Modeling and Verification of Probabilistic Systems

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http://moves.rwth-aachen.de/teaching/ws-1819/movep18/

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Overview

Geometric distribution

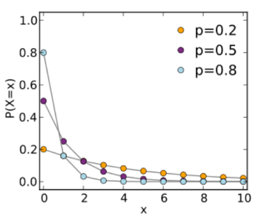
Geometric distribution

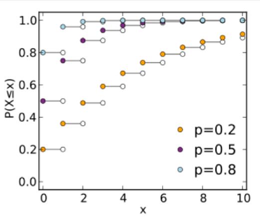
Let X be a discrete random variable, natural $k > 0$ and $0 < p \leqslant 1$. The mass function of a *geometric distribution* is given by:

$$Pr\{ X = k \} = (1-p)^{k-1} \cdot p$$

We have $E[X] = \frac{1}{p}$ and $Var[X] = \frac{1-p}{p^2}$ and cdf $Pr\{ X \leqslant k \} = 1 - (1-p)^k$.

Geometric distributions and their cdf's





Memoryless property

Theorem

1. For any random variable X with a geometric distribution:

$$Pr\{X = k + m \mid X > m\} \;=\; Pr\{X = k\} \quad \text{for any} \quad m \in T, k \geqslant 1$$

This is called the memoryless property, and X is a memoryless r.v..

$$\Pr\{T = k+m \mid T > m\}$$

$$= \Pr\{T = k+m \wedge \cancel{T > m}\} \,/\, \Pr\{T > m\}$$

$$= \frac{\cancel{p} \cdot (1-p)^{k+m-1}}{\sum_{i=m+1}^{\infty} \cancel{p} \cdot (1-p)^{i-1}}$$

$$= \frac{(1-p)^{k+\cancel{m-1}}}{\cancel{(1-p)^{m}} \cdot \underbrace{\sum_{i=0}^{\infty} (1-p)^{i}}_{\text{geometric series}}} = \frac{(1-p)^{k-1}}{\frac{1}{1-(1-p)}}$$

$$= p \cdot (1-p)^{k-1}$$

$$= \Pr\{T = k\}$$

Memoryless property

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2. Any discrete random variable which is memoryless is geometrically distributed.

Proof:

Exercise.

Andrei Andrejewitsch Markow

Markov property

The conditional probability distribution of future states of a Markov process only depends on the current state and not on its further history.

Markov process

A discrete-time stochastic process $\{ X(t) \mid t \in T \}$ over state space $\{ d_0, d_1, \ldots \}$ is a *Markov process* if for any $t_0 < t_1 < \ldots < t_n < t_{n+1}$:

$$Pr\{ X(t_{n+1}) = d_{n+1} \mid X(t_0) = d_0, X(t_1) = d_1, \ldots, X(t_n) = d_n \}$$

$$=$$

$$Pr\{ X(t_{n+1}) = d_{n+1} \mid X(t_n) = d_n \}$$

The distribution of $X(t_{n+1})$, given the values $X(t_0)$ through $X(t_n)$, only depends on the current state $X(t_n)$.

Invariance to time-shifts

Time homogeneity

Markov process $\{ X(t) \mid t \in T \}$ is *time-homogeneous* iff for any $t' < t$:

$$Pr\{ X(t) = d \mid X(t') = d' \} = Pr\{ X(t - t') = d \mid X(0) = d' \}.$$

A time-homogeneous stochastic process is invariant to time shifts.

Discrete-time Markov chain

A *discrete-time Markov chain* (DTMC) is a time-homogeneous Markov process with discrete parameter T and discrete state space.

Discrete-time Markov chain

Discrete-time Markov chain

A *discrete-time Markov chain* (DTMC) is a time-homogeneous Markov process with discrete parameter T and discrete state space S.

Transition probabilities

The *(one-step) transition probability* from $s \in S$ to $s' \in S$ at epoch $n \in \mathbb{N}$ is given by:

$$p^{(n)}(s, s') \;=\; Pr\{\, X_{n+1} = s' \mid X_n = s \,\} \;=\; Pr\{\, X_1 = s' \mid X_0 = s \,\}$$

where the last equality is due to time-homogeneity.

Since $p^{(n)}(\cdot) = p^{(k)}(\cdot)$, the superscript (n) is omitted, and we write $p(\cdot)$.

Transition probability matrix

Discrete-time Markov chain

A *discrete-time Markov chain* (DTMC) is a time-homogeneous Markov process with discrete parameter T and discrete state space S.

Transition probability matrix

Let $\mathbf{P}$ be a function with $\mathbf{P}(s_i, s_j) = p(s_i, s_j)$. For finite state space S, function $\mathbf{P}$ is called the *transition probability matrix* of the DTMC with state space S.

Properties

1. $\mathbf{P}$ is a (right) *stochastic* matrix, i.e., it is a square matrix, all its elements are in $[0, 1]$, and each row sum equals one.
2. $\mathbf{P}$ has an eigenvalue of one, and all its eigenvalues are at most one.

$P \cdot \underline{x} = c \cdot \underline{x}$ eigenvalue $c \leq 1$

Transition probability matrix

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2. **P** has an eigenvalue of one, and all its eigenvalues are at most one.
3. For all $n \in \mathbb{N}$, $\mathbf{P}^n$ is a stochastic matrix.

DTMCs — A transition system perspective

Discrete-time Markov chain

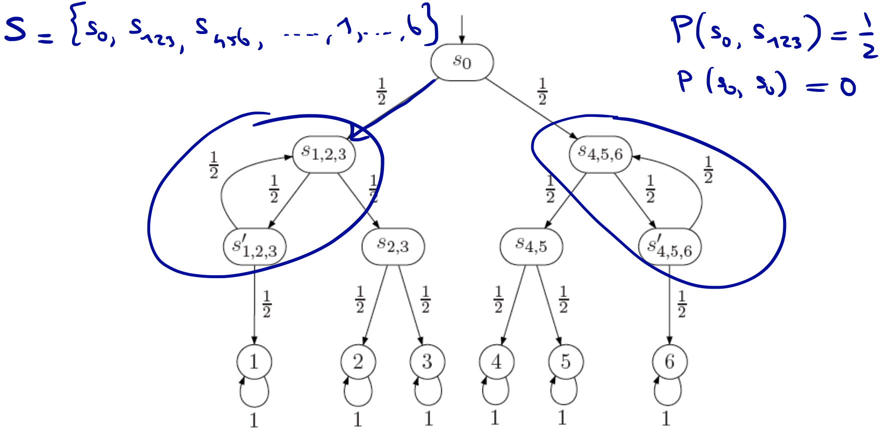
A DTMC $\mathcal{D}$ is a tuple $(S, \mathbf{P}, \iota_{\text{init}}, AP, L)$ with:

- S is a countable nonempty set of states
- $\mathbf{P} : S \times S \to [0, 1]$, transition probability function s.t. $\sum_{s'} \mathbf{P}(s, s') = 1$
- $\iota_{\text{init}} : S \to [0, 1]$, the initial distribution with $\sum_{s \in S} \iota_{\text{init}}(s) = 1$
- AP is a set of atomic propositions.
- $L : S \to 2^{AP}$, the labeling function, assigning to state s, the set $L(s)$ of atomic propositions that are valid in s.

Initial states

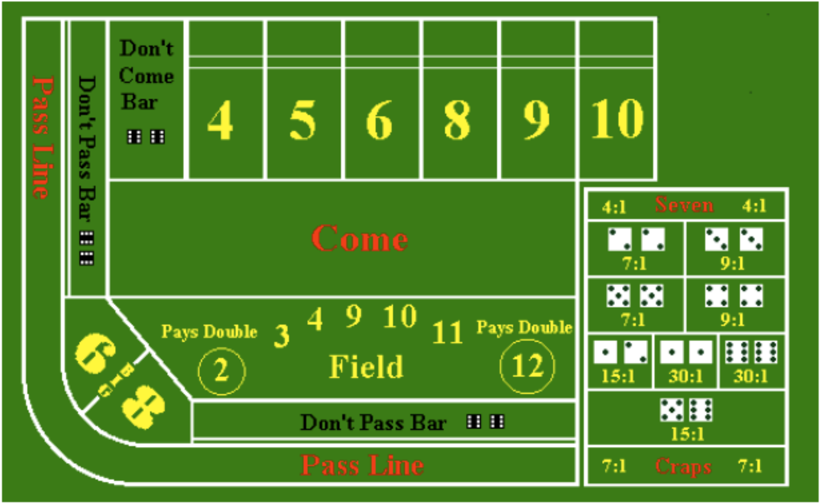
- $\iota_{\text{init}}(s)$ is the probability that DTMC $\mathcal{D}$ starts in state s
- the set $\{ s \in S \mid \iota_{\text{init}}(s) > 0 \}$ are the possible initial states.

Simulating a die by a fair coin [Knuth & Yao]



Heads = "go left"; tails = "go right".

Craps


Pass Line
Don't Pass Bar
Don't Come Bar
4
5
6
8
9
10
Come
4:1 Seven 4:1
7:1
9:1
7:1
9:1
15:1
30:1
30:1
15:1
7:1 Craps 7:1
Pays Double
2
3
4
9
10
11
Field
Pays Double
12
6
BIG
8
Don't Pass Bar
Pass Line

Craps

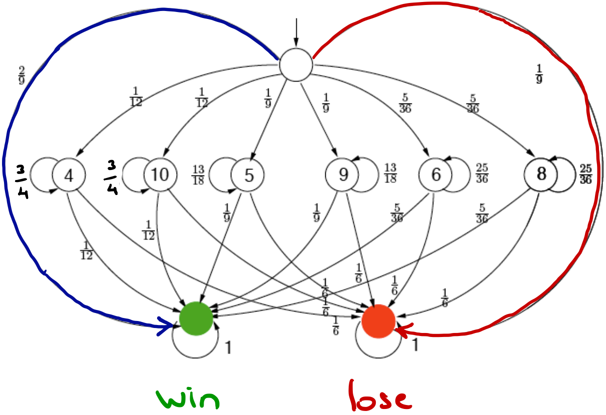
- Roll two dice and bet
- Come-out roll ("pass line" wager):
  - outcome 7 or 11: win
  - outcome 2, 3, or 12: lose ("craps")
  - any other outcome: roll again (outcome is "point")
- Repeat until 7 or the "point" is thrown:
  - outcome 7: lose ("seven-out")
  - outcome the point: win

Craps

- Roll two dice and bet
- Come-out roll ("pass line" wager):
  - outcome 7 or 11: win
  - outcome 2, 3, or 12: lose ("craps")
  - any other outcome: roll again (outcome is "point") 4
- Repeat until 7 or the "point" (= 4) is thrown:
  - outcome 7: lose ("seven-out")
  - outcome the point: win
  - any other outcome: roll again

A DTMC model of Craps

- Come-out roll:
  - 7 or 11: win
  - 2, 3, or 12: lose
  - else: roll again
- Next roll(s):
  - 7: lose
  - point: win
  - else: roll again



Overview

State residence time distribution

Let T_s be the number of epochs of DTMC $\mathcal{D}$ to stay in state s:

$$
\begin{aligned}
Pr\{\, T_s = 1 \,\} &= 1 - \mathbf{P}(s,s) \\
Pr\{\, T_s = 2 \,\} &= \mathbf{P}(s,s) \cdot (1 - \mathbf{P}(s,s)) \\
\ldots\ldots &\ \ldots\ \ \ldots\ldots \\
Pr\{\, T_s = n \,\} &= \mathbf{P}(s,s)^{n-1} \cdot (1 - \mathbf{P}(s,s))
\end{aligned}
$$

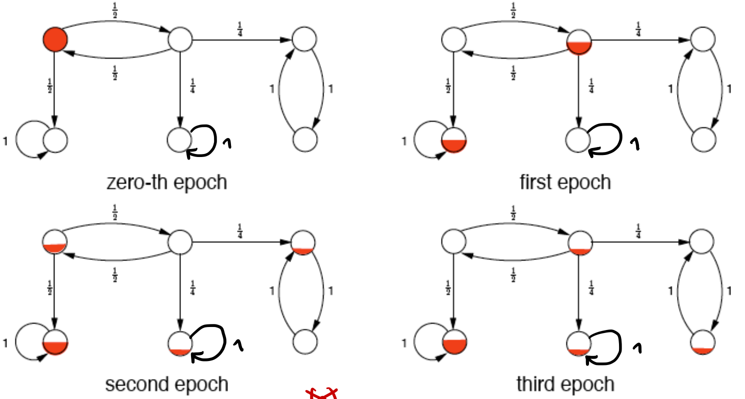
So, the state residence times in a DTMC obey a *geometric* distribution.

The expected number of time steps to stay in state s equals $E[T_s] = \frac{1}{1-\mathbf{P}(s,s)}$.
The variance of the residence time distribution is $Var[T_s] = \frac{\mathbf{P}(s,s)}{(1-\mathbf{P}(s,s))^2}$.

Recall: the geometric distribution is the only discrete probability distribution that is memoryless.

Overview

Evolution of an example DTMC



We want to determine $p_{s,s'}(n) = Pr\{\, X(n) = s' \mid X(0) = s \,\}$ for $n \in \mathbb{N}$.

Determining n-step transition probabilities

n-step transition probabilities

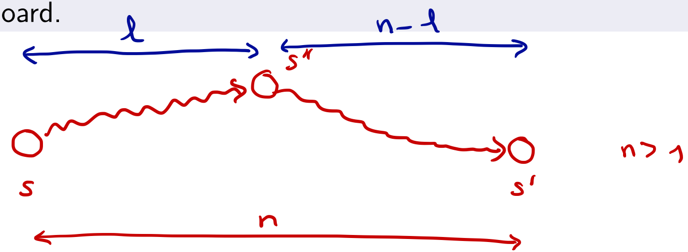
The probability to move from s to s' in $n \in \mathbb{N}$ steps is inductively defined:

$$p_{s,s'}(0) \; = \; 1 \quad \text{if } s = s', \quad \text{and 0 otherwise,}$$

$p_{s,s'}(1) = \mathbf{P}(s, s')$, and for $n > 1$ by the Chapman-Kolmogorov equation:

$$p_{s,s'}(n) = \sum_{s''} p_{s,s''}(l) \cdot p_{s'',s'}(n{-}l) \quad \text{for some } 0 < l < n$$

Proof: see black board.



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Proof: see black board.

for $n > 0$: $$p_{s,s'}(n) = \sum_{s''} p_{s,s''}(\ell) \cdot p_{s'',s'}(n-\ell)$$

Proof:

$$p_{s,s'}(n) = \Pr\{X(n)=s' \mid X(0)=s\}$$

$$= \Pr\{X(n)=s' \wedge X(0)=s\} / \Pr\{X(0)=s\}$$

$$= \sum_{s''} \frac{\Pr\{X(n)=s' \wedge X(\ell)=s'' \wedge X(0)=s\}}{\Pr\{X(0)=s\}}$$

$$= \quad (* \ \Pr(A \wedge B) = \Pr(A \mid B) \cdot \Pr(B) \ *)$$

$$\sum_{s''} \Pr\{X(n)=s' \mid X(\ell)=s'' \wedge X(0)=s\} \cdot \frac{\Pr\{X(\ell)=s'' \wedge X(0)=s\}}{\Pr\{X(0)=s\}}$$

$$= \quad (* \text{ Markov property + def. cond. prob. } *)$$

$$\sum_{s''} \Pr\{X(n)=s' \mid X(\ell)=s''\} \cdot \Pr\{X(\ell)=s'' \mid X(0)=s\}$$

$$= \quad (* \text{ time homogeneity } *)$$

$$\sum_{s''} \Pr\{X(n-\ell)=s' \mid X(0)=s''\} \cdot \Pr\{X(\ell)=s'' \mid X(0)=s\}$$

$$= \sum_{s''} p_{s'',s}(n-\ell) \cdot p_{s,s'}(\ell)$$

$\boxtimes$

Determining n-step transition probabilities

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For $l = 1$ and $n > 0$ we obtain: $p_{s,s'}(n) = \sum_{s''} p_{s,s''}(1) \cdot p_{s'',s'}(n-1)$

Determining n-step transition probabilities

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$\mathbf{P}^{(n)} = \mathbf{P}^{(1)} \cdot \mathbf{P}^{(n-1)} = \mathbf{P} \cdot \mathbf{P}^{(n-1)}$ is the n-step transition probability matrix

Repeating this scheme: $\mathbf{P}^{(n)} = \mathbf{P} \cdot \mathbf{P}^{(n-1)} = \ldots = \mathbf{P}^{n-1} \cdot \mathbf{P}^{(1)} = \mathbf{P}^n$.

Transient probability distribution

Transient distribution

$\mathbf{P}^n(s, t)$ equals the probability of being in state t after n steps given that the computation starts in s.

The probability of DTMC $\mathcal{D}$ being in state t after exactly n transitions is:

$$\Theta_n^{\mathcal{D}}(t) \;=\; \sum_{s \in S} \iota_{\text{init}}(s) \cdot \mathbf{P}^n(s, t)$$

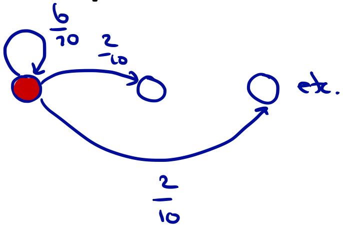
$\Theta_n^{\mathcal{D}}(t)$ is called the *transient state probability* at epoch n for state t. The function $\Theta_n^{\mathcal{D}}$ is the *transient state distribution* at epoch n of DTMC $\mathcal{D}$.

When considering $\Theta_n^{\mathcal{D}}$ as vector $(\Theta_n^{\mathcal{D}})_{t \in S}$ we have:

$$\boxed{\Theta_n^{\mathcal{D}} \;=\; \iota_{\text{init}} \cdot \underbrace{\mathbf{P} \cdot \mathbf{P} \cdot \ldots \cdot \mathbf{P}}_{n \text{ times}} \;=\; \iota_{\text{init}} \cdot \mathbf{P}^n.}$$

Transient probability distribution: example

$$P = \frac{1}{10}\begin{pmatrix} 6 & 2 & 2 \\ 1 & 8 & 1 \\ 6 & 0 & 4 \end{pmatrix}$$



$$\underline{p}(0) = (1,0,0) = \iota_{init}$$

$$\underline{p}'(0) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

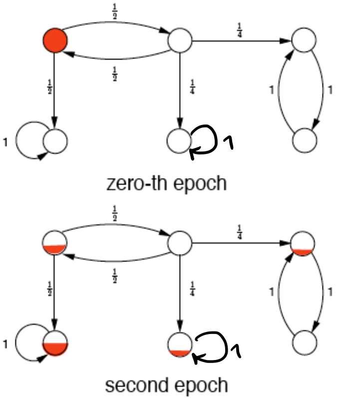
$$\underline{p}(1) = (1,0,0)\cdot P = \frac{1}{10}(6,2,2)$$

$$\underline{p}(2) = (\underline{p}(0)\cdot P)\cdot P = \frac{1}{10}(6,2,2)\cdot P = \frac{1}{100}(50,28,22)$$

$$\underline{p}(3) = \frac{1}{1000}(460, 324, 216)$$

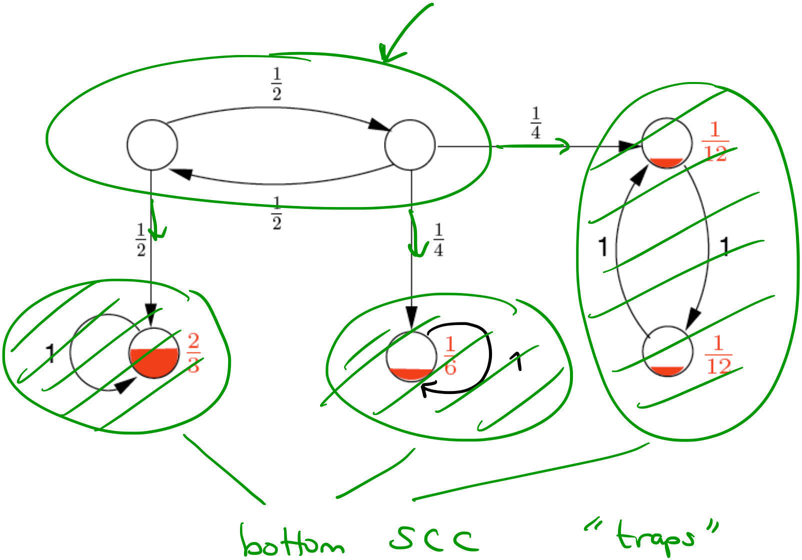
Overview

Evolution of an example DTMC

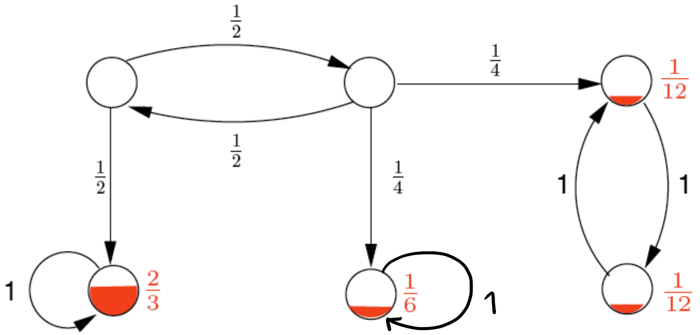


We want to determine the probability to be in a state on the long run.

On the long run



On the long run

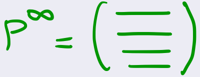


The probability mass on the long run is only left in bottom SCCs.

Limiting distribution

Ergodic stochastic matrix

Stochastic matrix **P** is called *ergodic* if:



$$\mathbf{P}^{\infty} \;=\; \lim_{n \to \infty} \mathbf{P}^{n} \quad \text{exists and has identical rows}$$

Ergodicity theorem

If the transition probability matrix **P** of a DTMC is ergodic, then:

1. $\underline{p}(n)$ converges to a limiting distribution $\underline{v}$ independent from $\underline{p}(0)$
2. each row of $\mathbf{P}^{\infty}$ equals the limiting distribution

Proof.

$$\lim_{n \to \infty} \underline{p}(0) \cdot \mathbf{P}^{n} = \underline{p}(0) \cdot \underbrace{\lim_{n \to \infty} \mathbf{P}^{n}}_{\mathbf{P}^{\infty}} = \underline{p}(0) \cdot \begin{pmatrix} v_{s_0} & \ldots & v_{s_n} \\ \ldots & \ldots & \ldots \\ v_{s_0} & \ldots & v_{s_n} \end{pmatrix} = \underline{v} \qquad \square$$

Example limiting distribution

(A)

$$P = \frac{1}{10}\begin{pmatrix} 6 & 2 & 2 \\ 1 & 8 & 1 \\ 6 & 0 & 4 \end{pmatrix}$$

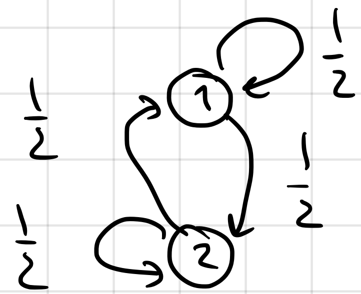
$$P^{\infty} = \lim_{n\to\infty} P^n = \begin{pmatrix} \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

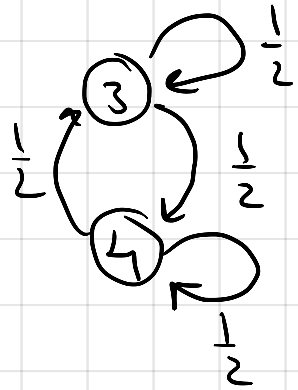
So: P is ergodic

(B)

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

<u>not</u> ergodic





$P^2 = P$ so $\lim_{n\to\infty} P^n = P$, but no identical rows

Example

$$\underline{v} = \underline{v} \cdot P \qquad \sum_i v_i = 1$$

$$\underline{v} = (v_1, v_2, v_3)$$

$$P = \frac{1}{10} \begin{pmatrix} 6 & 2 & 2 \\ 1 & 8 & 1 \\ 6 & 0 & 4 \end{pmatrix}$$

$$\left.\begin{aligned} v_1 &= \frac{1}{10}(6v_1 + 2v_2 + 2v_3) \\ v_2 &= \frac{1}{10}(v_1 + 8v_2 + v_3) \\ v_3 &= \frac{1}{10}(6v_1 + 4v_3) \\ v_1 + v_2 + v_3 &= 1 \end{aligned}\right\} \qquad \begin{aligned} v_1 &= \frac{2}{5} \\ v_2 &= \frac{2}{5} \\ v_3 &= \frac{1}{5} \end{aligned}$$

Limiting distribution

- We also have:

$$\underline{v} = \lim_{n\to\infty} \underline{p}(n+1) = \lim_{n\to\infty} \underline{p}(0) \cdot \mathbf{P}^{n+1} = \left(\lim_{n\to\infty} \underline{p}(0) \cdot \mathbf{P}^{n} \right) \cdot \mathbf{P} = \underline{v} \cdot \mathbf{P}$$

- Thus, limiting probabilities can be obtained by solving the (homogeneous) system of linear equations:

$$\underline{v} = \underline{v} \cdot \mathbf{P} \quad \text{or} \quad \underline{v} \cdot (\mathbf{I} - \mathbf{P}) = \underline{0} \quad \text{under } \textstyle\sum_i \underline{v}(i) = 1$$

  - vector $\underline{v}$ is the left Eigenvector of $\mathbf{P}$ with Eigenvalue 1
- $\underline{v}$ is called the *limiting* state-probability vector

Two interpretations of $\underline{v}(s)$:

- the long-run proportion of time that the DTMC "spends" in state s
- the probability the DTMC is in s when making a snapshot after a very long time

Limiting distributions should not be confused with <u>stationary</u> <u>distributions</u>

vector $\underline{\pi}$ is a stationary distribution of a DTMC with transition prob matrix P whenever:

$$\underline{\pi} = \underline{\pi} \cdot P$$

Equivalently:

$$\underline{\pi}(s) = \sum_{s'} \underline{\pi}(s') \cdot P(s', s)$$

or equivalently

$$\underbrace{\underline{\pi}(s) \cdot \big(1 - P(s,s)\big)}_{\text{the "outflux" of } s} = \underbrace{\sum_{s' \neq s} \underline{\pi}(s') \cdot P(s', s)}_{\text{the "influx" of } s}$$

Periodic Markov chains may have a stationary distribution but do <u>not</u> have a limiting distribution.

Summary

What are Markov chains?

- A discrete-time Markov chain (DTMC) is a time-homogeneous Markov process with discrete parameter T and discrete state space S.
- State residence times are geometrically distributed.
- Alternative: a DTMC $\mathcal{D}$ is a tuple $(S, \mathbf{P}, \iota_{\text{init}}, AP, L)$

What are transient probabilities?

- $\Theta_n^{\mathcal{D}}(s)$ is the probability to be in state s after n steps.
- These transient probabilities satisfy: $\Theta_n^{\mathcal{D}} = \iota_{\text{init}} \cdot \mathbf{P}^n$.

What are long-run probabilities?

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What are long-run probabilities?

- $\underline{v}(s)$ is the probability to be in state s after infinitely many steps.

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What are long-run probabilities?

- $\underline{v}(s)$ is the probability to be in state s after infinitely many steps.
- long-run probabilities satisfy: $\underline{v} \cdot (\mathbf{I} - \mathbf{P}) = \underline{0}$ under $\sum_i \underline{v}(i) = 1$.