while (x > 0) {
    x--
}
```

What is the weakest pre-condition on `x`
such that on termination `x` is non-negative?

Approximating while-loops

Let:

$$\begin{aligned} \text{while}^0(G)\{P\}) &= \text{diverge} \\ \text{while}^{n+1}(G)\{P\}) &= \text{if } (G) \text{ then } P; \text{while}^n(G)\{P\}) \text{ else skip} \end{aligned}$$

Let $\Phi(X) = ((G \wedge wp(P, X)) \vee (\neg G \wedge F))$. Then for all $n \in \mathbb{N}$ it holds:

$$\Phi^n(\text{false}) = wp(\text{while}^n(G)\{P\}, F)$$

Proof.

By induction on n using the inductive definition of wp. □

Overview

Weakest liberal preconditions

Weakest liberal precondition

For program P and $E, F \in \mathbb{P}$, the predicate transformer $wlp(P, \cdot) : \mathbb{P} \to \mathbb{P}$ is defined by $wlp(P, F) = E$ if and only if when P starts in an initial state satisfying E it holds:

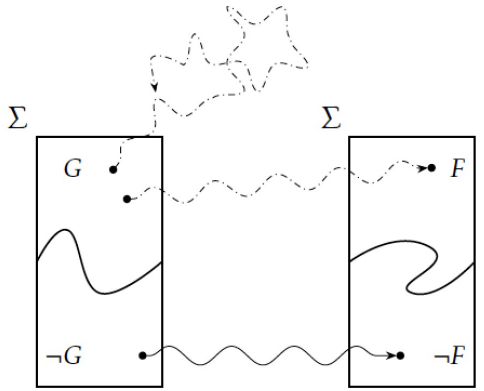
1. either P diverges or P terminates in a state satisfying F, and
2. for any $H \in \mathbb{P}$ such that P either diverges or terminates in a state satisfying F, $H \Rightarrow E$.

$wlp(P, F)$ is called the weakest liberal precondition on the initial state of P such that P either diverges or terminates in a final state satisfying the postcondition F.

Weakest liberal preconditions correspond to so-called partial correctness: a program is either correct, or diverges.

Examples.

Weakest liberal precondition G w.r.t. F



This holds for every deterministic program.

Weakest liberal preconditions

Consider the program P and postcondition $F \in \mathbb{P}$.

Then $wlp(P, F) = E$ means:

1. From any state $s \models E$,
   - 1.1 either the program P terminates in a state $t \models F$
   - 1.2 or does not terminate at all.
2. From any state $s \not\models E$, the program P terminates in a state $t \not\models F$

Weakest liberal preconditions for Dijkstra's GCL

Syntax	Semantics $wlp(P, F)$
`skip`	F
`diverge`	$true$
`x := E`	$F[x := E]$
`P ; Q`	$wlp(P, wlp(Q, F))$
`if (G) P else Q`	$(G \wedge wlp(P, F)) \vee (\neg G \wedge wlp(Q, F))$
`while (G)P`	$\text{gfp}\ X.\ ((G \wedge wlp(P, X)) \vee (\neg G \wedge F))$

gfp is the greatest fixed point wrt. the ordering $\sqsubseteq\ = \Rightarrow$ on the set $\mathbb{P}$ of predicates.

Loops

$$wlp(\text{while } (G)\{\, P \,\}, F) \;=\; \text{gfp}\, X.\ \underbrace{((G \wedge wlp(P, X)) \vee (\neg G \wedge F))}_{\Phi(X)}$$

Scott continuity of Φ

The function $\Phi : \mathbb{P} \to \mathbb{P}$ (defined as above) is continuous on $(\mathbb{P}, \sqsubseteq)$.

Corollary

By Kleene's fixpoint theorem, it follows $\text{gfp}\ \Phi = \inf_{n \in \mathbb{N}} \Phi^n(\text{true})$.

$\Phi^n(\text{true})$ denotes the wp of running while $(G)\{\, P \,\}$ exactly n times starting from the entire set $\mathbb{S}$ of states.

Elementary properties of Dijkstra's wp and wlp

- Monotonicity: $F \Rightarrow G$ implies $wp(P, F) \Rightarrow wp(P, G)$
- Duality: $wlp(P, F) = wp(P, F) \vee \neg wp(P, true)$
- Strictness: $wp(P, false) = false$ and $wlp(P, true) = true$
- Distribution $wp(P, F \vee G) = wp(P, F) \vee wp(P, G)$

$wp(P, true)$ = weakest precondition under which P terminates

$wp(P, F) \neq wlp(P, F)$ implies that P may diverge.