Modeling and Verification of Probabilistic Systems

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http://moves.rwth-aachen.de/teaching/ws-1819/movep18/

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Overview

$Pr(\Diamond G)$

$Pr(s \models \Box\Diamond G)$

$Pr(s \models \Diamond\Box G)$

ω-regular properties

$\Box\Diamond F \wedge \Diamond\Box H$

Summary of previous lectures

Reachability probabilities

Can be obtained as a unique solution of a linear equation system.

Reachability probabilities are pivotal

1. Repeated reachability
   - = Reachability of the BSCCs containing a goal state
2. Persistence
   - = Reachability of the BSCCs only containing goal states

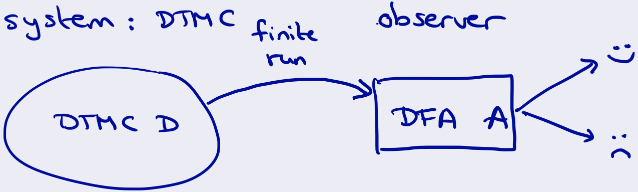
$\Diamond\Box G$

Aim of this lecture

Reachability probabilities = key to determine the probability of any ω-regular property. (This includes all linear temporal logic formulas.)

Major steps for Markov chain $\mathcal{D}$

1. Consider first a simple class of properties: regular safety properties.
2. All traces refuting such property P are recognized by a deterministic finite-state automaton $\mathcal{A}$.



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Major steps for Markov chain $\mathcal{D}$

1. Consider first a simple class of properties: regular safety properties.
2. All traces refuting such property P are recognized by a deterministic finite-state automaton $\mathcal{A}$.
3. Probability of P = reachability probability in a product of $\mathcal{D}$ and $\mathcal{A}$.
4. What are ω-regular properties?
5. All traces satisfying such property P are recognized by a deterministic Rabin automaton $\mathcal{A}$.

$$\text{DTMC } D \xrightarrow{\text{infinite run}} \boxed{\text{D}\underline{\text{R}}\text{A } A}$$

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Major steps for Markov chain $\mathcal{D}$

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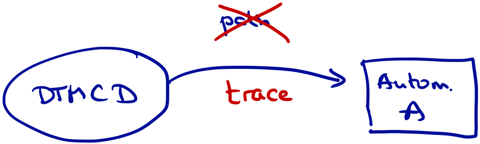
Overview

Paths and traces

Paths

A *path* in DTMC $\mathcal{D}$ is an infinite sequence of states $s_0 s_1 s_2 \ldots\ldots$ with $\mathbf{P}(s_i, s_{i+1}) > 0$ for all i.

Let $Paths(\mathcal{D})$ denote the set of paths in $\mathcal{D}$, and $Paths^*(\mathcal{D})$ the set of finite prefixes thereof.



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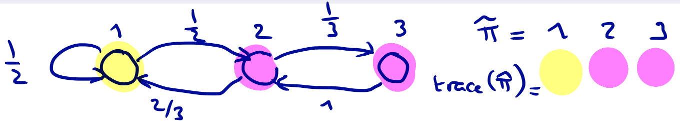
Traces

The *trace* of path $\pi = s_0\, s_1\, s_2 \ldots$ is $trace(\pi) = \underbrace{L(s_0)}_{\text{set of elts. in AP}}\, L(s_1)\, L(s_2) \ldots$.

$L : S \longrightarrow 2^{AP}$



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Traces

The *trace* of path $\pi = s_0\, s_1\, s_2 \ldots$ is $trace(\pi) = L(s_0)\, L(s_1)\, L(s_2) \ldots$.
The trace of finite path $\widehat{\pi} = s_0\, s_1 \ldots s_n$ is $trace(\widehat{\pi}) = L(s_0)\, L(s_1) \ldots L(s_n)$.

The *set of traces* of a set Π of paths: $trace(\Pi) = \{\, trace(\pi) \mid \pi \in \Pi \,\}$.

LT properties

P is a set of infinite traces

Linear-time property

A *linear-time property* (LT property) over AP is a subset of $\left(2^{AP}\right)^{\omega}$.

LT properties

Linear-time property

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LT properties

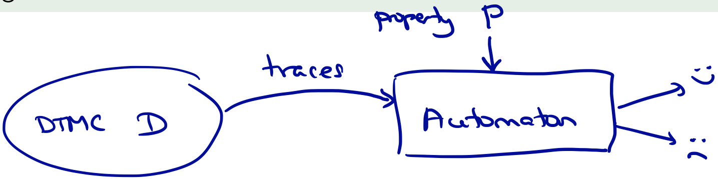
P

Linear-time property

A *linear-time property* (LT property) over AP is a subset of $\left(2^{AP}\right)^{\omega}$. An LT-property is thus a set of infinite traces over 2^{AP}.

Intuition

An LT-property gives the admissible behaviours of the DTMC at hand.



Probability of LT properties

Probability of LT properties

The *probability* for DTMC $\mathcal{D}$ to exhibit a trace in property P (over AP) is:

$$Pr^{\mathcal{D}}(P) \;=\; Pr^{\mathcal{D}}\{\, \pi \in Paths(\mathcal{D}) \mid trace(\pi) \in P \,\}.$$

cylinder sets

1 2 3

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For state s in $\mathcal{D}$, let $Pr(s \models P) \;=\; Pr_s\{\, \pi \in Paths(s) \mid trace(\pi) \in P \,\}$.

D with initial state s

Probability of LT properties

assume this is measurable

Probability of LT properties

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We do not address measurability of P yet. We will later identify a rich set P of LT-properties—those that include all LTL formulas—for which the set of paths $\{\,\pi \in Paths(\mathcal{D}) \mid trace(\pi) \in P\,\}$ is measurable.

Safety properties

Safety properties

set of infinite traces

Safety property

LT property P_{safe} over AP is a *safety property* if for all $\sigma \in \underline{(2^{AP})^\omega \setminus P_{safe}}$ there exists a finite prefix $\widehat{\sigma}$ of σ such that:

$\overline{P_{safe}}$

$$P_{safe} \cap \underbrace{\left\{ \sigma' \in (2^{AP})^\omega \mid \widehat{\sigma} \text{ is a prefix of } \sigma' \right\}}_{\text{all possible extensions of } \widehat{\sigma}} = \varnothing.$$

no extension of $\widehat{\sigma}$ is a trace in P_{safe}

$\widehat{\sigma}$ = bad prefix

Safety properties

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Any such finite word $\widehat{\sigma}$ is called a *bad prefix* for P_{safe}.

Safety properties

DTMC D finite trace regular safety prop. P

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Regular safety property

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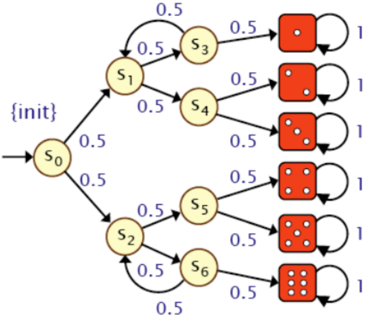
Regular safety property

A safety property is *regular* if its set of bad prefixes constitutes a regular language (over the alphabet 2^{AP}). Thus, the set of all bad prefixes of a regular safety property can be represented by a finite-state automaton.

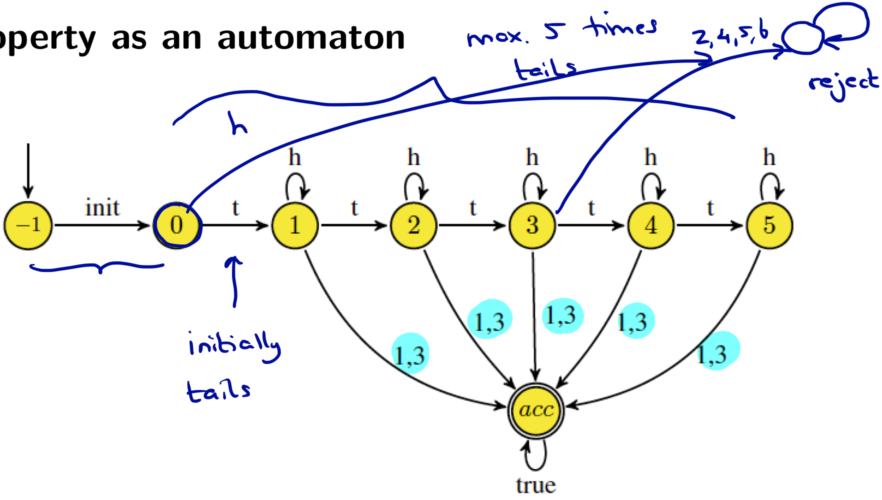
Property of Knuth's die



Property of Knuth's die

After initial tails, yield 1 or 3 but with maximally five time tails.

Property as an automaton



After initial tails, yield 1 or 3 but with at most five times tails in total

Overview

Probability of a regular safety property

Let $\mathcal{A} = (Q, 2^{AP}, \delta, q_0, F)$ be a deterministic finite-state automaton (DFA) for the bad prefixes of regular safety property P_{safe}:

$$P_{safe} = \{ \underbrace{A_0\, A_1\, A_2 \ldots \in (2^{AP})^{\omega}}_{\text{traces}} \mid \underbrace{\forall n \geqslant 0.\, A_0\, A_1 \ldots A_n \notin \mathcal{L}(\mathcal{A})}_{\text{no prefix is bad}} \}.$$

alphabet of the automaton = 2^{AP}

bad prefixes

finite trace

DTMC $\mathcal{D}$

DFA $\mathcal{A}$

$\underbrace{L(s_1)}_{} L(s_2) \ldots L(s_n)$

$\in 2^{AP}$ = input symbol of $\mathcal{A}$

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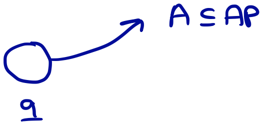
Let δ be total, i.e., $\underline{\delta(q, A)}$ is defined for each $A \subseteq AP$ and state $q \in Q$.



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Let δ be total, i.e., $\delta(q, A)$ is defined for each $A \subseteq AP$ and state $q \in Q$. Furthermore, let $\mathcal{D} = (S, \mathbf{P}, \iota_{\mathrm{init}}, AP, L)$ be a finite DTMC. Our interest is to compute the probability

$$\underbrace{Pr^{\mathcal{D}}(P_{safe})} \; = \; 1 \; - \underbrace{\sum_{s \in S} \iota_{\mathrm{init}}(s) \cdot \overbrace{Pr(s \models \mathcal{A})}^{\text{Pr (A accepts if D starts in s)}}}_{\text{1 − pr. A is accepting}} \quad \text{where}$$

Probability of a regular safety property

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$$Pr(s \models \mathcal{A}) \quad = \quad Pr^{\mathcal{D}}_{s}\{ \pi \in Paths(s) \mid trace(\pi) \notin P_{safe} \}.$$

These probabilities can be obtained by considering a product of DTMC $\mathcal{D}$ with DFA $\mathcal{A}$.

Probability of a regular safety property

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$$Pr(s \models \mathcal{A}) \;\;=\;\; Pr_s^{\mathcal{D}}\{\, \pi \in Paths(s) \mid trace(\pi) \notin P_{safe} \,\}.$$

Remark

The value $Pr(s \models \mathcal{A})$ can be written as the (possibly infinite) sum:

$$Pr(s \models \mathcal{A}) \;=\; \sum_{\widehat{\pi}} \mathbf{P}(\widehat{\pi})$$

where $\widehat{\pi}$ ranges over all finite path prefixes $s_0\, s_1 \ldots s_n$ with $s_0 = s$ and:

1. $trace(s_0\, s_1 \ldots s_n) \;=\; L(s_0)\, L(s_1) \ldots L(s_n) \;\in \mathcal{L}(\mathcal{A})$, and
2. the length of $\widehat{\pi}$ is minimal, i.e., $trace(s_0\, s_1 \ldots s_i) \notin \mathcal{L}(\mathcal{A})$ for all $0 \leqslant i < n$.

Product construction: intuition

DTMC $\mathcal{D}$
with state space S

F
D~~R~~A $\mathcal{A}$
with state space Q

$\in 2^{AP}$

s_0	$L(s_0)=A_0$	$q_0 \in Q_0$
$\downarrow$		$\downarrow A_0$
s_1	$L(s_1)=A_1$	q_1
$\downarrow$		$\downarrow A_1$
s_2	$L(s_2)=A_2$	q_2
$\downarrow$		$\downarrow A_2$
$\vdots$	$\vdots$	$\vdots$
$\downarrow$		$\downarrow$
s_n	$L(s_n)=A_n$	q_n
		$\downarrow A_n$
		q_{n+1}
path	trace	

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D~~R~~A $\mathcal{A}$
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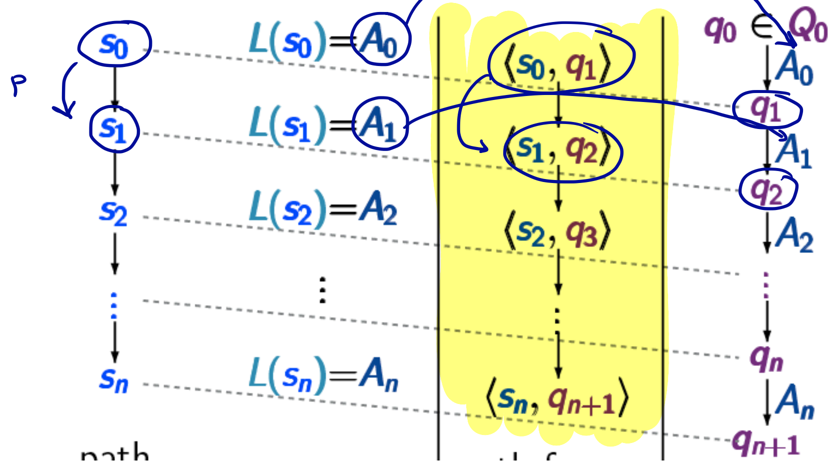

product $\mathcal{D} \otimes \mathcal{A}$

Product Markov chain

Product Markov chain

Let $\mathcal{D} = (S, \mathbf{P}, \iota_{\text{init}}, AP, L)$ be a DTMC and $\mathcal{A} = (Q, 2^{AP}, \delta, q_0, F)$ be a DFA. The *product* $\mathcal{D} \otimes \mathcal{A}$ is the DTMC:

$$\mathcal{D} \otimes \mathcal{A} \;=\; (S \times Q, \mathbf{P}', \iota'_{\text{init}}, \{\, accept \,\}, L')$$

where $L'(\langle s, q \rangle) = \{\, accept \,\}$ if $q \in F$ and $L'(\langle s, q \rangle) = \varnothing$ otherwise,

$L' : S \times A \longrightarrow$

$q \notin F$

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$$\iota'_{\mathrm{init}}(\langle s, q \rangle) \;=\; \begin{cases} \iota_{\mathrm{init}}(s) & \text{if } q = \delta(q_0, L(s)) \\ 0 & \text{otherwise.} \end{cases}$$

state of D

Product Markov chain

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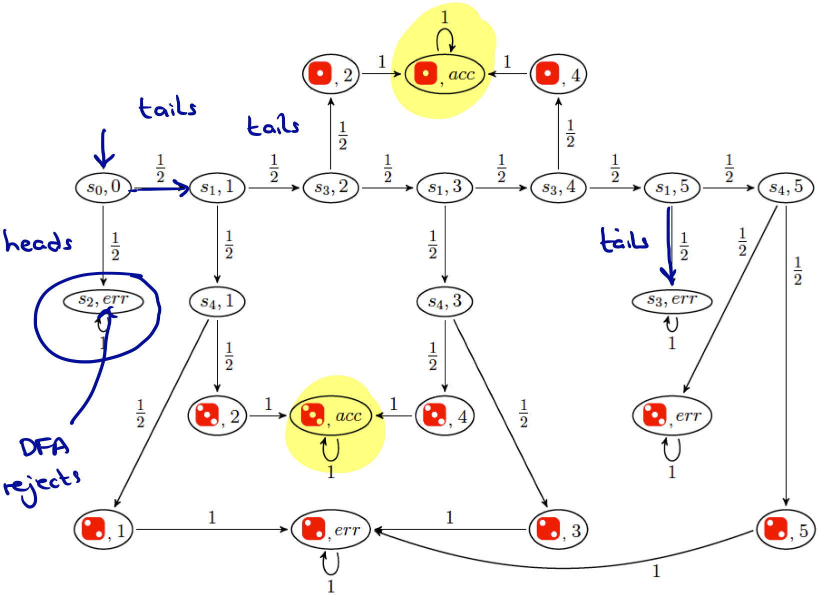
The transition probabilities in $\mathcal{D} \otimes \mathcal{A}$ are given by:

$$\mathbf{P}'(\langle s, q \rangle, \langle s', q' \rangle) \;=\; \begin{cases} \mathbf{P}(s, s') & \text{if } q' = \delta(q, L(s')) \\ 0 & \text{otherwise.} \end{cases}$$

Example product: Knuth-Yao's die



Product Markov chain

Product Markov chain

Some observations

- For each path $\pi = s_0\, s_1\, s_2 \ldots$ in DTMC $\mathcal{D}$ there exists a unique run $q_0\, q_1\, q_2 \ldots$ in DFA $\mathcal{A}$ for $trace(\pi) = L(s_0)\, L(s_1)\, L(s_2) \ldots$ and $\pi^+ = \langle s_0, q_1 \rangle \langle s_1, q_2 \rangle \langle s_2, q_3 \rangle \ldots$ is a path in $\mathcal{D} \otimes \mathcal{A}$.
- The DFA $\mathcal{A}$ does not affect the probabilities, i.e., for each measurable set Π of paths in $\mathcal{D}$ and state s:

$$Pr_s^{\mathcal{D}}(\Pi) \;=\; Pr^{\mathcal{D}\otimes\mathcal{A}}_{\langle s, \delta(q_0, L(s))\rangle} \underbrace{\{\, \pi^+ \mid \pi \in \Pi \,\}}_{\Pi^+}$$

- For $\Pi = \{\, \pi \in Paths^{\mathcal{D}}(s) \mid pref(trace(\pi)) \cap \mathcal{L}(\mathcal{A}) \neq \varnothing \,\}$, the set Π^+ is given by:

$$\Pi^+ = \{\, \pi^+ \in Paths^{\mathcal{D}\otimes\mathcal{A}}(\langle s, \delta(q_0, L(s))\rangle) \mid \pi^+ \models \Diamond accept \,\}.$$

Quantitative analysis of regular safety properties

Quantitative analysis of regular safety properties

Theorem for analysing regular safety properties

Let P_{safe} be a regular safety property, $\mathcal{A}$ a DFA for the set of bad prefixes of P_{safe}, $\mathcal{D}$ a DTMC, and s a state in $\mathcal{D}$. Then:

label

$$\underbrace{Pr^{\mathcal{D}}(s \models P_{safe})}_{\text{D starts in s and satisfies } P_{safe}} \quad = \quad \underbrace{Pr^{\mathcal{D}\otimes\mathcal{A}}(\langle s, q_s \rangle \not\models \lozenge accept)}_{\text{in the product } D \otimes A}$$

all bad prefixes of Psafe = regular language

= language accepted

by a DFA A

Quantitative analysis of regular safety properties

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$$
\begin{aligned}
Pr^{\mathcal{D}}(s \models P_{safe}) &= Pr^{\mathcal{D}\otimes\mathcal{A}}(\langle s, q_s\rangle \not\models \Diamond accept) \\
&= 1 - \underline{Pr^{\mathcal{D}\otimes\mathcal{A}}(\langle s, q_s\rangle \models \Diamond accept)}
\end{aligned}
$$

where $q_s = \delta(q_0, L(s))$.

DTMC $\mathcal{D}$

$s = L(s)$

DFA $\mathcal{A}$

q_0

q_s

Quantitative analysis of regular safety properties

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then $\mathcal{D} \otimes \mathcal{A}$ is again a Markov chain

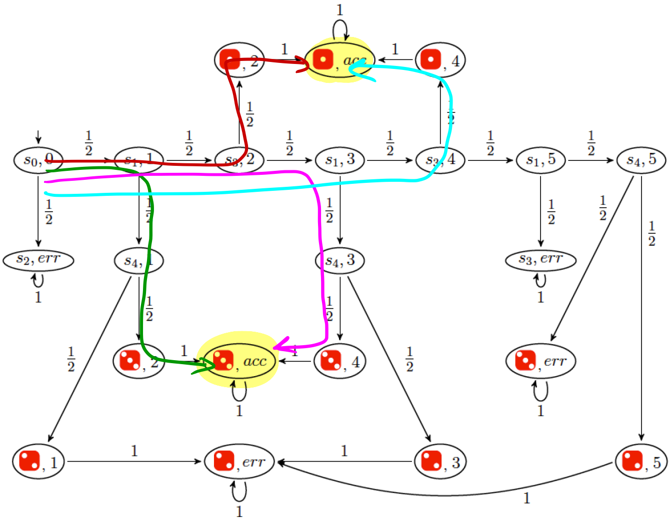
$$\begin{aligned} Pr^{\mathcal{D}}(s \models P_{safe}) &= Pr^{\mathcal{D}\otimes\mathcal{A}}(\langle s, q_s \rangle \not\models \Diamond accept) \\ &= 1 - Pr^{\mathcal{D}\otimes\mathcal{A}}(\langle s, q_s \rangle \models \Diamond accept) \end{aligned}$$

where $q_s = \delta(q_0, L(s))$.

Remarks

1. For finite DTMCs, $Pr^{\mathcal{D}}(s \models P_{safe})$ can thus be computed by determining reachability probabilities of *accept* states in $\mathcal{D} \otimes \mathcal{A}$. This amounts to solving a linear equation system.
2. For qualitative regular safety properties, i.e., $Pr^{\mathcal{D}}(s \models P_{safe}) > 0$ and $Pr^{\mathcal{D}}(s \models P_{safe}) = 1$, a graph analysis of $\mathcal{D} \otimes \mathcal{A}$ suffices.

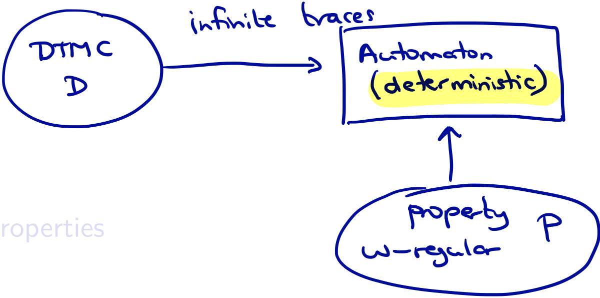
Determining the property's probability



$$Pr^{\mathcal{D}\otimes\mathcal{A}}(\langle s, q_s\rangle \models \Diamond accept) \text{ equals } \frac{1}{8} + \frac{1}{8} + \frac{1}{32} + \frac{1}{32} = \frac{5}{16}.$$

Overview

1. Introduction
2. Preliminaries
3. Verifying regular safety properties
4. ω-regular properties
5. Verifying DBA objectives
6. Verifying ω-regular properties
7. Summary



$$Pr(D \models P)$$

ω-regular languages

ω-regular languages

Infinite repetition of languages

Let Σ be a finite alphabet.

$\Sigma = \{a, b, c\}$

ω-regular languages

finite words over Σ

Infinite repetition of languages

Let Σ be a finite alphabet. For language $\mathcal{L} \subseteq \Sigma^*$, let $\mathcal{L}^\omega$ be the set of words in $\Sigma^* \cup \Sigma^\omega$ that arise from the infinite concatenation of (arbitrary) words in Σ,

$\Sigma = \{a, b, c\}$

$aba, \; abb, \; acccb \in \Sigma^*$

$acabaaab \, aaab \ldots \rightsquigarrow (aaab)^\omega$

ω-regular languages

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$$\mathcal{L}^\omega \;=\; \left\{ w_1 w_2 w_3 \ldots \mid \underbrace{w_i \in \mathcal{L}}_{\text{finite words}}, i \geqslant 1 \right\}.$$

ω-regular languages

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The result is an *ω-language*, i.e., $\mathcal{L} \subseteq \Sigma^*$, provided that $\mathcal{L} \subseteq \Sigma^+$, i.e., $\varepsilon \notin \mathcal{L}$.

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ω-regular expression

An *ω-regular expression* G over the Σ has the form:

$\{a, b, c\}$

ω-regular languages

Infinite repetition of languages

Let Σ be a finite alphabet. For language $\mathcal{L} \subseteq \Sigma^*$, let $\mathcal{L}^\omega$ be the set of words in $\Sigma^* \cup \Sigma^\omega$ that arise from the infinite concatenation of (arbitrary) words in Σ, i.e.,

$$\mathcal{L}^\omega = \left\{ w_1 w_2 w_3 \ldots \mid w_i \in \mathcal{L}, i \geqslant 1 \right\}.$$

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$E_1 \cdot F_1^\omega + E_2 \cdot F_2^\omega + \ldots + E_n \cdot F_n^\omega$

regular expr

regular expr (but not ε)

ω-regular languages

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The *semantics* of G is defined by $\mathcal{L}_\omega(\mathsf{G}) \;=\; \mathcal{L}(\mathsf{E}_1).\mathcal{L}(\mathsf{F}_1)^\omega \cup \ldots \cup \mathcal{L}(\mathsf{E}_n).\mathcal{L}(\mathsf{F}_n)^\omega$ where $\mathcal{L}(\mathsf{E}) \subseteq \Sigma^*$ denotes the language (of finite words) induced by the regular expression E.

ω-regular expressions

$E_1 . F_1^{\omega} + E_2 . F_2^{\omega}$ $n=2$

$\underbrace{A(B+C)}_{E_1}$ $\underbrace{A}_{F_1}$ B $F_2 = A+C$

ω-regular expression

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Example

Examples for ω-regular expressions over the alphabet $\Sigma = \{ A, B, C \}$ are

$$\underbrace{(A+B)^* A(AAB+C)^{\omega}} \quad \text{or} \quad A(B+C)^* A^{\omega} + B(A+C)^{\omega}.$$

$n=1$ $E_1 \cdot F_1^{\omega}$ $E_1 = (A+B)^* \cdot A$

$F_1 = AAB+C$

ω-regular properties

ω-regular properties

ω-regular property

LT property P over AP is called *ω-regular* if $P = \mathcal{L}_\omega(\mathrm{G})$ for some ω-regular expression G over the alphabet 2^{AP}.

set of infinite traces

P can be represented by an ω-regular expression

↳ ω-regular

ω-regular properties

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Example

Let $AP = \{ a, b \}$. Then some ω-regular properties over AP are:

- always a, i.e., $(\{ a \} + \{ a, b \})^{\omega}$.

$$\begin{array}{ccc} E_1 & \cdot & F_1^{\omega} \\ \downarrow & & \downarrow \\ \varepsilon & & \{a\} + \{a,b\} \end{array}$$

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- eventually a, i.e., $\underbrace{\underbrace{(\varnothing + \{ b \})^*}_{\text{no a's}}.\underbrace{(\{ a \} + \{ a, b \})}_{\text{one a}}}_{E_1}.(\underbrace{\underbrace{2^{AP}}_{\text{anything}}}_{F_1})^\omega$.

$\Diamond a$

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- eventually a, i.e., $(\varnothing + \{ b \})^* . (\{ a \} + \{ a, b \}) . (2^{AP})^\omega$.
- infinitely often a, i.e., $(\underbrace{(\varnothing + \{ b \})^* . (\{ a \} + \{ a, b \})}_{F_1})^\omega$. $E_1 = \varepsilon$

$\Box\Diamond a$

$\underbrace{\longleftrightarrow}_{\text{no } a\text{'s}}$ a $\underbrace{\longleftrightarrow}_{\text{no } a\text{'s}}$ a $\underbrace{\longleftrightarrow}_{\text{no } a\text{'s}}$ a

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- always a, i.e., $(\{ a \} + \{ a, b \})^\omega$.
- eventually a, i.e., $(\varnothing + \{ b \})^*.(\{ a \} + \{ a, b \}).(2^{AP})^\omega$.
- infinitely often a, i.e., $((\varnothing + \{ b \})^*.(\{ a \} + \{ a, b \}))^\omega$.
- from some moment on, always a, i.e., $(2^{AP})^*.(\{ a \} + \{ a, b \})^\omega$.

$\Diamond \Box a$

ω-regular properties

ω-regular property

LT property P over AP is called *ω-regular* if $P = \mathcal{L}_\omega(\mathrm{G})$ for some ω-regular expression G over the alphabet 2^{AP}.

Example

Any regular safety property P_{safe} is an ω-regular property. This follows from the fact that the complement language

$$\left(2^{AP}\right)^\omega \setminus P_{safe} \;=\; \underbrace{\overbrace{BadPref(P_{safe})}^{F_1}}_{\text{regular}} . \overbrace{\left(2^{AP}\right)^\omega}^{F_1}$$

is an ω-regular language, and ω-regular languages are closed under complement.

ω-regular properties

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LT property P over AP is called *ω-regular* if $P = \mathcal{L}_\omega(\mathrm{G})$ for some ω-regular expression G over the alphabet 2^{AP}.

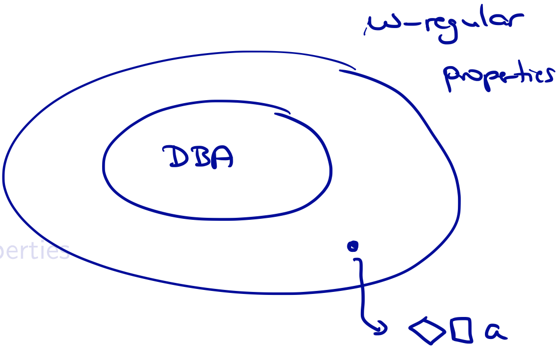
Example

Starvation freedom in the sense of "whenever process $\mathcal{P}$ is waiting then it will enter its critical section eventually" is an ω-regular property as it can be described by

$$\big((\neg wait)^*.wait.\mathrm{true}^*.crit\big)^\omega \;+\; \big((\neg wait)^*.wait.\mathrm{true}^*.crit\big)^*.(\neg wait)^\omega$$

Intuitively, the first summand stands for the case where $\mathcal{P}$ requests and enters its critical section infinitely often, while the second summand stands for the case where $\mathcal{P}$ is in its waiting phase only finitely many times.

Overview



Deterministic Büchi automata

Deterministic Büchi automata

Deterministic Büchi Automaton (DBA)

A *deterministic Büchi automaton* (DBA) $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ with

- Q is a finite set of states with initial state $q_0 \in Q_0$,
- Σ is an alphabet,
- $\delta : Q \times \Sigma \to Q$ is a transition function,
- $F \subseteq Q$ is a set of *accept* (or: final) states.

à la DFAs

A *run* for $\sigma = A_0 A_1 A_2 \ldots \in \Sigma^\omega$ denotes an infinite sequence $q_0\, q_1\, q_2 \ldots$ of states in $\mathcal{A}$ such that $q_0 \in Q_0$ and $q_i \xrightarrow{A_i} q_{i+1}$ for $i \geqslant 0$.

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Run $q_0\, q_1\, q_2 \ldots$ is *accepting* if $q_i \in F$ for infinitely many indices $i \in \mathbb{N}$.

The infinite *language* of $\mathcal{A}$ is

$$\mathcal{L}_\omega(\mathcal{A}) \;=\; \{\, \sigma \in \Sigma^\omega \mid \text{there exists an accepting run for } \sigma \text{ in } \mathcal{A} \,\}.$$

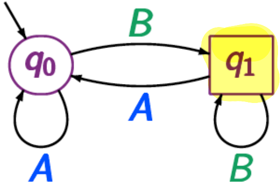
Deterministic Büchi automata for LT properties

$B A A^{10} B$

$Q = \{ q_0, q_1 \}$

$L_\omega(A) = \cancel{\{ B^\omega \}}$

$(A^* . B)^\omega$



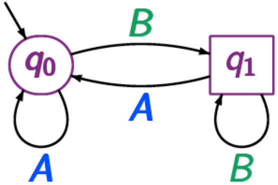
runs =

$\{ q_0 q_0 q_0 \ldots$

$q_0 q_0 q_1 q_1 \ldots q_1 \ldots$

DBA over $\underbrace{\{ A, B \}}_{= \Sigma}$ with $F = \{ q_1 \}$ and initial state q_0

Deterministic Büchi automata for LT properties



DBA over $\{ A, B \}$ with $F = \{ q_1 \}$ and initial state q_0 accepting the LT property "infinitely often B".

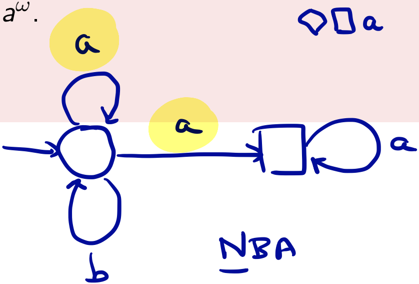
Some facts about DBA

Expressiveness of DBA

For any DBA $\mathcal{A}$, the language $\mathcal{L}_\omega(\mathcal{A})$ is ω-regular.

There does not exist a DBA over the alphabet $\Sigma = \{ a, b \}$ for the ω-regular expression $(a + b)^*.a^\omega$.

$\Diamond \Box a$



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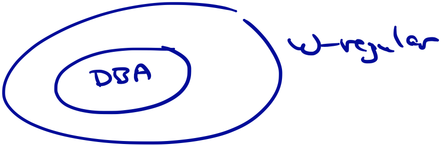
The class of DBA-recognizable languages is a proper subclass of the class of ω-regular languages



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The class of DBA-recognizable languages is a proper subclass of the class of ω-regular languages and is not closed under complementation.

An ω-language is recognizable by a DBA iff it is the limit language of a regular language. (Details: see lecture Applications of Automata Theory.)

let $L \subseteq \Sigma^*$ for alphabet Σ

$w \in \Sigma^\omega$ is in the <u>Limit</u> of L if and only if

$$|\mathrm{pref}(w) \cap L| = \infty$$

Thus: for arbitrary n, there is a $u \in L$ such that $|u| > n$ with $u \in \mathrm{pref}(w)$

<u>Lemma</u> $L = L_\omega(A)$ for some DBA A

if and only if

L is the <u>Limit</u> of some regular lang.

<u>Proof</u>: let A be a DBA and A' the corresponding DFA. Claim $L_\omega(A) =$ Limit of $L(A')$.

$w \in \Sigma^\omega$ is accepted by DBA A iff some final state in A is visited infinitely often. This holds iff ∞ many prefixes of w are accepted by A'. Hence,

$L_\omega(A) =$ Limit of $L(A')$ $\boxtimes$

Quantitative analysis of DBA properties

Quantitative analysis of DBA properties

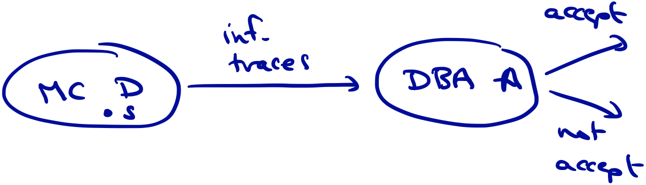
DFA $\Diamond$ accept

Quantitative Analysis for DBA-Definable Properties

Let $\mathcal{A}$ be a DBA and $\mathcal{D}$ a DTMC. Then, for all states s in $\mathcal{D}$:

$$Pr^{\mathcal{D}}(s \models \mathcal{A}) \;=\; \underbrace{Pr^{\mathcal{D}\otimes\mathcal{A}}}_{\text{product MC}}(\langle s, q_s\rangle \models \Box\Diamond accept)$$

where $q_s = \delta(q_0, L(s))$.



Quantitative analysis of DBA properties

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where $q_s = \delta(q_0, L(s))$.

Algorithm

Recall that for finite DTMCs, the probability of $\Box\Diamond accept$ can be obtained in polynomial time by first determining the BSCCs of $\mathcal{D} \otimes \mathcal{A}$.

Quantitative analysis of DBA properties

Quantitative Analysis for DBA-Definable Properties

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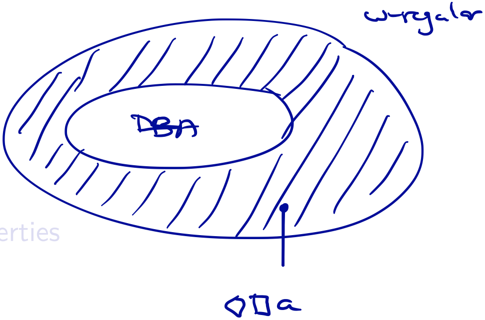
where $q_s = \delta(q_0, L(s))$.

a BSCC that contains
≥ 1 accept state.

Algorithm

Recall that for finite DTMCs, the probability of $\Box\Diamond accept$ can be obtained in polynomial time by first determining the BSCCs of $\mathcal{D} \otimes \mathcal{A}$. For each BSCC B that contains a state $\langle s, q \rangle$ with $q \in F$, determine the probability of eventually reaching B. Its sum is the required probability. Thus this amounts to solve a linear equation system for each "accepting" BSCC in $\mathcal{D}$.

Overview



Beyond DBA properties

Beyond DBA properties

Remarks

- Since DBAs do not have the full power of ω-regular languages, this approach is not capable of handling arbitrary ω-regular properties.
- To overcome this deficiency, Büchi automata will be replaced by an alternative automaton model for which their deterministic counterparts are as expressive as ω-regular languages.
- Such automata have the same components as DBA (finite set of states, and so on) except for the acceptance sets. We consider *deterministic Rabin automata*.

alternative
Muller /
Street

Beyond DBA properties

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- Since DBAs do not have the full power of ω-regular languages, this approach is not capable of handling arbitrary ω-regular properties.
- To overcome this deficiency, Büchi automata will be replaced by an alternative automaton model for which their deterministic counterparts are as expressive as ω-regular languages.
- Such automata have the same components as DBA (finite set of states, and so on) except for the acceptance sets. We consider *deterministic Rabin automata*. There are alternatives, e.g., Muller automata.
- Determinism is important to stay within the realm of Markov chains; a product of an MC with a deterministic automaton yields a MC.

Deterministic Rabin automata

Michael ~~Robin~~

Deterministic Rabin automata

$\mathcal{F} \subseteq 2^Q \times 2^Q$

Deterministic Rabin automaton

A *deterministic Rabin automaton* (DRA) $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F})$ with

- Q, $q_0 \in Q_0$, Σ is an alphabet, and $\delta : Q \times \Sigma \to Q$ as before
- $\mathcal{F} = \{ (L_i, K_i) \mid 0 < i \leqslant k \}$ with $L_i, K_i \subseteq Q$, is a set of *accept pairs*

A *run* for $\sigma = A_0 A_1 A_2 \ldots \in \Sigma^\omega$ denotes an infinite sequence $q_0\, q_1\, q_2 \ldots$ of states in $\mathcal{A}$ such that $q_0 \in Q_0$ and $q_i \xrightarrow{A_i} q_{i+1}$ for $i \geqslant 0$.

Run $q_0\, q_1\, q_2 \ldots$ is *accepting* if for some pair (L_i, K_i), the states in L_i are visited finitely often and the states in K_i infinitely often. That is, an accepting run should satisfy

$$\bigvee_{0<i\leqslant k} (\underbrace{\Diamond\Box\neg L_i}_{\text{finitely often } L_i} \wedge \underbrace{\Box\Diamond K_i}_{\text{infinitely often } K_i}).$$

When does a DRA accept an infinite word?

Acceptance condition

A run of a word in Σ^ω on a DRA is accepting if and only if:

for some $(L_i, K_i) \in \mathcal{F}$, the states in L_i are visited finitely often

and (some of) the states in K_i are visited infinitely often

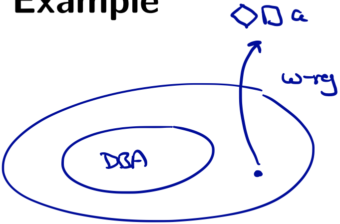
Stated in terms of an LTL formula:

$$\bigvee_{0<i\leqslant k} (\Diamond\Box\neg L_i \wedge \Box\Diamond K_i)$$

accepting set of the DBA

A deterministic Büchi automaton is a DRA with acceptance condition $\{(\varnothing, F)\}$.

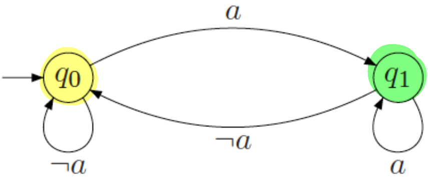
Deterministic Rabin automaton: Example



Deterministic Rabin automaton: Example

Acceptance condition

A run of a word in Σ^ω on a DRA is accepting iff $\bigvee_{0<i\leqslant k} (\Diamond \Box \neg L_i \wedge \Box \Diamond K_i)$.



For $\mathcal{F} = \{ (L, K) \}$ with $L = \{ q_0 \}$ and $K = \{ q_1 \}$, this DRA accepts $\Diamond \Box a$

Deterministic Rabin automaton: Example

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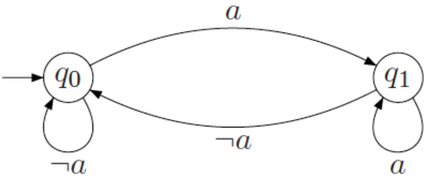


For $\mathcal{F} = \{ (L, K) \}$ with $L = \{ q_0 \}$ and $K = \{ q_1 \}$, this DRA accepts $\Diamond \Box a$

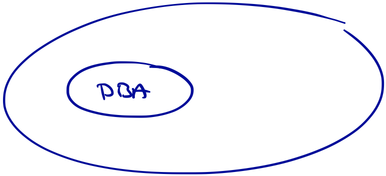
Recall that there does not exist a deterministic Büchi automaton for $\Diamond \Box a$.

Deterministic Rabin automata

Deterministic Rabin automata

DRA are ω-regular

A language on infinite words is ω-regular iff there exists a DRA that generates it.



ω-regular

=

DRA recognizeble languages

Deterministic Rabin automata

DRA are ω-regular

A language on infinite words is ω-regular iff there exists a DRA that generates it.

- DRA are thus equally expressive as nondeterministic Büchi automata.
- They are more expressive than deterministic Büchi automata.
- Any nondeterministic Büchi automata of n states can be converted to a DRA of size $2^{\mathcal{O}(n \cdot \log n)}$. (Details omitted.)

Verifying DRA properties

Verifying DRA properties

Product of a Markov chain and a DRA

The product of DTMC $\mathcal{D}$ and DRA $\mathcal{A}$ is defined as the product of a Markov chain and a DFA, except that the labeling is defined differently.

Let the acceptance condition of $\mathcal{A}$ is $\mathcal{F} = \{ (L_1, K_1), \ldots, (L_k, K_k) \}$. Then the sets L_i, K_i serve as atomic propositions in $\mathcal{D} \otimes \mathcal{A}$. The labeling function L' in $\mathcal{D} \otimes \mathcal{A}$ is the obvious one: if $H \in \{ L_1, \ldots, L_k, K_1, \ldots, K_k \}$, then $H \in L'(\langle s, q \rangle)$ iff $q \in H$.

$q \in L_3$ then $L_3 \in L'(\langle s, q \rangle)$

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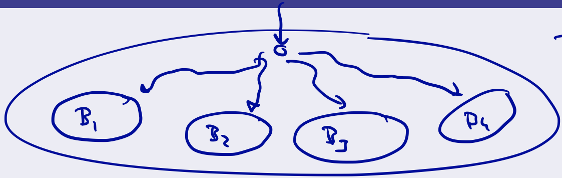
Accepting BSCC



$\mathcal{D} \otimes \mathcal{A}$

finite

$L_i \not\subseteq B_j$

$K_i \cap B_j \neq \emptyset$

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Accepting BSCC

A BSCC T in $\mathcal{D} \otimes \mathcal{A}$ is *accepting* iff for some index $i \in \{ 1, \ldots, k \}$ we have:

$$\underbrace{T \cap (S \times L_i) = \varnothing}_{\text{no } L_i\text{-state in } T} \quad \text{and} \quad \underbrace{T \cap (S \times K_i) \neq \varnothing}_{\geqslant 1\ K_i\text{-state in } T}.$$

Verifying DRA properties

Product of a Markov chain and a DRA

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$$T \cap (S \times L_i) = \varnothing \quad \text{and} \quad T \cap (S \times K_i) \neq \varnothing.$$

Thus, once such an accepting BSCC T is reached in $\mathcal{D} \otimes \mathcal{A}$, the acceptance criterion for the DRA $\mathcal{A}$ is fulfilled almost surely.

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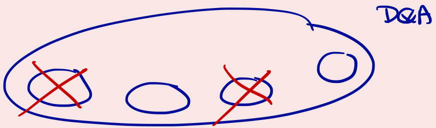

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$$Pr^{\mathcal{D}}(s \models \mathcal{A}) \; = \; Pr^{\mathcal{D} \otimes \mathcal{A}}(\, \langle s, q_s \rangle \models \Diamond U \,) \quad \text{where} \quad q_s = \delta(q_0, L(s)).$$

Proof

On the blackboard (if time permits).

Verifying DRA objectives

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DRA probabilities = reachability probabilities

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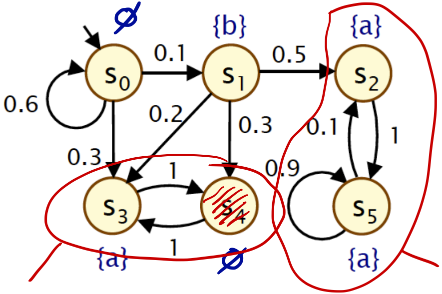
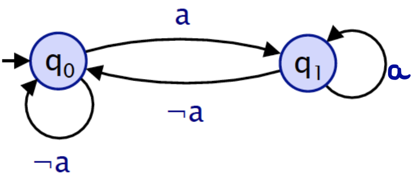
Probabilities for satisfying ω-regular properties are obtained by computing the reachability probabilities for accepting BSCCs in $\mathcal{D} \otimes \mathcal{A}$. Again, a graph analysis and solving systems of linear equations suffice. The time complexity is polynomial in the size of $\mathcal{D}$ and $\mathcal{A}$.

Example: verifying a DTMC versus a DRA

$\Box\Diamond a$





$Acc = \{ (\{q_0\},\{q_1\}) \}$

L K

BSCC B_1 "bad" = non-accepting

BSCC B_2 no $\neg a$-states $\Rightarrow$ B_2 is accepting

Single accepting BSCC: $\{ \langle s_2, q_1 \rangle, \langle s_5, q_1 \rangle \}$.

Example: verifying a DTMC versus a DRA

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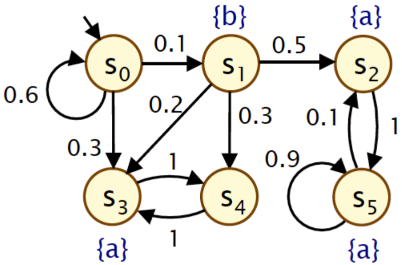

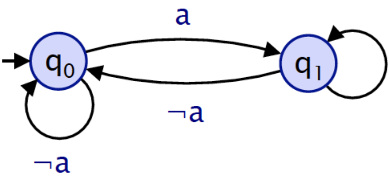

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Reachability probability is $\frac{1}{2} \cdot \frac{1}{10} \cdot \sum_{k=0}^{\infty} \left(\frac{3}{5}\right)^k = \frac{1}{8}$.

Measurability

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Measurability theorem for ω-regular properties **[Vardi 1985]**

For any DTMC $\mathcal{D}$ and DRA $\mathcal{A}$ the set

$$\{\, \pi \in \mathit{Paths}(\mathcal{D}) \mid \mathit{trace}(\pi) \in \mathcal{L}_\omega(\mathcal{A}) \,\}$$

is measurable.

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Let DRA $\mathcal{A}$ with accept sets $\{ (L_1, K_1), \ldots, (L_m, K_m) \}$. Let $\varphi_i = \Diamond \Box \neg L_i \wedge \Box \Diamond K_i$ and $\underbrace{\Pi_i \text{ the set of paths satisfying}}_{\text{accepted by } (L_i, K_i)}$ φ_i.

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↓

set of accepting paths

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$\pi \in \mathit{Paths}(\mathcal{D})$

π^+ is the corresponding path to π in $\mathcal{D} \otimes \mathcal{A}$

(this is needed, as π is regardless of $\mathcal{A}$)

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check.

Linear temporal logic

Linear temporal logic

Linear Temporal Logic: Syntax [Pnueli 1977]

LTL formulas over the set AP obey the grammar:

$$\varphi \ ::= \ a \ \Big| \ \neg\varphi \ \Big| \ \varphi_1 \wedge \varphi_2 \ \Big| \ \bigcirc \varphi \ \Big| \ \varphi_1 \, \mathsf{U} \, \varphi_2$$

where $a \in AP$ and φ, φ_1, and φ_2 are LTL formulas.

Example

On the blackboard.

LTL semantics

LTL semantics

The LT-property induced by LTL formula φ over AP is:

$$\underbrace{\mathit{Words}(\varphi)}_{\text{set of traces satisfying } \varphi} = \{ \underbrace{\sigma \in \left(2^{AP}\right)^{\omega}}_{\text{traces}} \mid \sigma \models \varphi \}, \text{ where } \models \text{ is the smallest relation satisfyin}$$

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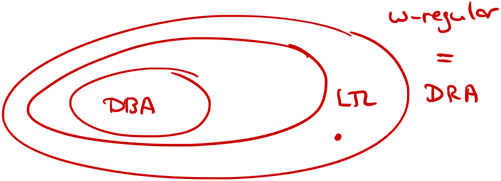
$$
\begin{array}{lllll}
\sigma & \models & \text{true} & & \\
\sigma & \models & a & \text{iff} & a \in A_0 \quad (\text{i.e., } A_0 \models a) \\
\sigma & \models & \varphi_1 \wedge \varphi_2 & \text{iff} & \sigma \models \varphi_1 \text{ and } \sigma \models \varphi_2 \\
\sigma & \models & \neg\varphi & \text{iff} & \sigma \not\models \varphi \\
\sigma & \models & \bigcirc\varphi & \text{iff} & \sigma^1 = A_1 A_2 A_3 \ldots \models \varphi \\
\sigma & \models & \varphi_1 \,\mathsf{U}\, \varphi_2 & \text{iff} & \exists j \geqslant 0.\ \sigma^j \models \varphi_2 \text{ and } \sigma^i \models \varphi_1,\ 0 \leqslant i < j
\end{array}
$$

for $\sigma = A_0 A_1 A_2 \ldots$ we have $\sigma^i = A_i A_{i+1} A_{i+2} \ldots$ is the suffix of σ from index i on.

Some facts about LTL



LTL is ω-regular

For any LTL formula φ, the set $Words(\varphi)$ is an ω-regular language.

LTL are DRA-definable

For any LTL formula φ, there exists a DRA $\mathcal{A}$ such that $\mathcal{L}_\omega = Words(\varphi)$

Some facts about LTL

LTL is ω-regular

For any LTL formula φ, the set *Words*(φ) is an ω-regular language.

LTL are DRA-definable

For any LTL formula φ, there exists a DRA $\mathcal{A}$ such that $\mathcal{L}_\omega = \mathit{Words}(\varphi)$ where the number of states in $\mathcal{A}$ lies in $2^{2^{|\varphi|}}$.

$\Longrightarrow$ size of A is double exponential in $|\varphi|$

Verifying a DTMC against LTL formulas

Complexity of LTL model checking **[Vardi 1985]**

The qualitative model-checking problem for finite DTMCs against LTL formula φ is PSPACE-complete, i.e., verifying whether $Pr(s \models \varphi) > 0$ or $Pr(s \models \varphi) = 1$ is PSPACE-complete.

Recall that the LTL model-checking problem for finite transition systems is PSPACE-complete.

Overview

Summary

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- Verifying a DTMC $\mathcal{D}$ against a DFA $\mathcal{A}$, i.e., determining $Pr(\mathcal{D} \models \mathcal{A})$, amounts to computing reachability probabilities of accept states in $\mathcal{D} \otimes \mathcal{A}$.
- For DBA objectives, the probability of infinitely often visiting an accept state in $\mathcal{D} \otimes \mathcal{A}$.
- DBA are strictly less powerful than ω-regular languages.
- Deterministic Rabin automata are as expressive as ω-regular languages.
- Verifying DTMC $\mathcal{D}$ agains DRA $\mathcal{A}$ amounts to computing reachability probabilities of accepting BSCCs in $\mathcal{D} \otimes \mathcal{A}$.

Take-home message

Model checking a DTMC against various automata models reduces to computing reachability probabilities in a product.