Bonus Exercise Sheet

General remarks:

- This is an **optional** assignment, which you may solve in order to get up to 100 bonus points in order to be admitted to the exam.

- All tasks in this assignment are comparable in difficulty to exam questions.

- **Due date:** January 11$^{\text{th}}$ (before the exercise class).

- Solutions must be written in English; please submit in groups of three.

Exercise 1 (Probabilistic Guarded Command Language) $\boxed{\textbf{10\%}}$

Write a PGCL program implementing the following specification of a robot moving an object in two-dimensional space.

- The object's position is determined by two integers x and y.
- The initial position of the object is given by $x = 10$ and $y = 20$.
- The robot can perform four actions:
  - (a) The x-coordinate is reduced by five.
  - (b) The x-coordinate is doubled.
  - (c) The y-coordinate is reduced by two.
  - (d) The y-coordinate is increased by one.
- The robot moves the object until $x \leq 0$ and $y \leq 0$.
- In every move, exactly one action is performed such that in total
  - action (a) is performed with probability $1/2$,
  - action (b) is performed with probability $1/3$,
  - action (c) is performed with probability $1/10$, and
  - action (d) is performed with probability $1/15$.

Exercise 2 (Weakest Pre-expectations) 20%
    Consider the following PGCL program P.

```
x := 1;
y := 0;
while(y = 0) {
    x := 2·x;
    { y := 0 } [1/4] { y := 1 }
}
```
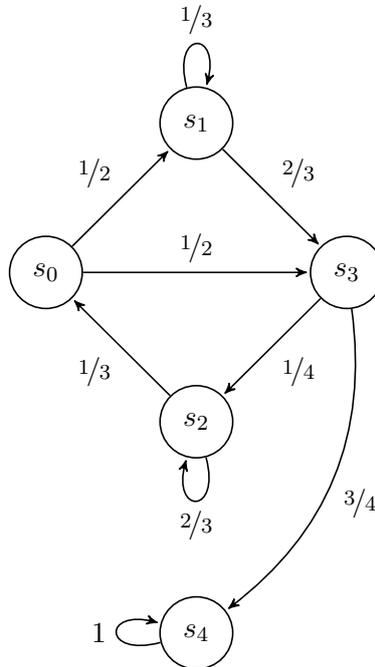
Prove that $wp[P](x) = 3$.

Hint: Recall that for all $0 < p < 1$, we have $\sum_{k=0}^{\infty} 1/p^k = 1/1-p$.

Exercise 3 (Markov Chains)   10%

Consider the following Markov chain.

Compute the probability of eventually reaching state s_2 when starting in state s_0.

Exercise 4 (Operational Semantics)   20%

Consider the following probabilistic program P:

$$x := 0; y := 0;$$

```
while(x < 2) {
    { y := 1 } [1/2] { y := 2 };
    { x := 1 } [1/4] { x := 2 };
    observe(x > y)
}
```

(a) Construct the Markov chain corresponding to P.

(b) Compute the expected value of $x + y$ after termination of P.

Exercise 5 (Invariants) $\boxed{20\%}$

Consider the following PGCL program P_{game}:

```
1 :    while (c = 1) {
2 :        {c := 0}[1/3]{c := 1};
3 :        if (c = 1) {w := w + s; } else {skip}
4 :    }
```

where s is a fixed natural number.

(a) Show that

$$I \triangleq [c \neq 1] \cdot w + 1/3 \cdot [c = 1] \cdot \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^i \cdot (w + i \cdot s)$$

is a wp-superinvariant of P_{game} with respect to expectation w. What does this tell you about the expected value of w after execution of P_{game}?

(b) Propose an invariant to derive a (non-trivial) lower bound of $wp(P_{game}, w)$. Justify your proposal. A formal proof is not required.

Exercise 6 (Conditioning) 20%

Use the `cwp` calculus to compute the expected value of y after executing the following PGCL program C:

$$
\begin{aligned}
&1: \quad a := 0;\ b := 0;\ x := 2;\ y := 0; \\
&2: \quad \texttt{if } (x > 0)\ \{ \\
&3: \qquad\quad \{a := 0\}[1/3]\{a := 1\}; \\
&4: \qquad\quad \{b := 1\}[1/4]\{b := 2\}; \\
&5: \qquad\quad \texttt{observe}(a + b > 1); \\
&6: \qquad\quad x := x - 1;\ y := y + a + b; \\
&7: \quad \}\ \texttt{else } \{ \\
&8: \qquad\quad \texttt{skip} \\
&9: \quad \}
\end{aligned}
$$

Hint: You may write C_{ij} to refer to the subprogram of C from line i to j.