Exercise Sheet 3

General remarks:

- **Due date:** November 9th (before the exercise class).
- You can hand in your solutions at the start of the exercise class or via L2P. Please remember to provide your matriculation number. We kindly ask you to hand in your solutions in groups of **three**.
- Solutions must be written in English.
- While we will publish sketches of exercise solutions, we do *not* guarantee that these sketches contain all details that are necessary to properly solve an exercise. Hence, it is recommended to attend the exercise classes.
- If you have any questions regarding the lecture or the exercise, feel free to write us an email or visit us at the chair.

Exercise 1 (Complete lattices, complete lattices everywhere!) **20%**

Which of the pairs below are complete lattices? Justify your answer.

(a) [5%] $(\mathbb{N}, \leq)$.

(b) [5%] $(2^D, \subseteq)$ for every non-empty set D.

(c) [5%] $(D, \sqsupseteq)$ for every complete lattice $(D, \sqsubseteq)$.

(d) [5%] $(\mathbb{Z} \times \mathbb{Z} \cup \{-\infty, \infty\}, \sqsubseteq)$, where, for all $u, v, u', v' \in \mathbb{Z}$, $-\infty \sqsubseteq (u, v)$, $(u, v) \sqsubseteq \infty$, and $(u, v) \sqsubseteq (u', v')$ iff $u \leq u'$ and $v \leq v'$.

Exercise 2 (Monotonicity on Chains) **15%**

Let $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$ be complete lattices, $F\colon D \to D'$ monotonic, and $S \subseteq D$ a chain in D.

(a) [5%] Show that $F(S) = \{F(d) \mid d \in S\}$ is a chain in D'.

(b) [10%] Show that $\bigsqcup F(S) \sqsubseteq' F\left(\bigsqcup S\right)$.

Exercise 3 (Continuity and Weakest Preconditions) **20%**

Recall from lecture 5, slide 19, the function $\Phi\colon \mathbb{P} \to \mathbb{P}$ given by

$$\Phi(X) \;=\; (G \wedge \mathtt{wp}(P, X)) \vee (\neg G \wedge F) \;,$$

where P is a GCL program. Show that Φ is continuous on the complete lattice $(\mathbb{P}, \sqsubseteq)$.

Exercise 4 (Pointwise Ordering) **45%**

Let $(D, \subseteq)$ be a complete lattice. Moreover, we denote the set of all functions $f : D \to D$ by $D \to D$. We define an order $\sqsubseteq$ on functions in $D \to D$ by setting

$$f \sqsubseteq g \text{ if and only if } \forall d \in D\colon f(d) \subseteq g(d) \;.$$

(a) [10%] Show that $(D \to D, \sqsubseteq)$ is a partial order.

(b) [10%] Show that $(D \to D, \sqsubseteq)$ is a complete lattice.

(c) [25%] Show that computing least fixed points of chains is "continuous". That is, show that for every non-empty chain $\mathcal{F}$ of continuous functions in $D \to D$ it holds

$$\mathrm{lfp}\left(\bigsqcup \mathcal{F}\right) \;=\; \bigcup \{\mathrm{lfp}(f) \mid f \in \mathcal{F}\} \;.$$