Probabilistic Programming

Lecture #16+#17: Expected Runtime Analysis

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Overview

Overview

The runtime of a probabilistic program

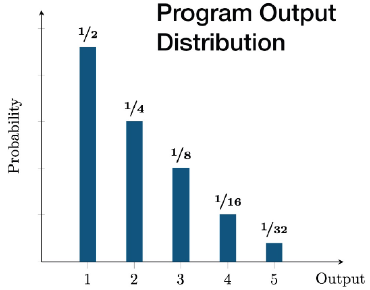
The runtime of a probabilistic program depends

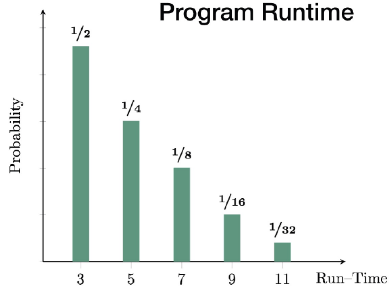
on the input and

on the internal randomness of the program.

The runtime of a probabilistic program is random

```
int i := 0;
repeat {i++; (c := false [0.5] c := true)}
until (c)
```





The expected runtime is $1 + 3 \cdot 1/2 + 6 \cdot 1/4 + \ldots (3n+1) \cdot 1/2^n \; = \; 5$.

Expected runtimes

Expected run-time of program P on input s:

$$\sum_{i=1}^{\infty} i \cdot Pr\left(\begin{array}{c} \text{“}P \text{ terminates after} \\ i \text{ steps on input } s\text{”} \end{array} \right)$$

Efficiency of randomised algorithms

Quicksort:

```
QS(A) =
  if |A| <= 1 { return A; }
  i := ceil(|A|/2);
  A< := {a in A | a < A[i]};
  A> := {a in A | a > A[i]};
  return QS(A<) ++ A[i] ++ QS(A>)
```

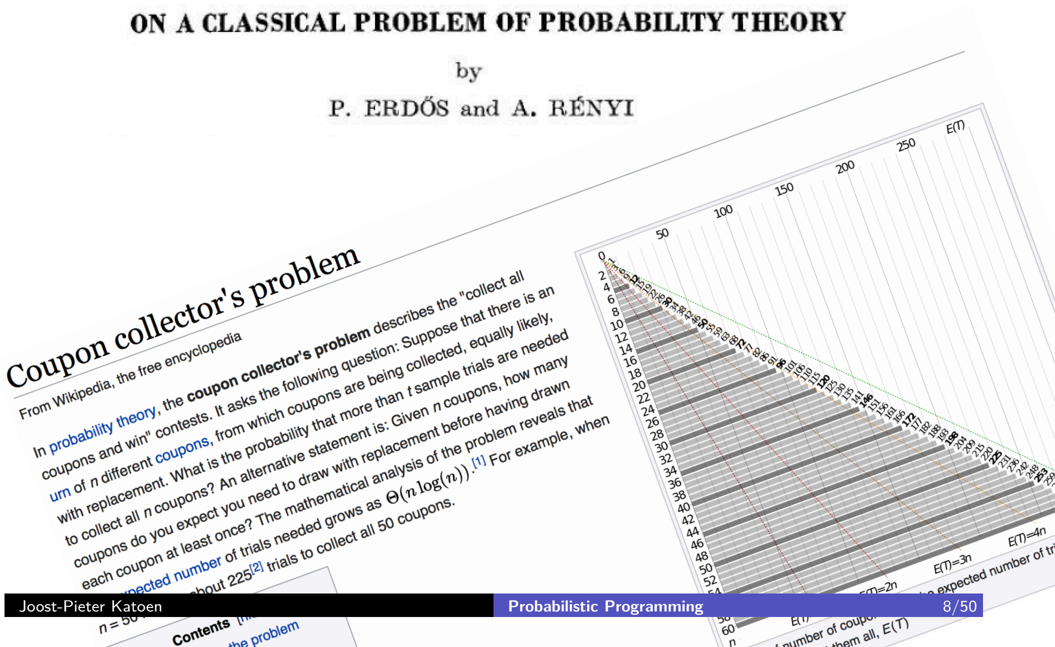
Worst case complexity:
$O(N^2)$ comparisons

Randomised Quicksort:

```
rQS(A) =
  if |A| <= 1 { return A; }
  i := Unif[1...|A|];
  A< := {a in A | a < A[i]};
  A> := {a in A | a > A[i]};
  return rQS(A<) ++ A[i] ++ rQS(A>)
```

Worst case complexity:
O(N log N) expected comparisons

Coupon collector's problem

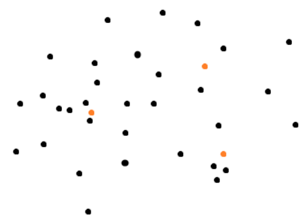


Coupon collector's problem

```
cp := [0,...,0]; // no coupons yet
i := 1; // coupon to be collected next
x := 0: // number of coupons collected
while (x < N) {
   while (cp[i] != 0) {
      i := uniform(1..N) // next coupon
   }
   cp[i] := 1;  // coupon i obtained
   x++;   // one coupon less to go
}
```

The expected runtime of this program is in $\Theta(N \cdot \log N)$.

Closest-pair problem



Closest-pair problem: find two distinct points $u, v \in \mathbb{R}^2$ among N points in the plane that minimise the Euclidean distance among all pairs of these points.

A naive deterministic approach takes $O(N^2)$. More efficient version in $O(N \cdot \log N)$.

Rabin's randomised algorithm has an expected runtime in $O(N)$.

Randomised primality test

Problem: is N prime or not?

Basic structure of a randomised primality test:

1. Randomly pick a number a, say
2. Do the primality test: Check some equality involving a and N
3. If equality fails, N is composite (with witness a)
4. Otherwise repeat the process.

If after $K > 0$ iterations, N is not found to be composite, then N is probably prime.

Some primality tests

- Fermat primality test:
  Select $a \in \mathbb{Z}$ relative prime to N. If $a^{N-1} \bmod N \neq 1$, then N is composite.
- Rabin-Miller test:
  Select $0 < a < N$. Let $2^s \cdot d = N{-}1$ where d is odd. If $a^d \neq 1 \pmod N$ and $a^{2^r \cdot d} \neq -1 \pmod N$ for all $0 \leq r \leq s{-}1$, then N is composite.
- Solovay and Strassen test:
  For N odd, pick $a < N$. If $a^{N-1/2} \neq \ldots\ldots$, then N is composite.

Adleman and Huang (1992) provided a randomised primality test that terminates with expected polynomial runtime and certainly provides the correct answer.[1]

[1] Decision problems with this characteristic constitute the complexity class ZPP (zero-error probabilistic polynomial time).

The aim of this lecture

A wp-calculus to reason about runtimes at the source code level.

No "descend" into the underlying probabilistic model.

The calculus should be compositional.

Proving positive almost-sure termination

- What? AST+termination in finite expected time
- Generalise. How?
  - Provide an weakest-precondition calculus
  - ...... for expected runtimes
- Why?
  - Reason about the efficiency of randomised algorithms
  - Reason about simulation efficiency of Bayesian networks
  - Is compositional and reasons at the program's code

Hurdles in runtime analysis

1. Programs may admit diverging runs while still having a finite expected runtime

```
while (x > 0) { x-- [1/2] skip }
```

   admits a diverging run but has expected runtime $O(x)$.
2. Having a finite expected time is not compositional w.r.t. sequencing
3. Expected runtimes are extremely sensitive to variations in probabilities

```
while (x > 0) { x-- [1/2+p] x++ } // 0 <= p <= 1/2
```

   - For p=0, the expected runtime is infinite.
   - For arbitrary small $p > 0$, the expected runtime is $^1/_2{\cdot}p{\cdot}x$, linear in x.

Overview

Re-use weakest preconditions?

Idea: equip the program with a counter `rc` and use standard wp-reasoning to determine its expected value.

Determine $wp(P, \texttt{rc})$ for program P.

Dexter Kozen
A probabilistic PDL
1983

An example

Consider the program P:

```
x := 1;
while (x > 0) { x := 0 [1/2] skip }
```

Equipping P with a runtime counter yields P_{rc}:

```
x := 1; rc := 0;
while (x > 0) { rc++; (x := 0 [1/2] skip) }
```

It follows $\Phi(I) \leq I$ for $I = \texttt{rc} + [x > 0] \cdot 2$.

In total, we thus obtain $wp(P_{rc}, \texttt{rc}) = 2$.

An example

Consider the program Q:

```
x := 1;
while (x > 0) { x := 0 [1/2] while(true) { skip } }
```

Equipping Q with a runtime counter yields Q_{rc}:

```
x := 1; rc := 0;
while (x > 0) {
   rc++;
   (x := 0 [1/2] while(true) { rc++ ; skip})
}
```

As $wp(\text{inner loop}, f) = 0$ for every f, it follows $\Phi_{Q_{rc}} \leq \Phi_{P_{rc}}$.

Thus, $\Phi_{Q_{rc}}(I) \leq \Phi_{P_{rc}}(I) \leq I$ for $I = \texttt{rc} + [x > 0] \cdot 2$.

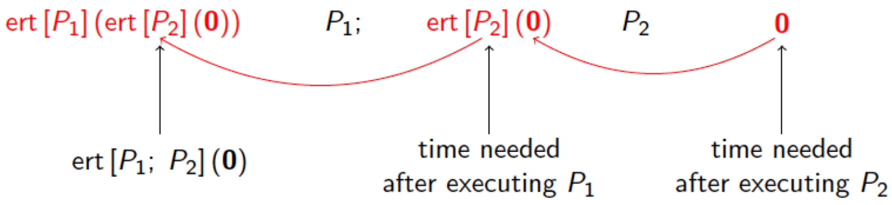
This contradicts the fact that the true expected runtime of Q is ∞.

Overview

The basic idea

Let $ert() : \text{pGCL} \rightarrow (\mathbb{T} \rightarrow \mathbb{T})$ where:

- $ert(P, t)(s)$ is the expected runtime of P on input state s if t captures the runtime of the computation following P.
- $ert(P, \mathbf{0})(s)$ is the expected runtime of P on input state s.



Runtimes

Expectations

A expectation $f : \mathbb{S} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$.

Let $\mathbb{E}$ be the set of all expectations and let $\sqsubseteq$ be defined for $f, g \in \mathbb{E}$ by:

$$f \sqsubseteq g \quad \text{if and only if} \quad f(s) \leq g(s) \quad \text{for all } s \in \mathbb{S}.$$

Runtimes

A runtime $t : \mathbb{S} \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$.

Let $\mathbb{T}$ denote the set of all runtimes and let $\preceq$ be defined for $t, u \in \mathbb{T}$ by:

$$t \preceq u \quad \text{if and only if} \quad t(s) \leq u(s) \quad \text{for all } s \in \mathbb{S}.$$

A runtime transformer is defined in a similar way as an expectation transformer

The runtime model

We assume the following runtimes:

- Executing a `skip`-statement takes a single time unit
- Executing an (ordinary or random) assignment takes a single time unit
- Evaluating a guard takes a single time unit
- Flipping a coin in a probabilistic choice takes a single time unit
- Sequential composition does not take time

The ert-calculus can be easily adapted to other runtime models.

Expected runtime transformer for pGCL

Syntax	Expected runtime $ert(P, t)$
`skip`	$\mathbf{1} + t$
`diverge`	∞
`x := E`	$\mathbf{1} + t[x := E]$
`x :r= mu`	$\mathbf{1} + \lambda s. \int_{\mathbb{Q}} (\lambda v. t(s[x := v]))\, d\mu_s$
`P1 ; P2`	$ert(P_1, ert(P_2, t))$
`if (G) P1 else P2`	$\mathbf{1} + [G] \cdot ert(P_1, t) + [\neg G] \cdot ert(P_2, t)$
`P1 [p] P2`	$\mathbf{1} + p \cdot ert(P_1, t) + (1-p) \cdot ert(P_2, t)$
`while(G)P`	$\text{lfp}\, X.\ (\mathbf{1} + [G] \cdot ert(P, X) + [\neg G] \cdot t)$

lfp is the least fixed point operator wrt. the ordering $\preceq$ on runtimes

Examples

Overview

Elementary properties

- Continuity: $ert(P, t)$ is continuous on $(\mathbb{T}, \leq)$
- Monotonicity: $t \leq t'$ implies $ert(P, t) \leq ert(P, t')$
- Constant propagation: $ert(P, \mathbf{k} + t) = \mathbf{k} + ert(P, t)$
- Preservation of ∞: $ert(P, \infty) = \infty$
- Connection to wp: $ert(P, t) = ert(P, \mathbf{0}) + wp(P, t)$
- Affinity: $ert(P, a \cdot t + t') \;=\; ert(P, \mathbf{0}) + r \cdot ert(P, t) + ert(P, t')$

(Positive) almost-sure termination

For every pGCL program P and input state s:

$$\underbrace{ert(P, \mathbf{0})(s) \;<\; \infty}_{\text{positive a.s-termination on } s} \quad \text{implies} \quad \underbrace{wp(P, \mathbf{1})(s) \;=\; \mathbf{1}}_{\text{almost-sure termination on } s}$$

Moreover:

$$\underbrace{ert(P, \mathbf{0}) \;\leq\; \infty}_{\text{universal positive a.s-termination}} \quad \text{implies} \quad \underbrace{wp(P, \mathbf{1}) \;=\; \mathbf{1}}_{\text{universal almost-sure termination}}$$

A Markov chain perspective on runtimes

- Consider $ert(P, t)$ for pCGL program P
- Consider the Markov chain $[\![P]\!]$ of program P
- Attach rewards to each Markov chain state in $[\![P]\!]$:
  - State $\langle \downarrow, s \rangle$ gets reward $t(s)$
  - State $\langle \texttt{skip}, s \rangle$ gets reward one
  - State $\langle \texttt{diverge}, s \rangle$ gets reward ∞
  - State $\langle x := E, s \rangle$ gets reward one
  - State $\langle x :\approx \mu, s \rangle$ gets reward one
  - State $\langle \texttt{if}\ G \ldots, s \rangle$ gets reward one
  - State $\langle P[p]Q, s \rangle$ gets reward one
  - State $\langle \texttt{while}(G)P' \ldots, s \rangle$ gets reward one
  - All other states get reward zero

Example

Correspondence between $ert()$ and Markov chains

Compatibility theorem

For every pGCL program P and input s:

$$ert(P, \mathbf{0})(s) \quad = \quad ER^{[\![P]\!]}(s, \Diamond sink)$$

In words: the $ert(P, \mathbf{0})$ for input s equals the expected reward to reach final state *sink* in MC $[\![P]\!]$ where reward function r in $[\![P]\!]$ is defined as defined on the previous slide.

Backward compatibility

Deterministic programs

For any GCL program P, $ert(P, \mathbf{0})$ equals the number of executed computational steps[2] of P until P terminates.

[2] This equals the number of skip statements, guard evaluations and assignments.

Overview

Loops

Reasoning about loops requires — like for wp — invariants.

Runtime invariants

Runtime invariants

Let Φ_t be the wp-characteristic function of $P' = \texttt{while}(G)\{P\}$ with respect to post-runtime $t \in \mathbb{T}$ and let $I \in \mathbb{T}$. Then:

1. I is a runtime-superinvariant of P' w.r.t. t iff $\Phi_t(I) \leq I$.
2. I is a runtime-subinvariant of P' w.r.t. t iff $I \leq \Phi_t(I)$.

If I is a runtime-superinvariant of $\texttt{while}(G)\{P\}$ with respect to $t \in \mathbb{T}$, then:

$$ert(\texttt{while}(G)\{P\}, t) \ \leq \ I$$

Example

A wrong proof rule for lower bonds

Probabilistic programs do not satisfy:

if $I \preceq \Phi_t(I)$ then $I \preceq ert(\texttt{while}(G)\,P, t)$.

These "metering" functions I do work for ordinary programs

[Frohn *et al.*, IJCAR 2016]

A counterexample

```
while (true) { skip [1/2] x++ }
```

- Characteristic functional $F(X) = \mathbf{1} + {}^1\!/_2\,(\mathbf{1} + \mathbf{1} + X[x/x{+}1])$
- Least fixed point is $\mathbf{4}$ as $F(\mathbf{4}) = \mathbf{2} + {}^1\!/_2 \cdot \mathbf{4} = \mathbf{4}$
- $\mathbf{4} + \mathbf{2}^i$ is a fixed point of F too:

$$F\left(\mathbf{4} + \mathbf{2}^i\right) \;=\; \mathbf{2} + \frac{1}{2}\left(\mathbf{4} + \mathbf{2}^{i+1}\right) \;=\; \mathbf{4} + \mathbf{2}^i$$

- Thus: $\mathbf{4} + \mathbf{2}^i \;\preceq\; F\left(\mathbf{4} + \mathbf{2}^i\right)$ but $\mathbf{4} + \mathbf{2}^i \;\not\preceq\; \mathbf{4} \;=\; \mathrm{lfp}\; F$
- In fact, $\mathbf{4} + \mathbf{2}^{i+c}$ is a fixed point of F for any c:

$$F\left(\mathbf{4} + \mathbf{2}^{i+c}\right) \;=\; \mathbf{2} + \frac{1}{2}\left(\mathbf{4} + \mathbf{2}^{i+c+1}\right) \;=\; \mathbf{4} + \mathbf{2}^{i+c}$$

Runtime ω-invariants

Runtime ω-invariants

Let $n \in \mathbb{N}$, $t \in \mathbb{T}$ and Φ_t the ert-characteristic function of $\texttt{while}(G)\{P\}$.

The monotonically increasing[3] sequence $(I)_{n \in \mathbb{N}}$ is a runtime-ω-subinvariant of the loop w.r.t. runtime t iff

$$I_0 \preceq \Phi_t(\mathbf{0}) \quad \text{and} \quad I_{n+1} \preceq \Phi_t(I_n) \quad \text{for all } n.$$

In a similar way, runtime ω-superinvariants can be defined, but we will not use them here.

[3] But not necessarily strictly increasing.

Lower bounds

Runtime lower bounds

If I_n is a runtime ω-subinvariant of $\texttt{while}(G)\{P\}$ with respect to t, then:

$$\sup_n I_n \;\preceq\; ert(\texttt{while}(G)\,P, t)$$

Example

Consider the same program as for proving an upper bound on the expected runtime.

Overview

1. Motivation
2. An unsound approach
3. The expected runtime transformer
4. Properties
5. Proof rules for runtimes of loops
6. Proving positive almost-sure termination
7. Case studies

PAST is not compositional

Consider the two probabilistic programs:

```
int x := 1;
bool c := true;
while (c) {
    c := false [0.5] c := true;
    x := 2*x
}
```

Finite expected termination time

```
while (x > 0) {
    x--
}
```

Finite termination time

Running the right after the left program
yields an infinite expected termination time

Proving that PAST is not compositional (1)

```
while (x > 0) { x := x-1 }
```

It is easy to check that a lower ω-invariant is:

$$J_n = \mathbf{1} + \underbrace{[0 < x < n]\cdot 2x}_{\text{on iteration}} + \underbrace{[x \geq n]\cdot(2n-1)}_{\text{on termination}}$$

Thus we obtain that:

$$\lim_{n\to\infty} \left(\mathbf{1} + [0 < x < n]\cdot 2x + [x \geq n]\cdot(2n-1)\right) = \mathbf{1} + [x > 0]\cdot 2x$$

is a lower bound on the runtime of the above program.

Proving that PAST is not compositional (2)

```
while (c) { {c := false [0.5] c := true}; x := 2*x};
while (x > 0) { x := x-1 }
```

Template for a lower ω-invariant of composed program:

$$I_n = \mathbf{1} + \underbrace{[c \neq 1]\cdot(\mathbf{1} + [x > 0]\cdot 2x)}_{\text{on termination}} + \underbrace{[c = 1]\cdot(a_n + b_n\cdot[x > 0]\cdot 2x)}_{\text{on iteration}}$$

The constraints on being a lower ω-invariant yield:

$$a_0 \leq 2 \quad \text{and} \quad a_{n+1} \leq 7/2 + 1/2\cdot a_n \quad \text{and} \quad b_0 \leq 0 \quad \text{and} \quad b_{n+1} \leq 1 + b_n$$

This admits the solution $a_n = 7 - 5/2^n$ and $b_n = n$. Then: $\lim_{n\to\infty} I_n = \infty$.

Proving PAST

The ert-transformer enables to prove

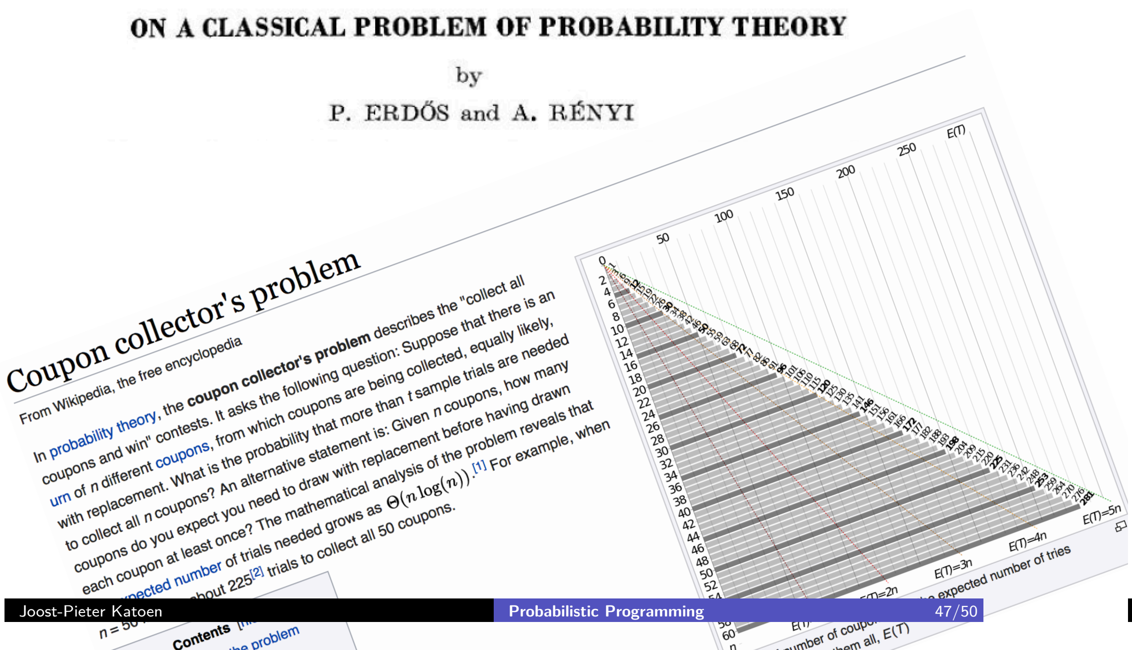
that a program is positively almost-surely terminating

in a compositional manner,

although PAST itself is not a compositional property.

Overview

Coupon collector's problem



Coupon collector's problem

```
cp := [0,...,0]; i := 1; x := 0; // no coupons yet
while (x < N) {
   while (cp[i] != 0) {
      i := uniform(1..N) // next coupon
   }
   cp[i] := 1;  // coupon i obtained
   x++;   // one coupon less to go
}
```

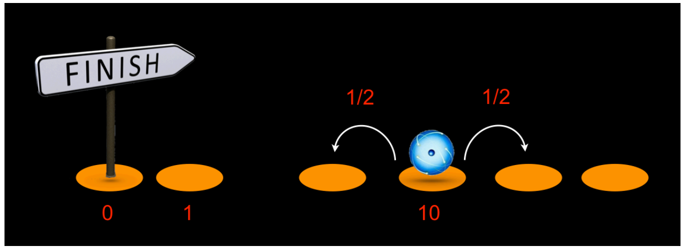
Using the ert-calculus one can prove that:

$$ert(cpcl, \mathbf{0}) \ = \ \mathbf{4} + [N > 0] \cdot 2N \cdot (2 + H_{N-1}) \ \in \ \Theta(N \cdot \log N)$$

As Harmonic number $H_{N-1} \in \Theta(\log N)$.

By systematic program verification. Machine checkable.

Random walk



Using the ert-calculus one can prove that its expected runtime is ∞.

By systematic formal verification. Machine checkable.

Randomised binary search

```
proc BinSearch {
 mid := Unif(left, right); // pick mid uniformly
 if (left < right) {
     if (A[mid] < val) {
         left := min(mid+1, right);
         call BinSearch
     } else {
         if (A[mid] > val) {
         right := max(mid-1, left);
         call BinSearch
     } else { skip }
 } else { skip }
}
```

Using the ert-calculus one can prove that its expected runtime is $\Theta(\log N)$.

By systematic formal verification. Machine checkable.