Modeling and Verification of Probabilistic Systems

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http://moves.rwth-aachen.de/teaching/ws-1819/movep18/

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Overview

Recapitulating reachability probabilities

Problem statement

Let $\mathcal{D}$ be a DTMC with finite state space S, $s \in S$ and $G \subseteq S$.

Aim: determine $Pr(s \models \Diamond G) \;=\; Pr_s\{\, \pi \in Paths(s) \mid \pi \models \Diamond G \,\}$

where Pr_s is the probability measure in $\mathcal{D}$ with single initial state s.

Approach

1. Determine by a graph analysis $S_{=0} = \{\, s \in S \mid Pr(s \models \Diamond G) = 0 \,\}$ and $S_{=1} = \{\, s \in S \mid Pr(s \models \Diamond G) = 1 \,\}$
2. Introduce a variable x_s for any state $s \in S_? = S \setminus (S_{=0} \cup S_{=1})$
3. Solve a linear equation system $\mathbf{x} = \mathbf{A} \cdot \mathbf{x} + \mathbf{b}$
4. . . . . . . using one of your favourite techniques, e.g., iterative methods
5. Intermediate results $\mathbf{x}^{(i)}$ represent the vector $\left(Pr(s \models \Diamond^{\leqslant i} G) \right)_{s \in S_?}$

Overview

Qualitative properties

Quantitative properties

Comparing the probability of an event such as $\Box G$, $\Diamond\Box G$ and $\Box\Diamond G$ with a threshold $\sim p$ with $p \in (0,1)$ and $\sim$ a binary comparison operator $(=, <, \leqslant, \geqslant, >)$ yields a quantitative property.

Example quantitative properties

$Pr(s \models \Diamond\Box G) \;>\; \frac{1}{2}$ or $Pr(s \models \Diamond^{\leqslant n} G) \;\leqslant\; \frac{\pi}{5}$

Qualitative properties

Comparing the probability of an event such as $\Box G$, $\Diamond\Box G$ and $\Box\Diamond G$ with a threshold > 0 or $= 1$ yields a qualitative property. Any event E with $Pr(E) = 1$ is called almost surely.

Example qualitative properties

$Pr(s \models \Diamond\Box G) \;>\; 0$ or $Pr(s \models \Diamond^{\leqslant n} G) \;=\; 1$

Aim of today's lecture

Take-home message

For finite DTMCs, qualitative properties do only depend on their state graph and not on the transition probabilities! For infinite DTMCs, this does not hold.

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Remark

In the following we will concentrate on almost sure events, i.e., events E with $Pr(E) = 1$. This suffices, as $Pr(E) > 0$ if and only if not $Pr(\overline{E}) = 1$.

Overview

Fairness

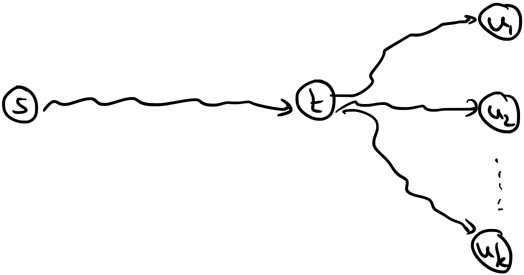
Fairness theorem

Let $\mathcal{D}$ be a (possibly infinite) DTMC and s, t states in $\mathcal{D}$. Then:

$$Pr(s \models \Box\Diamond t) \;=\; Pr(s \models \bigwedge_{u \in Post^*(t)} \Box\Diamond u).$$

infinitely often visit t from s

visit all successors of t ∞ often from s



Fairness

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Let $\mathcal{D}$ be a (possibly infinite) DTMC and s, t states in $\mathcal{D}$. Then:

$$\underbrace{Pr(s \models \Box\Diamond t)}_{=1} \; = \; \underbrace{Pr(s \models \bigwedge_{u \in Post^*(t)} \Box\Diamond u)}_{=1}.$$

When infinite branching, this is an infinitary conjunction (countable intersection).

In particular, if t is visited infinitely often almost surely, then this property carries over to any successor u of t.

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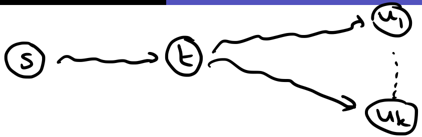
Corollary

For any state s in a (possibly infinite) DTMC we have:

$$Pr(s \models \bigwedge_{t \in S} \bigwedge_{u \in Post^*(t)} (\Box\Diamond t \; \Rightarrow \; \Box\Diamond u)) \; = \; 1.$$

Proof (1)



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This result follows directly from the following claim that we will prove below.

Claim

The probability to infinitely often visit state t equals the probability to take any finite path $\hat{\pi}$ emanating from state t infinitely often.

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Proof (2)

Claim

Let $\mathcal{D}$ be a (possibly infinite) DTMC and s, t states in $\mathcal{D}$. Then:

$$\underbrace{Pr(s \models \Box\Diamond t)} = Pr_s\left(\bigwedge_{\underline{\hat{\pi} \in Paths^*(t)}} \Box\Diamond\hat{\pi} \right)$$

where $\Box\Diamond\hat{\pi}$ denotes the set of paths π such that $\hat{\pi}$ occurs infinitely in π.

$$\pi = (\ldots \hat{\pi} \ldots \hat{\pi} \ldots \hat{\pi} \ldots)^\omega$$

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Claim

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where $\Box\Diamond\hat{\pi}$ denotes the set of paths π such that $\hat{\pi}$ occurs infinitely in π.

Proof:

This claim is proven in three steps:

1. For any $\hat{\pi} \in Paths^*(t)$, it holds $Pr(s \models \Box\Diamond t) = Pr(s \models \Diamond\hat{\pi})$.
2. For any $\hat{\pi} \in Paths^*(t)$, it holds $Pr(\Box\Diamond t \wedge \Diamond\Box\neg\hat{\pi}) = 0$.
3. $Pr(\Box\Diamond t \wedge \bigwedge_{\hat{\pi} \in Paths^*(t)} \Diamond\Box\neg\hat{\pi}) = 0$.

Proof (3)

$$\underbrace{Pr(s \models \Box\Diamond t)}_{A} = Pr_s\Big(\underbrace{\bigwedge_{\hat{\pi} \in Paths^*_s(t)} \Box\Diamond \hat{\pi}}_{B}\Big)$$

Observe: $Pr(A \mid B) = 1$ $\quad (+*)$

We will show: $\boxed{Pr(A \wedge \overline{B}) = 0}$

$$Pr(A) = Pr(A \wedge B) + \underbrace{Pr(A \wedge \overline{B})}_{=0} = Pr(A \wedge B) \quad (*)$$

$$Pr(A \mid B) = \frac{Pr(A \wedge B)}{Pr(B)} = \frac{Pr(A)}{Pr(B)} = 1$$

To show: $\Pr(A \wedge \overline{B}) = 0$.

Three steps: first take a single $\hat{\pi} \in Paths^*(t)$

(a) $\Pr(\underbrace{\Box\Diamond t}_{A} \wedge \text{never } \hat{\pi}) = 0$

(b) $\Pr(\Box\Diamond t \wedge \underbrace{\Diamond\Box\neg\hat{\pi}}_{\text{finitely often } \hat{\pi}}) = 0$

(c) $\Pr(\underbrace{\Box\Diamond t}_{A} \wedge \underbrace{(\Diamond\Box\neg\hat{\pi}_1 \vee \ldots \vee \Diamond\Box\neg\hat{\pi}_k)}_{\overline{B}}) = 0$

(a) $Pr(\Box\Diamond t \wedge \text{never } \hat{\pi}) = 0$

Let $p = Pr(\hat{\pi})$. As $\hat{\pi} \in paths^*(t)$, $p > 0$.

let $E_n(\hat{\pi})$ = "visit $t \geq n$ times but never $\hat{\pi}$"

Then $Pr(E_n(\hat{\pi})) \leq (1-p)^n$ (*)

let $E(\hat{\pi}) = \bigcap_{n>0} E_n(\hat{\pi})$

= "visit t ∞ often but never $\hat{\pi}$"

$E_1(\hat{\pi}) \supseteq E_2(\hat{\pi}) \supseteq E_3(\hat{\pi})$ .... it follows

$$Pr_s(E(\hat{\pi})) = \lim_{n\to\infty} Pr(E_n(\hat{\pi})) \overset{(*)}{\leq} \lim_{n\to\infty} (1-p)^n = 0 \quad \boxtimes$$

(b) $Pr(\Box\Diamond t \wedge \underbrace{\Diamond\Box\neg\hat{\pi}}_{\text{finitely often}}) = 0$

let $F_n(\hat{\pi})$ = "$\Box\Diamond t$ but never $\hat{\pi}$ from position n on"

$$Pr_s(F_n(\hat{\pi})) = \sum_{s' \in S} \underbrace{Pr(s \models \bigcirc^n s')}_{\text{after } n \text{ steps the MC in in state } s'} \cdot \underbrace{Pr_{s'}(\underbrace{E(\hat{\pi})}_{\text{proof (a)}})}_{=0}$$

$$= 0$$

let $F(\hat{\pi}) = \bigcup_{n \geq 0} F_n(\hat{\pi})$

It follows $Pr_s(F(\hat{\pi})) = 0$ ⊠

(c) $$F = \bigcup_{\hat{\pi} \in Paths^*(t)} F(\hat{\pi})$$

$$F = \text{" } \Box\Diamond t \text{ and } (\underbrace{\Diamond\Box\neg\hat{\pi}_1 \wedge \ldots \Diamond\Box\neg\hat{\pi}_k}_{\text{all } \hat{\pi}_i \in Paths^*(t)})$$

As $Paths^*(t)$ is countable,

$$Pr_s(F) \leq \sum_{\hat{\pi} \in Paths^*(t)} \underbrace{Pr_s(F(\hat{\pi}))}_{(b)=0} = 0$$

☒

Overview

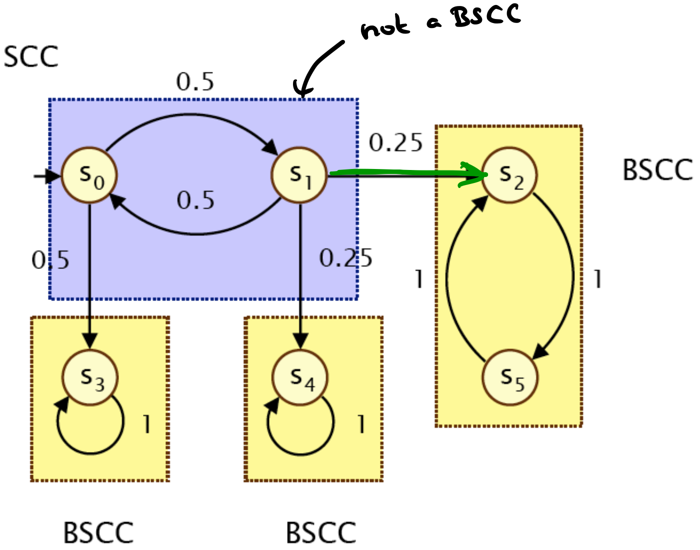
Graph notions

Let $\mathcal{D} = (S, \mathbf{P}, \iota_{\text{init}}, AP, L)$ be a (possibly infinite) DTMC.

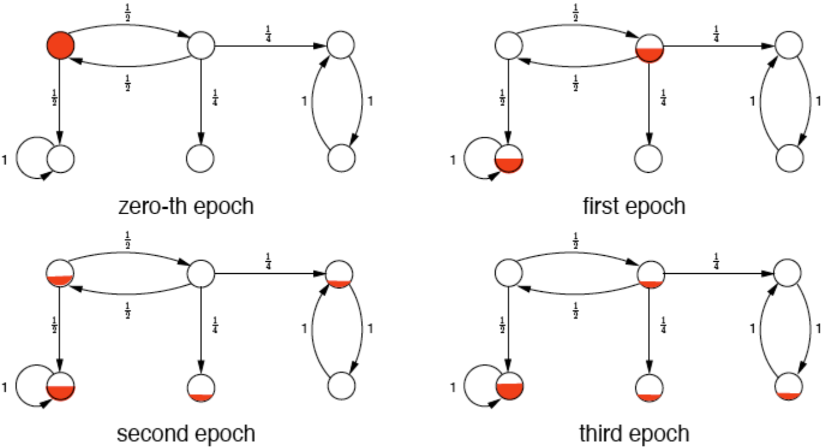
Strongly connected component

- $T \subseteq S$ is *strongly connected* if for any $s, t \in T$, states s and $t \in T$ are mutually reachable via edges in T.
- T is a *strongly connected component* (SCC) of $\mathcal{D}$ if it is strongly connected and no proper superset of T is strongly connected.
- SCC T is a *bottom SCC* (BSCC) if no state outside T is reachable from T, i.e., for any state $s \in T$, $\mathbf{P}(s, T) = \sum_{t \in T} \mathbf{P}(s, t) = 1$.
- Let $BSCC(\mathcal{D})$ denote the set of BSCCs of DTMC $\mathcal{D}$.

Example

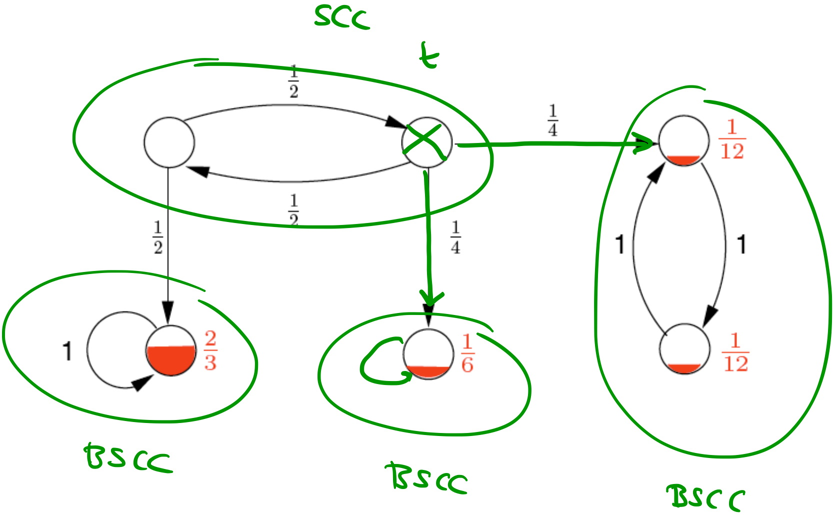

not a BSCC
SCC
0.5
0.25
BSCC
s_0
s_1
s_2
0.5
0.5
0.25
1
1
s_3
s_4
s_5
1
1
BSCC
BSCC

Evolution of an example DTMC

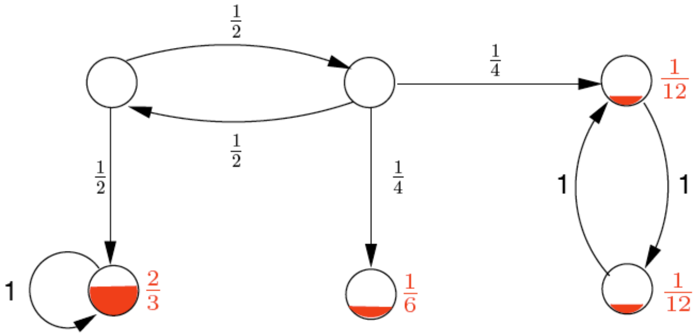


Which states have a probability > 0 when repeating this on the long run?

On the long run



On the long run



The probability mass on the long run is only left in BSCCs.

Measurability

Lemma

For any state s in (possibly infinite) DTMC $\mathcal{D}$:

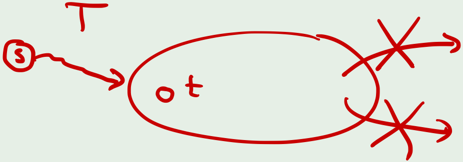
$$\{\, \pi \in \mathit{Paths}(s) \mid \inf(\pi) \in \mathit{BSCC}(\mathcal{D}) \,\} \text{ is measurable}$$

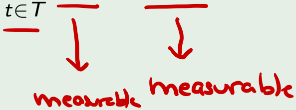
where $\inf(\pi)$ is the set of states that are visited infinitely often along π.

Proof:

1. For BSCC T, $\{\, \pi \in \mathit{Paths}(s) \mid \inf(\pi) = T \,\}$ is measurable as:

$$\{\, \pi \in \mathit{Paths}(s) \mid \inf(\pi) = T \,\} \;=\; \bigcap_{t \in T} \Box\Diamond t \;\cap\; \Diamond\Box T.$$





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2. As $BSCC(\mathcal{D})$ is countable, we have:

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2. As $\mathit{BSCC}(\mathcal{D})$ is countable, we have:

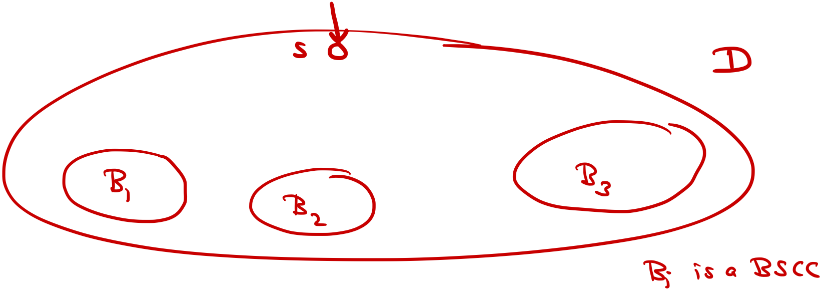
$$\{ \pi \in \mathit{Paths}(s) \mid \inf(\pi) \in \mathit{BSCC}(\mathcal{D}) \} = \bigcup_{T \in \mathit{BSCC}(\mathcal{D})} \bigcap_{t \in T} \Box\Diamond t \wedge \Diamond\Box T.$$

Fundamental result

Long-run theorem

For each state s of a finite Markov chain $\mathcal{D}$:

$$Pr_s\{\, \pi \in Paths(s) \mid \inf(\pi) \in BSCC(\mathcal{D}) \,\} = 1.$$



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Intuition

Almost surely any finite DTMC eventually reaches a BSCC and visits all its states infinitely often.

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- As $\mathcal{D}$ is finite, $\inf(\pi)$ is strongly connected, i.e., part of SCC T, say.
- Hence, $\sum_{\text{SCC}\, T} Pr_s\{\, \pi \in Paths(s) \mid \inf(\pi) = T \,\} = 1$ $(*)$
- Assume $Pr_s\{\, \pi \in Paths(s) \mid \inf(\pi) = T \,\} > 0$. for a given T

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- Assume $Pr_s\{\, \pi \in Paths(s) \mid \inf(\pi) = T \,\} > 0$.
- By the fairness theorem, almost all paths π with $\inf(\pi) = T$ fulfill

$$Post^*(T) = Post^*(\inf(\pi)) \subseteq \inf(\pi) = T.$$

$$\bigcup_{t \in T} Post^*(t)$$

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$$Post^*(T) \;=\; Post^*(\inf(\pi)) \;\subseteq\; \inf(\pi) \;=\; T.$$

- Hence, $T = Post^*(T)$, i.e., T is a BSCC. The claim follows from $(*)$.

Zeroconf example

Aim of the Zeroconf protocol

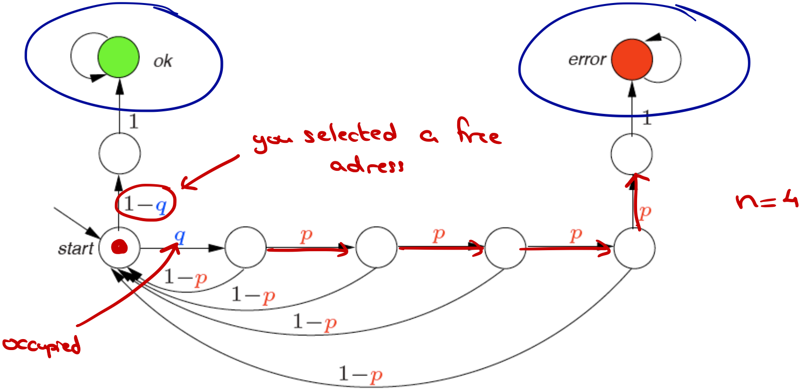
- IPv4 is aimed at plug-and-play networks for domestic appliances.
- New devices must get a unique IP address in an automated way.
- This is done by the IPv4 zeroconf protocol (proposed by IETF).

Basic functioning of the Zeroconf protocol

1. Randomly select one of the 65,024 possible addresses.
2. Loop: as long as number of sent probes $< n$.
3. Broadcast probe "who is using my current address?"
4. Receive reply? Goto step 1.
5. Receive no reply within $r > 0$ time units, then
   - 5.1 number of sent probes $= n$? Exit, and use selected address.
   - 5.2 number of sent probes $< n$? Goto step 2.

Let p be probability that no reply is received on a probe.

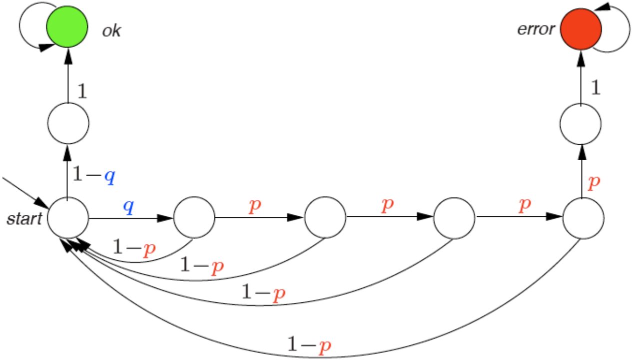
Zeroconf example



p = probability of message loss; q = probability of selecting occupied address

6s...

Zeroconf example



p = probability of message loss; q = probability of selecting occupied address

By the long-run theorem, the probability of acquiring an address infinitely often is zero.

Almost sure reachability

Recall: an absorbing state in a DTMC is a state with a self-loop with probability one.

Almost sure reachability theorem

For finite DTMC with state space S, $s \in S$ and $G \subseteq S$ a set of absorbing states:

$$Pr(s \models \Diamond G) = 1 \quad \text{iff} \quad s \in S \setminus Pre^*(S \setminus Pre^*(G)).$$

Note: $S \setminus Pre^*(S \setminus Pre^*(G))$ are states that cannot reach states from which G cannot be reached.

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Note: $S \setminus Pre^*\big(S \setminus Pre^*(G)\big)$ are states that cannot reach states from which G cannot be reached.

Proof:

Show that both sides of the equivalence are equivalent to $Post^*(t) \cap G \neq \varnothing$ for each state $t \in Post^*(s)$. Rather straightforward.

Computing almost sure reachability properties

Aim:

For finite DTMC $\mathcal{D}$ and $G \subseteq S$, determine $\underbrace{\{ s \in S \mid Pr(s \models \Diamond G) = 1 \}}_{S_{=1}}$.

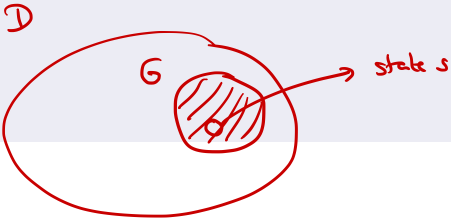
Computing almost sure reachability properties

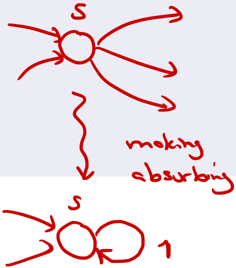
Aim:

For finite DTMC $\mathcal{D}$ and $G \subseteq S$, determine $\{ s \in S \mid Pr(s \models \Diamond G) = 1 \}$.

Algorithm

1. Make all states in G absorbing yielding $\mathcal{D}[G]$.





Computing almost sure reachability properties

Aim:

For finite DTMC $\mathcal{D}$ and $G \subseteq S$, determine $\{ s \in S \mid Pr(s \models \Diamond G) = 1 \}$.

Algorithm

1. Make all states in G absorbing yielding $\mathcal{D}[G]$.
2. Determine $S \setminus Pre^*(\, S \setminus Pre^*(G)\,)$ by a graph analysis:
   2.1 do a backward search from G in $\mathcal{D}[G]$ to determine $Pre^*(G)$.
   2.2 followed by a backward search from $S \setminus Pre^*(G)$ in $\mathcal{D}[G]$.

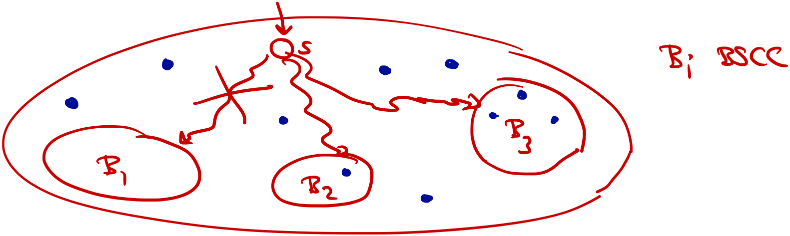
This yields a time complexity which is linear in the size of the DTMC $\mathcal{D}$.

Repeated reachability

Almost sure repeated reachability theorem

For finite DTMC with state space S, $G \subseteq S$, and $s \in S$:

$Pr(s \models \Box\Diamond G) = 1$ iff for each BSCC $T \subseteq Post^*(s)$. $T \cap G \neq \varnothing$.



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Proof:

Immediate consequence of the long-run theorem.

Almost sure repeated reachability

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For finite DTMC with state space S, $G \subseteq S$, and $s \in S$:

$Pr(s \models \Box\Diamond G) = 1$ iff for each BSCC $T \subseteq Post^*(s)$. $T \cap G \neq \varnothing$.

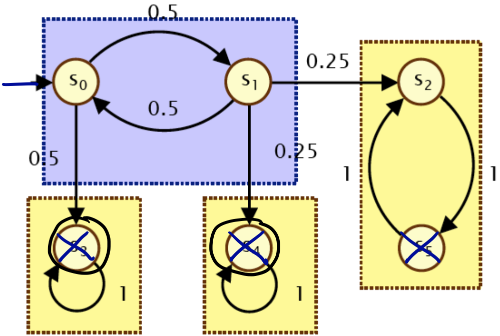
$Pr(s_0 \models \Box\Diamond G) = 1$

$Pr(s_0 \models \Box\Diamond G') < 1$

Example:

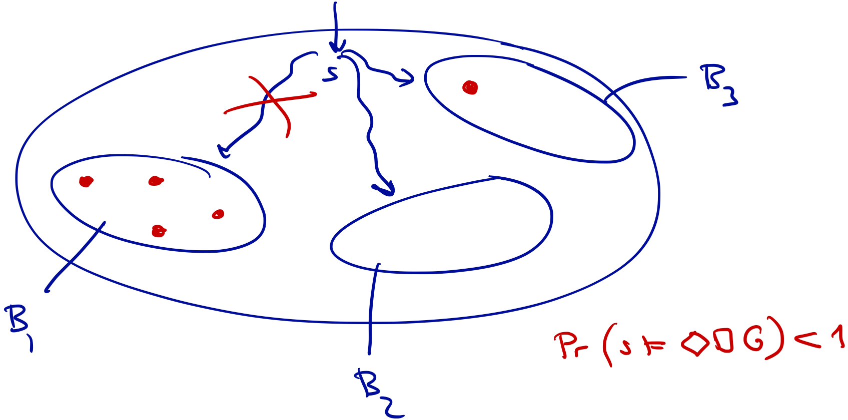
$G = \{ s_3, s_4, s_5 \}$

$G' = \{ s_3, s_4 \}$



Almost sure persistence

$\Pr(s \models \Diamond\Box G) = 1$



Almost sure persistence

Almost sure persistence theorem

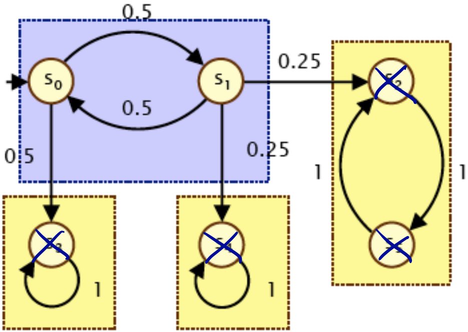
For finite DTMC with state space S, $G \subseteq S$, and $s \in S$:

$Pr(s \models \Diamond\Box G) = 1$ if and only if $T \subseteq G$ for any BSCC $T \subseteq Post^*(s)$

Example:

$G = \{ s_2, s_3, s_4, s_5 \}$

$Pr(s_0 \models \Diamond\Box G) = 1$



A remark on infinite Markov chains

Graph analysis for infinite DTMCs does not suffice!

finite MCs:

$Pr(s \models \Diamond G) = 1 \rightsquigarrow$ graph analysis

$Pr(s \models \Diamond \Box G) = 1 \rightsquigarrow$ graph analysis

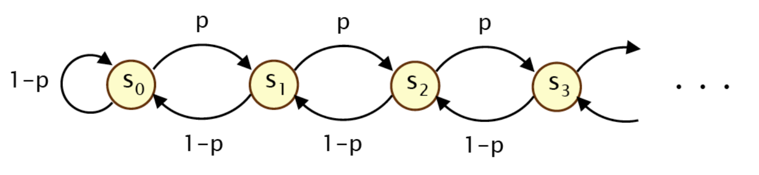
BSCCs + reachability

$Pr(s \models \Box \Diamond G) = 1 \rightsquigarrow$ graph analysis

A remark on infinite Markov chains

Graph analysis for infinite DTMCs does not suffice!

Consider the following infinitely countable DTMC, known as random walk:



The value of rational probability p does affect qualitative properties:

$$Pr(s \models \Diamond s_0) = \begin{cases} 1 & \text{if } p \leqslant \frac{1}{2} \\ < 1 & \text{if } p > \frac{1}{2} \end{cases} \quad \text{and}$$

$$Pr(s \models \Box\Diamond s_0) = \begin{cases} 1 \text{ if } p \leqslant \frac{1}{2} \\ 0 \text{ if } p > \frac{1}{2} \end{cases}$$

Quantitative properties

$Pr(s \models \Box\Diamond G) = 1 \longrightarrow$ graph analysis

ditto for $\Diamond\Box G$

now: $Pr(s \models \Box\Diamond G)$: what is the value?

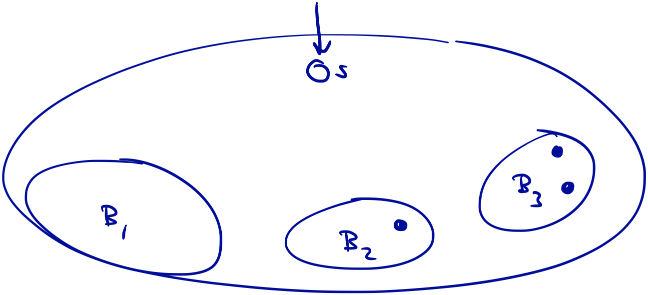
Quantitative properties

Quantitative repeated reachability theorem

For finite DTMC with state space S, $G \subseteq S$, and $s \in S$:

$$Pr(s \models \Box\Diamond G) = Pr(s \models \Diamond U)$$

where U is the union of all BSCCs T with $T \cap G \neq \varnothing$.



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Remark

Thus probabilities for $\Box\Diamond G$ and $\Box\Diamond G$ are reduced to reachability probabilities. These can be computed by solving a linear equation system.

Overview

Summary

- Executions of a DTMC are strongly fair with respect to all probabilistic choices.
- A finite DTMC almost surely ends up in a BSCC on the long run.
- Almost sure reachability = double backward search.
- Almost sure $\Box\Diamond G$ and $\Diamond\Box G$ properties can be checked by BSCC analysis and reachability.
- Probabilities for $\Box\Diamond G$ and $\Diamond\Box G$ reduce to reachability probabilities.

Take-home message

For finite DTMCs, qualitative properties do only depend on their state graph and not on the transition probabilities! For infinite DTMCs, this does not hold.