Modeling and Verification of Probabilistic Systems

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http://moves.rwth-aachen.de/teaching/ws-1819/movep18/

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Overview

Overview

Summary of previous lecture

Reachability probabilities

Can be obtained as a unique solution of a linear equation system.

Reachability probabilities are pivotal

The probability of satisfying an ω-regular property P in a Markov chain $\mathcal{D}$ $=$ reachability probability of accepting BSCCs in the product of $\mathcal{D}$ with a DRA for P.

Aim of this lecture

Introduce probabilistic CTL. Provide a polynomial-time model-checking algorithm for verifying a finite Markov chain against a PCTL formula.

Set up of this lecture

1. Syntax and formal semantics of probabilistic CTL.
2. Model checking algorithm for probabilistic CTL on Markov chains.
3. Time complexity analysis.

Overview

Probabilistic Computation Tree Logic

- PCTL is a language for formally specifying properties over DTMCs.

LTL formula φ $\qquad$ $Pr(D \models \varphi)$

Probabilistic Computation Tree Logic

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- It is a branching-time temporal logic (based on CTL).

↳ LTL : infinite traces (ω-regular)
PCTL : infinite trees

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- It is a branching-time temporal logic (based on CTL).
- Formula interpretation is Boolean, i.e., a formula is satisfied or not.

$Pr(\varphi) > \frac{1}{2} \quad \leq \frac{4}{5}$

Probabilistic Computation Tree Logic

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- The main operator is $\mathbb{P}_J(\varphi)$
  - where φ constrains the paths and J is a threshold on the probability.

$\varphi = \Diamond a \qquad J = [0, \frac{1}{2}]$

$\mathbb{P}_{[0,\frac{1}{2}]}(\Diamond a) =$ "$Pr\{$ all paths $\models \Diamond a \} \in [0, \frac{1}{2}]$"

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- The main operator is $\mathbb{P}_J(\varphi)$
  - where φ constrains the paths and J is a threshold on the probability.
  - it is the probabilistic counterpart of $\exists$ and $\forall$ path-quantifiers in CTL.

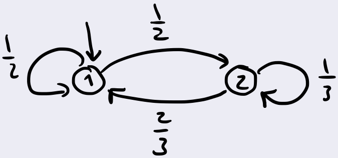
$\approx \mathbb{P}_{>0}(\varphi)$ $\approx \mathbb{P}_{=1}(\varphi)$

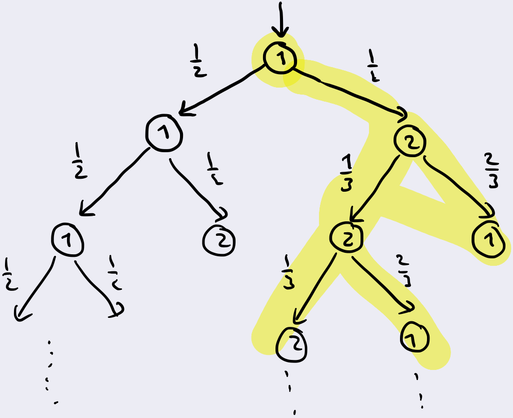
PCTL syntax

[Hansson & Jonsson, 1994]

Probabilistic Computation Tree Logic: Syntax

PCTL consists of state- and path-formulas.





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PCTL consists of state- and path-formulas.

- PCTL *state formulas* over the set AP obey the grammar:

$$\Phi \;::=\; \mathrm{true} \;\Big|\; a \;\Big|\; \Phi_1 \wedge \Phi_2 \;\Big|\; \neg\Phi \;\Big|\; \mathbb{P}_J(\varphi)$$

  where $a \in AP$, φ is a path formula and $J \subseteq [0,1]$, $J \neq \varnothing$ is a non-empty interval.

- PCTL *path formulae* are formed according to the following grammar:

$$\varphi \;::=\; \bigcirc\Phi \;\Big|\; \Phi_1 \,\mathrm{U}\, \Phi_2 \;\Big|\; \Phi_1 \,\mathrm{U}^{\leqslant n}\, \Phi_2$$

$J = (\frac{1}{2}, 1]$

  where Φ, Φ_1, and Φ_2 are state formulae and $n \in \mathbb{N}$.

$\Diamond\Phi = \mathrm{true}\,\mathrm{U}\,\Phi$

$\mathbb{P}_{>\frac{1}{2}}(\Diamond a)$ $\qquad$ $\mathbb{P}_{=1}(a \,\mathrm{U}^{\leq 10}\, \mathbb{P}_{>\frac{1}{2}}(\Diamond b))$

Probabilistic Computation Tree Logic

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$$\varphi \ ::= \ \bigcirc\Phi \ \Big| \ \Phi_1 \, \mathsf{U} \, \Phi_2 \ \Big| \ \Phi_1 \, \mathsf{U}^{\leqslant n} \, \Phi_2 \quad \text{where } n \in \mathbb{N}.$$

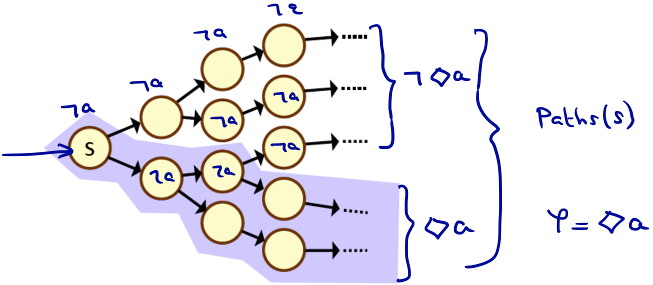
Intuitive semantics

- $s_0 s_1 s_2 \ldots \models \Phi \, \mathsf{U}^{\leqslant n} \, \Psi$ if Φ holds until Ψ holds within n steps.

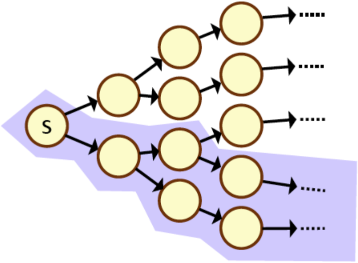
$s_0 s_1 s_2 s_3 \ldots s_k$. ① $s_k \models \Psi$ ③ $k \leq n$ ② $\forall i < k.\ s_i \models \Phi$

Overview

Semantics of $\mathbb{P}$-operator



Semantics of $\mathbb{P}$-operator



- $s \models \mathbb{P}_J(\varphi)$ if:
  - the probability of all paths starting in s fulfilling φ lies in J.
- Example: $s \models \mathbb{P}_{>\frac{1}{2}}(\lozenge a)$ if
  - the probability to reach an a-labeled state from s exceeds $\frac{1}{2}$.
- Formally:
  - $s \models \mathbb{P}_J(\varphi)$ if and only if $Pr_s\{\, \pi \in Paths(s) \mid \pi \models \varphi \,\} \in J$.

Derived operators

$$\Diamond \Phi \;=\; \text{true} \, \mathsf{U} \, \Phi$$

$$\Diamond^{\leqslant n} \Phi \;=\; \text{true} \, \mathsf{U}^{\leqslant n} \Phi$$

$$\mathbb{P}_{\leqslant p}(\Box \Phi) \;=\; \mathbb{P}_{>1-p}(\Diamond \neg \Phi)$$

$$\Box \Phi \equiv \neg \Diamond \neg \Phi$$

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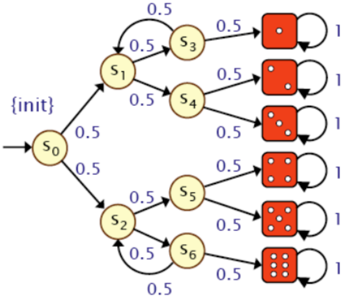
$$\mathbb{P}_{\leqslant p}(\Box \Phi) \ = \ \mathbb{P}_{>1-p}(\Diamond \neg \Phi)$$

$$\mathbb{P}_{(p,q)}(\Box^{\leqslant n} \Phi) \ = \ \mathbb{P}_{[1-q,1-p]}(\Diamond^{\leqslant n} \neg \Phi)$$

Correctness of Knuth's die



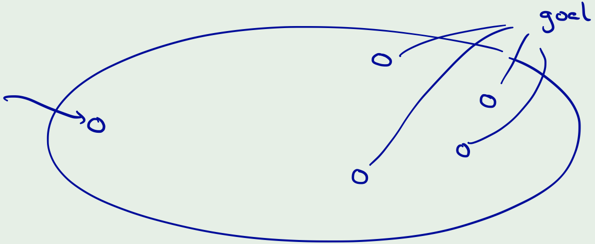
Correctness of Knuth's die

$$\mathbb{P}_{=\frac{1}{6}}(\lozenge 1) \;\wedge\; \mathbb{P}_{=\frac{1}{6}}(\lozenge 2) \;\wedge\; \mathbb{P}_{=\frac{1}{6}}(\lozenge 3) \;\wedge\; \mathbb{P}_{=\frac{1}{6}}(\lozenge 4) \;\wedge\; \mathbb{P}_{=\frac{1}{6}}(\lozenge 5) \;\wedge\; \mathbb{P}_{=\frac{1}{6}}(\lozenge 6)$$

Example properties

- Transient probabilities to be in *goal* state at the fourth epoch:

$$\mathbb{P}_{\geqslant 0.92}\left(\Diamond^{=4}\, goal\right)$$



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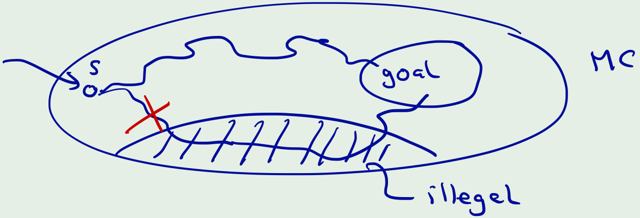

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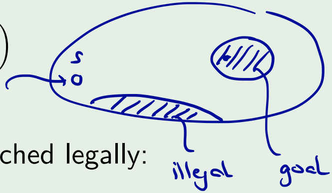

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- . . . in maximally 137 steps: $\mathbb{P}_{\geqslant 0.92}\left(\neg\ illegal\ \mathsf{U}^{\leqslant 137}\ goal\right)$
- . . . once there, remain there almost surely for the next 31 steps:

$$\mathbb{P}_{\geqslant 0.92}\left(\neg\ illegal\ \mathsf{U}^{\leqslant 137}\ \mathbb{P}_{=1}(\Box^{[0,31]}\ goal)\right)$$

PCTL semantics (1)

Notation

$\mathcal{D}, s \models \Phi$ iff state-formula Φ holds in state s of (possibly infinite) DTMC $\mathcal{D}$. As $\mathcal{D}$ is known from the context we simply write $s \models \Phi$.

Satisfaction relation for state formulas

The satisfaction relation $\models$ is defined for PCTL state formulas by:

$$
\begin{array}{lll}
s \models a & \text{iff} & a \in L(s) \\
s \models \neg\,\Phi & \text{iff} & \text{not } (s \models \Phi) \\
s \models \Phi \,\wedge\, \Psi & \text{iff} & (s \models \Phi) \text{ and } (s \models \Psi)
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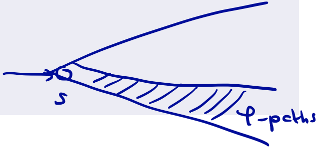
s \models \mathbb{P}_J(\varphi) & \text{iff} & Pr(s \models \varphi) \in J
\end{array}
$$

where $Pr(s \models \varphi) = Pr_s\{\, \pi \in Paths(s) \mid \pi \models \varphi \,\}$



PCTL semantics (2)

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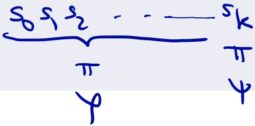
Satisfaction relation for path formulas

Let $\pi = s_0\, s_1\, s_2 \ldots$ be an infinite path in (possibly infinite) DTMC $\mathcal{D}$.
Recall that $\pi[i] = s_i$ denotes the $(i{+}1)$-st state along π.

The satisfaction relation $\models$ is defined for state formulas by:

$$\pi \models \bigcirc\Phi \quad \text{iff} \quad s_1 \models \Phi$$

$$\pi \models \Phi \,\mathsf{U}\, \Psi \quad \text{iff} \quad \exists k \geqslant 0. (\, \pi[k] \models \Psi \text{ and } \forall 0 \leqslant i < k.\, \pi[i] \models \Phi\,)$$



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$$\pi \models \Phi \,\mathsf{U}^{\leqslant n}\, \Psi \quad \text{iff} \quad \exists k \geqslant 0.(\, k \leqslant n \text{ and } \pi[k] \models \Psi \text{ and } \forall 0 \leqslant i < k.\, \pi[i] \models \Phi\,)$$

Examples

Measurability

PCTL measurability

For PCTL path formula φ and state s of DTMC $\mathcal{D}$,
$\{ \pi \in Paths(s) \mid \pi \models \varphi \}$ is measurable.

Proof (sketch):

Three cases:

1. $\bigcirc \Phi$:
   - cylinder sets constructed from paths of length one.
2. $\Phi \, \mathsf{U}^{\leqslant n} \, \Psi$:
   - (finite number of) cylinder sets from paths of length at most n.
3. $\Phi \, \mathsf{U} \, \Psi$:
   - countable union of paths satisfying $\Phi \, \mathsf{U}^{\leqslant n} \, \Psi$ for all $n \geqslant 0$.

Overview

PCTL model checking

PCTL model checking problem

Input: a finite DTMC $\mathcal{D} = (S, \mathbf{P}, \iota_{\text{init}}, AP, L)$, state $s \in S$, and PCTL state formula Φ

Output: yes, if $s \models \Phi$; no, otherwise.

Basic algorithm

In order to check whether $s \models \Phi$ do:

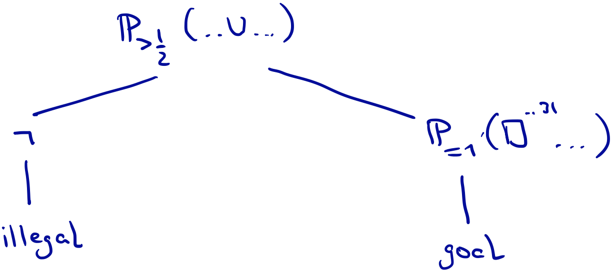
1. Compute the satisfaction set $Sat(\Phi) = \{ s \in S \mid s \models \Phi \}$.
2. This is done recursively by a bottom-up traversal of Φ's parse tree.
   - The nodes of the parse tree represent the subformulae of Φ.
   - For each node, i.e., for each subformula Ψ of Φ, determine $Sat(\Psi)$.
   - Determine $Sat(\Psi)$ as function of the satisfaction sets of its children:

     e.g., $Sat(\Psi_1 \wedge \Psi_2) = Sat(\Psi_1) \cap Sat(\Psi_2)$ and $Sat(\neg\Psi) = S \setminus Sat(\Psi)$.
3. Check whether state s belongs to $Sat(\Phi)$.

Example

$$\Phi = \mathbb{P}_{>\frac{1}{2}}\left(\neg \text{illegal} \; \mathsf{U} \; \mathbb{P}_{=1}\left(\Box^{[0,31]} \text{goal}\right)\right)$$

parse tree of Φ.



Core model-checking algorithm

Propositional formulas

$Sat(\cdot)$ is defined by structural induction as follows:

$$\begin{aligned} Sat(\text{true}) &= S \\ Sat(a) &= \{ s \in S \mid a \in L(s) \}, \text{ for any } a \in AP \\ Sat(\Phi \wedge \Psi) &= Sat(\Phi) \cap Sat(\Psi) \\ Sat(\neg\Phi) &= S \setminus Sat(\Phi). \end{aligned}$$

Probabilistic operator $\mathbb{P}$

In order to determine whether $s \in Sat(\mathbb{P}_J(\varphi))$, the probability $Pr(s \models \varphi)$ for the event specified by φ needs to be established. Then

$$Sat(\mathbb{P}_J(\varphi)) = \{ s \in S \mid Pr(s \models \varphi) \in J \}.$$

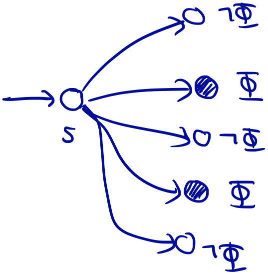
Let us consider the computation of $Pr(s \models \varphi)$ for all possible φ.

The next-step operator

Recall that: $s \models \mathbb{P}_J(\bigcirc \Phi)$ if and only if $Pr(s \models \bigcirc \Phi) \in J$.

Lemma

$Pr(s \models \bigcirc \Phi) = \sum_{s' \in Sat(\Phi)} \mathbf{P}(s, s')$.



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Lemma

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Algorithm

Considering the above equation for all states simultaneously yields:

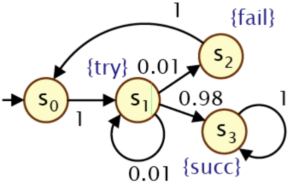
$$\big(Pr(s \models \bigcirc \Phi)\big)_{s \in S} = \mathbf{P} \cdot \mathbf{b}_\Phi$$

with $\mathbf{b}_\Phi$ the characteristic vector of $Sat(\Phi)$, i.e., $b_\Phi(s) = 1$ iff $s \in Sat(\Phi)$.

Checking the next-step operator reduces to a single matrix-vector multiplication.

Example

Consider DTMC:



and PCTL-formula:

$$\mathbb{P}_{\geqslant 0.9}\left(\bigcirc\left(\neg try \vee succ\right)\right)$$

1. $Sat(\neg try \vee succ) = (S \setminus Sat(try)) \cup Sat(succ) = \{ s_0, s_2, s_3 \}$
2. We know: $\left(Pr(s \models \bigcirc \Phi)\right)_{s \in S} = \mathbf{P} \cdot \mathbf{b}_\Phi$ where $\Phi = \neg try \vee succ$
3. Applying that to this example yields:

$$\left(Pr(s \models \bigcirc \Phi)\right)_{s \in S} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.01 & 0.01 & 0.98 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.99 \\ 1 \\ 1 \end{pmatrix}$$

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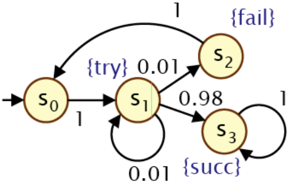

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4. Thus: $Sat(\mathbb{P}_{\geqslant 0.9}(\bigcirc(\neg try \vee succ)) = \{ s_1, s_2, s_3 \}$.

Bounded until (1)

Recall that: $s \models \mathbb{P}_J(\Phi \, \mathsf{U}^{\leqslant n} \, \Psi)$ if and only if $Pr(s \models \Phi \, \mathsf{U}^{\leqslant n} \, \Psi) \in J$.

Lemma

Let $S_{=1} = Sat(\Psi)$, $S_{=0} = S \setminus (Sat(\Phi) \cup Sat(\Psi))$, and $S_? = S \setminus (S_{=0} \cup S_{=1})$. Then:

$$Pr(s \models \Phi \, \mathsf{U}^{\leqslant n} \, \Psi) = \begin{cases} 1 & \text{if } s \in S_{=1} \\ 0 & \text{if } s \in S_{=0} \\ 0 & \text{if } s \in S_? \wedge n{=}0 \\ \sum\limits_{s' \in S} \mathbf{P}(s, s') \cdot Pr(s' \models \Phi \, \mathsf{U}^{\leqslant n-1} \, \Psi) & \text{otherwise} \end{cases}$$

Bounded until (2)

Let $S_{=1} = Sat(\Psi)$, $S_{=0} = S \setminus (Sat(\Phi) \cup Sat(\Psi))$, and $S_{?} = S \setminus (S_{=0} \cup S_{=1})$. Then:

$$Pr(s \models \Phi \, \mathsf{U}^{\leqslant n} \, \Psi) \;=\; \begin{cases} 1 & \text{if } s \in S_{=1} \\ 0 & \text{if } s \in S_{=0} \\ 0 & \text{if } s \in S_{?} \wedge n{=}0 \\ \sum\limits_{s' \in S} \mathbf{P}(s, s') \cdot Pr(s' \models \Phi \, \mathsf{U}^{\leqslant n-1} \, \Psi) & \text{otherwise} \end{cases}$$

$P^{n}_{\Phi,\Psi} \cdot b_{\Psi}$

Algorithm

1. Let $\mathbf{P}_{\Phi,\Psi}$ be the probability matrix of $\mathcal{D}[S_{=0} \cup S_{=1}]$.
2. Then $\left(Pr(s \models \Phi \, \mathsf{U}^{\leqslant 0} \, \Psi)\right)_{s \in S} \;=\; \mathbf{b}_{\Psi}$
3. And $\left(Pr(s \models \Phi \, \mathsf{U}^{\leqslant i+1} \, \Psi)\right)_{s \in S} \;=\; \mathbf{P}_{\Phi,\Psi} \cdot \left(Pr(s \models \Phi \, \mathsf{U}^{\leqslant i} \, \Psi)\right)_{s \in S}$.
4. This requires n matrix-vector multiplications in total.

Bounded until (3)

Algorithm

1. Let $\mathbf{P}_{\Phi,\Psi}$ be the probability matrix of $\mathcal{D}[S_{=0} \cup S_{=1}]$.
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Remarks

1. In terms of matrix powers: $\left(Pr(s \models \Phi \, \mathsf{U}^{\leqslant n} \, \Psi)\right)_{s \in S} \;=\; \mathbf{P}^{n}_{\Phi,\Psi} \cdot \mathbf{b}_{\Psi}$.
   - Computing $\mathbf{P}^{n}_{\Phi,\Psi}$ in $\log_2 n$ steps is inefficient due to fill-in.
   - That is to say, $\mathbf{P}^{n}_{\Phi,\Psi}$ is much less sparse than $\mathbf{P}_{\Phi,\Psi}$.
2. $\mathbf{P}^{n}_{\Phi,\Psi} \cdot \mathbf{b}_{\Psi} \;=\; \left(Pr(s \models \bigcirc^{=n} \Psi)\right)_{s \in S_?}$ in $\mathcal{D}[S_{=0} \cup S_{=1}]$.
   - Where $\bigcirc^{0} \, \Psi = \Psi$ and $\bigcirc^{i+1} \, \Psi = \bigcirc \, (\bigcirc^{i} \, \Psi)$.
   - This thus amounts to a transient analysis in DTMC $\mathcal{D}[S_{=0} \cup S_{=1}]$.

Optimization

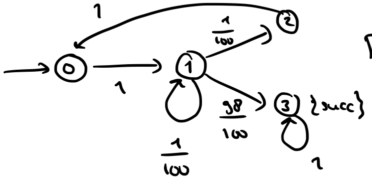
The above procedure used:

- $S_{=1} = Sat(\Psi)$, and
- $S_{=0} = S \setminus (Sat(\Phi) \cup Sat(\Psi)) = Sat(\neg\Phi \wedge \neg\Psi)$, and
- perform the matrix-vector multiplications on the remaining states

This can be optimized (in practice) by enlarging $S_{=0}$ and $S_{=1}$:

- $S_{=1} = Sat(\mathbb{P}_{=1}(\Phi \, \mathsf{U} \, \Psi))$, obtained by a graph analysis
- $S_{=0} = Sat(\mathbb{P}_{=0}(\Phi \, \mathsf{U} \, \Psi))$, obtained by a graph analysis too, and
- perform the matrix-vector multiplications on the remaining states.

Example



$$\mathbb{P}_{>0.98}(\Diamond^{\leq 2} succ) = true \; U^{\leq 2} \; succ$$

$S_{=1} = \{s_3\}$

$S_{=0} = \emptyset$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{1}{100} & \frac{1}{100} & \frac{98}{100} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^2 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{b_{succ}} = \begin{bmatrix} 0.98 \\ 0.9898 \\ 0 \\ 1 \end{bmatrix}$$

Until

Until

Recall that: $s \models \mathbb{P}_J(\Phi \,\mathsf{U}\, \Psi)$ if and only if $Pr(s \models \Phi \,\mathsf{U}\, \Psi) \in J$.

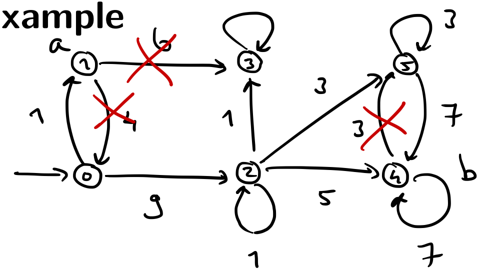
Algorithm

1. Determine $S_{=1} = Sat(\mathbb{P}_{=1}(\Phi \,\mathsf{U}\, \Psi))$ by a graph analysis.
2. Determine $S_{=0} = Sat(\mathbb{P}_{=0}(\Phi \,\mathsf{U}\, \Psi))$ by a graph analysis.
3. Then solve a linear equation system over all remaining states.

Importance of pre-computation using graph analysis

1. Ensures unique solution to linear equation system.
2. Reduces the number of variables in the linear equation system.
3. Gives exact results for the states in $S_{=1}$ and $S_{=0}$ (i.e., no round-off).
4. For qualitative properties, no further computation is needed.

Example



divide all numbers by 10.

$Sat(\mathbb{P}_{\geq \frac{4}{5}}(\neg a \cup b))$

0. $Sat(\neg a) = S \setminus \{1\}$
   $Sat(b) = \{4\}$
   } make 1 + 4 absorbing

1. $Sat_{=1}(\Diamond b) = \{4, 5\}$ graph analysis
2. $Sat_{=0}(\Diamond b) = \{1, 3\}$
3. $S_? = \{0, 2\}$

$x = Ax + b$

solve yields $x_0 = \frac{4}{5}$, $x_2 = \frac{8}{9}$

$$\bar{x} = \begin{bmatrix} x_0 \\ x_2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 0 & 9 \\ 0 & 1 \end{bmatrix}\bar{x} + \frac{1}{10}\begin{bmatrix} 0 \\ 8 \end{bmatrix}$$

Overview

Time complexity

Let $|\Phi|$ be the size of Φ, i.e., the number of logical and temporal operators in Φ.

Time complexity of PCTL model checking

For finite DTMC $\mathcal{D}$ and PCTL state-formula Φ, the PCTL model-checking problem can be solved in time

$$\mathcal{O}(\, poly(size(\mathcal{D})) \cdot n_{\max} \cdot |\Phi|\,)$$

where $n_{\max} = \max\{\, n \mid \Psi_1 \, \mathsf{U}^{\leqslant n} \Psi_2 \text{ occurs in } \Phi \,\}$ with and $n_{\max} = 1$ if Φ does not contain a bounded until-operator.

– bottom-up traversal on parse tree of Φ ⇝ linear in $|\Phi|$

Time complexity

Time complexity of PCTL model checking

For finite DTMC $\mathcal{D}$ and PCTL state-formula Φ, the PCTL model-checking problem can be solved in time

$$\mathcal{O}\big(\, poly(size(\mathcal{D})) \cdot n_{\max} \cdot |\Phi| \,\big).$$

Proof (sketch)

1. For each node in the parse tree, a model-checking is performed; this yields a linear complexity in $|\Phi|$.
2. The worst-case operator is (unbounded) until.

Time complexity

Time complexity of PCTL model checking

For finite DTMC $\mathcal{D}$ and PCTL state-formula Φ, the PCTL model-checking problem can be solved in time

$$\mathcal{O}\big(\, poly(size(\mathcal{D})) \,\cdot\, n_{\max} \cdot |\Phi| \,\big).$$

Proof (sketch)

1. For each node in the parse tree, a model-checking is performed; this yields a linear complexity in $|\Phi|$.
2. The worst-case operator is (unbounded) until.
   - 2.1 Determining $S_{=0}$ and $S_{=1}$ can be done in linear time.
   - 2.2 Direct methods to solve linear equation systems are in $\Theta(|S_?|^3)$.
3. Strictly speaking, $\mathsf{U}^{\leqslant n}$ could be more expensive for large n.

   But it remains polynomial, and n is small in practice.

Example: Lost passenger ticket problem

- N passengers are waiting to board an airplane.
- The plane is fully booked
- The first passenger lost his boarding pass; he randomly picks a seat
- All other passengers have their boarding pass.
  1. reserved seat free? ⟶ sit down
  2. occupied? ⟶ randomly pick a free seat

Q: what is the probability that the last passenger gets his reserved seat?

Verification results

storm model checker

(stormchecker.org)

N	ver. time (in seconds)
100	0.1
1000	0.1
10,000	0.2
1,000,000	6.4
10,000,000	66.8

Value iteration

$x = Ax + b \qquad x^{(0)} = 0$

- Reachability probabilities are typically obtained iteratively:

$$\mathbf{x}^{(n+1)} = \mathbf{A} \cdot \mathbf{x}^{(n)} + \mathbf{b}$$

- Then: reachability probability $Pr(\Diamond G)$ equals $\lim_{n\to\infty} \mathbf{x}^{(n)}$
- Question: when to halt this iterative process?
- Typical approach:

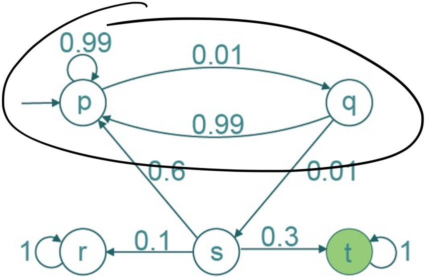
$Pr(\Diamond G) < \frac{1}{2}$

$$\left|\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)}\right| \leqslant \varepsilon$$

  for some ε, e.g.,10^{-6}
- Potential problem: premature convergence

  That is: iterations are stopped too early
- Verification results are obtained without guarantees

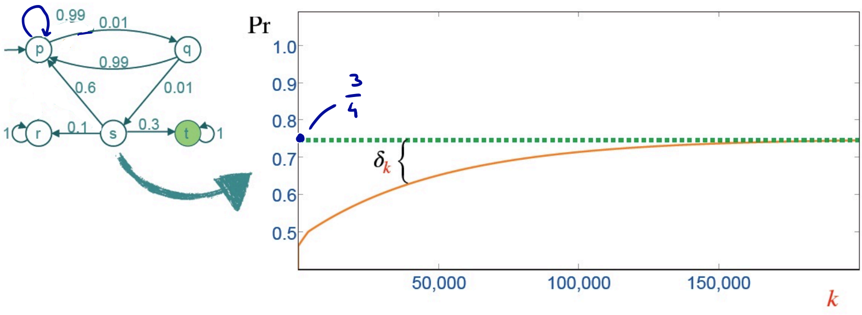
Example



- Exact answer: $Pr(\Diamond\, t) = \frac{3}{4}$
- Value iteration with $\varepsilon = 0,000001$ yields 0.7248
- True error: 0.0252

Value iteration

Idea: approach $Pr(\lozenge G)$ by computing $Pr(\lozenge^{\leqslant k} G)$ for increasing k



- Problem: δ_k is unknown
- Stopping criterion: $|Pr(\lozenge^{\leqslant k+1} G) - Pr(\lozenge^{\leqslant k} G)| \leqslant \varepsilon$
- But this is independent from the aim: $\underbrace{Pr(\lozenge G) - Pr(\lozenge^{\leqslant k} G)}_{\delta_k} \leqslant \varepsilon$

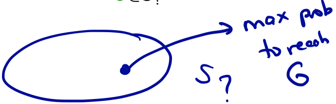
Remedy: bound $Pr(\Diamond G)$ from above too

error

Idea: provide bounds $\ell_k \leqslant \delta_k \leqslant u_k$ for $\delta_k = Pr(\Diamond G) - Pr(\Diamond^{\leqslant k} G)$

How to obtain these bounds? Towards an upper bound observe:

$$\delta_k \;=\; \underbrace{Pr(\Diamond G) - Pr(\Diamond^{\leqslant k} G)}_{\text{probability to reach } G \text{ in } > k \text{ steps}} \;\leqslant\; Pr(\Box^{\leqslant k} S_?) \cdot \max_{s \in S_?} Pr_s(\Diamond G)$$



Towards a lower bound observe:

$$\delta_k \;=\; \underbrace{Pr(\Diamond G) - Pr(\Diamond^{\leqslant k} G)}_{\text{probability to reach } G \text{ in } > k \text{ steps}} \;\geqslant\; Pr(\Box^{\leqslant k} S_?) \cdot \min_{s \in S_?} Pr_s(\Diamond G)$$

Let $s_{max} \in S_?$ be a state with maximal reachability probability to reach G, i.e.,

$$s_{max} = \arg\max_{s \in S_?} Pr_s(\Diamond G)$$

↓ probability measure in DTMC D_s (i.e. D with initial state s)

Then: $\delta_k = Pr(\Diamond G) - Pr(\Diamond^{\leq k} G)$

$$\leq Pr(\Box^{\leq k} S_?) \cdot Pr_{s_{max}}(\Diamond G) \qquad (*)$$

Thus for s_{max} we obtain:

$$Pr_{s_{max}}(\Diamond G) - Pr_{s_{max}}(\Diamond^{\leq k} G) \leq Pr_{s_{max}}(\Box^{\leq k} S_?) \cdot Pr_{s_{max}}(\Diamond G)$$

$$\Rightarrow \quad Pr_{s_{max}}(\Diamond G) \leq \frac{Pr_{s_{max}}(\Diamond^{\leq k} G)}{1 - Pr_{s_{max}}(\Box^{\leq k} S_?)}$$

$$\leq \max_{s \in S_?} \frac{Pr_s(\Diamond^{\leq k} G)}{1 - Pr_s(\Box^{\leq k} S_?)} \qquad (**)$$

$(*) + (**)$

$$\Rightarrow \quad \delta_k \leq Pr(\Box^{\leq k} S_?) \cdot \max_{s \in S_?} \frac{Pr_s(\Diamond^{\leq k} G)}{1 - Pr_s(\Box^{\leq k} S_?)}$$

Sound value iteration

Sound value iteration theorem

For DTMC $\mathcal{D}$, goal states $G \subseteq S$ and $k \in \mathbb{N}$:

$$Pr(\Diamond^{\leqslant k} G) + \ell_k \leqslant Pr(\Diamond G) \leqslant Pr(\Diamond^{\leqslant k} G) + u_k$$

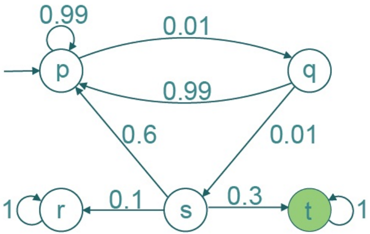
where:

$$u_k = Pr(\Box^{\leqslant k} S_?) \cdot \max_{s \in S_?} \frac{Pr_s(\Diamond^{\leqslant k} G)}{1 - Pr_s(\Box^{\leqslant k} S_?)}$$

and

$$\ell_k = Pr(\Box^{\leqslant k} S_?) \cdot \min_{s \in S_?} \frac{Pr_s(\Diamond^{\leqslant k} G)}{1 - Pr_s(\Box^{\leqslant k} S_?)}$$

Example sound value iteration



- Exact answer: $Pr(\Diamond\, t) = \frac{3}{4}$
- $S_? = \{ s_0, s_1, s_2 \}$
- We have $\mathbf{l}_3 = (0.00003, 0.003, 0.3)$
- and $\mathbf{u}_3 = (0.99996, 0.996, 0.6)$
- For all $s \in S_?$ we have $\frac{\ell_3(s)}{1-u_3(s)} = \frac{3}{4}$
- Thus $\ell_3 = u_3 = \frac{3}{4}$
- Three iterations suffice for the exact answer

Overview

Summary

- PCTL is a branching-time logic with key operator $\mathbb{P}_J(\varphi)$.
- Sets of paths fulfilling PCTL path-formula φ are measurable.
- PCTL model checking is performed by a recursive descent over Φ.
- The next operator amounts to a single matrix-vector multiplication.
- Bounded until $\mathsf{U}^{\leqslant n}$ amounts to n matrix-vector multiplications.
- The until-operator amounts to solving a linear equation system.
- Time complexity of $\mathcal{D} \models \Phi$ is polynomial in $|\mathcal{D}|$ and linear in $|\Phi|$.
- Value iteration is sound when upper bounding $Pr(\Diamond G)$
- Variations: long-run operator, conditional probabilities, expected reward until reaching a set of states.