Linear equation system

Reachability probabilities as linear equation system

- Let $S_? = Pre^*(G) \setminus G$, the states that can reach G by > 0 steps
- $\mathbf{A} = \big(\mathbf{P}(s,t)\big)_{s,t \in S_?}$, the transition probabilities in $S_?$
- $\mathbf{b} = \big(b_s\big)_{s \in S_?}$, the probs to reach G in 1 step, i.e., $b_s = \sum_{u \in G} \mathbf{P}(s,u)$

Then: $\mathbf{x} = (x_s)_{s \in S_?}$ with $x_s = Pr(s \models \Diamond G)$ is the unique solution of:

$$\mathbf{x} = \mathbf{A} \cdot \mathbf{x} + \mathbf{b} \quad \text{or} \quad (\mathbf{I} - \mathbf{A}) \cdot \mathbf{x} = \mathbf{b}$$

where $\mathbf{I}$ is the identity matrix of cardinality $|S_?| \times |S_?|$.

Proof (unique sol. eq. sys.)

Towards contradiction assume there are two solutions y, z with: $Ay + b = y$
$Az + b = z$
$y \neq z$

$\Rightarrow A(y-z) = (y-z)$

We show $Ax = x$ implies $x = 0$

Then: $A(y-z) = y-z$ implies $y - z = 0$
$\Leftrightarrow y = z$ ↯

Proof for ( ):

Towards contradiction assume $x = (x_s)_{s \in S_?}$ with $Ax = x$ and $x \neq 0$

Define $x_{max} := \max_{s \in S_?} |x_s|$ (The max. exists, because DTMC D is finite)

$x \neq 0 \Rightarrow x_{max} > 0$

$T := \{ s \in S_? \mid |x_s| = x_{max} \}$ (Observe: $T \neq \emptyset$)

- For all $s \in S_?$: $Ax = x$

$$\Rightarrow x_s = \sum_{t \in S_?} P(s,t) \cdot x_t$$

$$\Rightarrow |x_s| = \left| \sum_{t \in S_?} P(s,t) \cdot x_t \right|$$

triangle ineq.
$$\Rightarrow |x_s| \leq \sum_{t \in S_?} |P(s,t) \cdot x_t|$$

$$\Rightarrow |x_s| \leq \sum_{t \in S_?} P(s,t) \cdot |x_t|$$

• For all $s \in T$

$$x_{max} = |x_s| \leq \sum_{t \in S_?} P(s,t) \cdot \underbrace{|x_t|}_{\leq x_{max}}$$

$$\leq \sum_{t \in S_?} \underbrace{P(s,t)}_{\leq 1} \cdot x_{max}$$

$$\leq x_{max}$$

$$\Rightarrow x_{max} = |x_s| = \sum_{t \in S_?} P(s,t) \cdot |x_t| = \left(\sum_{t \in S_?} P(s,t)\right) \cdot x_{max}$$

$$\overset{x_{max} > 0}{\Rightarrow} \sum_{t \in S_?} P(s,t) = 1 \quad \Rightarrow Post(s) := \{t \in S \mid P(s,t) > 0\} \subseteq S_?$$

$\Rightarrow |x_t| = x_{max}$ for all $t \in Post(s)$

$\Rightarrow t \in T$ for all $t \in Post(s)$

$\Rightarrow Post(s) \subseteq T$

We conclude $Post^*(s) \subseteq T \subseteq S_?$

Hence, $Post^*(s) \cap G = \emptyset$ $\quad (G \cap S_? = \emptyset)$

It follows that G is not reachable from any state in T, i.e., $T \subseteq S_{=0}$

But as $T \subseteq S_?$ and $S_{=0} \cap S_? = \emptyset$, $T = \emptyset$

↯ contradiction

□