Modeling and Verification of Probabilistic Systems

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http://moves.rwth-aachen.de/teaching/ws-1819/movep18/

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Overview

Overview

Labeled transition system

Transition system

A *(labeled) transition system* TS is a structure $(S, Act, \longrightarrow, I_0, AP, L)$ where

- S is a (possibly infinitely countable) set of states.
- Act is a (possibly infinitely countable) set of actions.
- $\longrightarrow \subseteq S \times Act \times S$ is a transition relation.
- $I_0 \subseteq S$ the set of initial states.
- AP is a set of atomic propositions.
- $L : S \to 2^{AP}$ is the labeling function.

Notation

We write $s \xrightarrow{\alpha} s'$ instead of $(s, \alpha, s') \in \longrightarrow$.

Strong bisimulation

Strong bisimulation relation [Milner, 1980 & Park, 1981]

Let $TS = (S, Act, \longrightarrow, I_0, AP, L)$ be a transition system and $R \subseteq S \times S$.
Then R is a *strong bisimulation* on TS whenever for all $(s, t) \in R$:

1. $L(s) = L(t)$

Strong bisimulation

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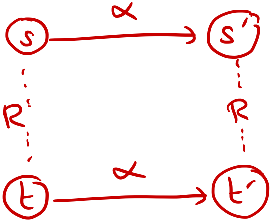
1. $L(s) = L(t)$
2. if $s \xrightarrow{\alpha} s'$ then there exists $t' \in S$ such that $t \xrightarrow{\alpha} t'$ and $(s', t') \in R$



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3. if $t \xrightarrow{\alpha} t'$ then there exists $s' \in S$ such that $s \xrightarrow{\alpha} s'$ and $(s', t') \in R$

$$\begin{tikzcd} t \arrow[r, "\alpha"] \arrow[d, dash, dotted, "R"] & t' \\ s & \end{tikzcd}$$

then

$$\begin{tikzcd} t \arrow[r, "\alpha"] \arrow[d, dash, dotted, "R"] & t' \arrow[d, dash, dotted, "R"] \\ s \arrow[r, "\alpha"] & s' \end{tikzcd}$$

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Strong bisimilarity

Let $TS = (S, Act, \longrightarrow, I_0, AP, L)$ be a transition system and $s, t \in S$.

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Strong bisimilarity

Let $TS = (S, Act, \longrightarrow, I_0, AP, L)$ be a transition system and $s, t \in S$.
Then: s is *strongly bisimilar* to t, notation $s \sim t$ if there *exists* a strong bisimulation R such that $(s,t) \in R$.

alternatively $\sim = \bigcup_{R} \{ R \text{ is a strong bisimulation} \}$

Strong bisimulation

$\sim$ is an equivalence, and is also a strong bisim.

Strong bisimulation relation [Milner, 1980 & Park, 1981]

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Remarks

Not every bisimulation relation is transitive.

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Not every bisimulation relation is transitive. But: $\sim$ is an equivalence.

Strong bisimulation

Pictorial representation

$$\begin{array}{ccc} s & \xrightarrow{\alpha} & s' \\ R & & \\ t & & \end{array} \quad \text{can be completed to} \quad \begin{array}{ccc} s & \xrightarrow{\alpha} & s' \\ R & & \color{red}{R} \\ t & \color{red}{\xrightarrow{\alpha}} & \color{red}{t'} \end{array}$$

and

$$\begin{array}{ccc} s & & \\ R & & \\ t & \xrightarrow{\alpha} & t' \end{array} \quad \text{can be completed to} \quad \begin{array}{ccc} s & \color{red}{\xrightarrow{\alpha}} & \color{red}{s'} \\ R & & \color{red}{R} \\ t & \xrightarrow{\alpha} & t' \end{array}$$

Strongly bisimilar transition systems

Bisimilar transition systems

Let TS_1, TS_2 be transition systems over the same set of atomic propositions with initial states $I_{0,1}$ and $I_{0,2}$, respectively.

Consider the transition system $TS = TS_1 \uplus TS_2$ that results from the disjoint union of TS_1 and TS_2.

Then: TS_1 and TS_2 are called strongly bisimilar if there exists a strong bisimulation R on $S_1 \uplus S_2$ such that:

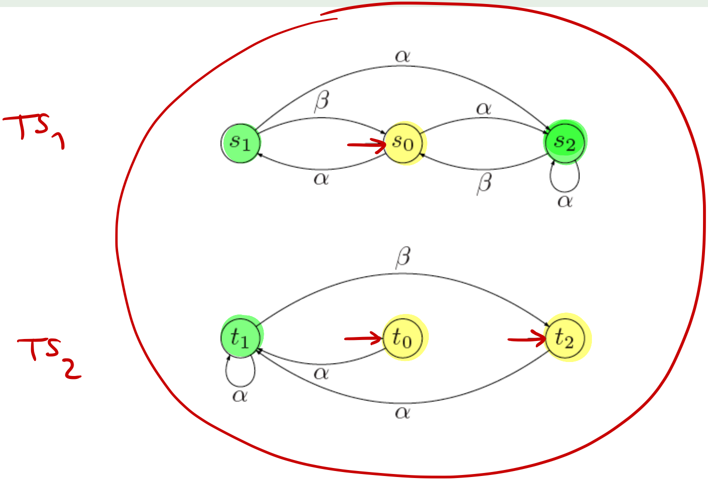
1. $\forall s \in I_{0,1}. \exists t \in I_{0,2}. (s, t) \in R$, and
2. $\forall t \in I_{0,2}. \exists s \in I_{0,1}. (s, t) \in R$.

Example (1)

$R \subseteq S \times S$

$$\begin{array}{ccc} t_0 & \xrightarrow{\alpha} & t_1 \\ \vdots & & \vdots \\ s_0 & \xrightarrow{\alpha} & s_1 \end{array}$$

Are these transition systems strongly bisimilar? (No propositions.)



S_1

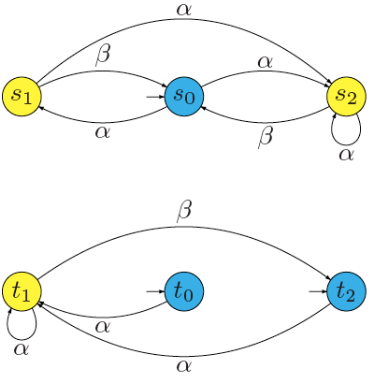
$\uplus$

S_2

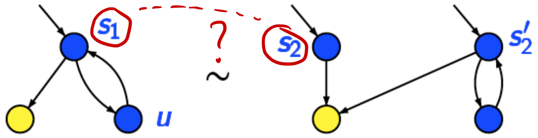
$= S$

Example (2)

Yes, they are!

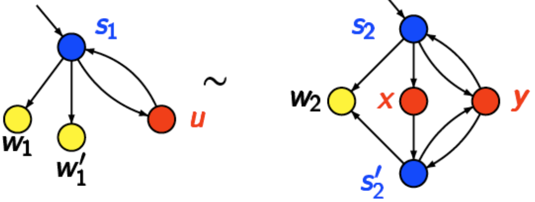


Correct or wrong?



wrong

$s_1 \longrightarrow u$, but $s_2 \not\longrightarrow$ *blue* (thus $s_1 \not\sim s_2$)



correct

bisimulation:

$$\{(w_1, w_2), (w_1', w_2), (s_1, s_2), (s_1, s_2'), (u, x), (u, y)\}$$

Quotient LTS under $\sim$

is an equivalence

Quotient transition system

For $TS = (S, Act, \longrightarrow, I_0, AP, L)$ and strong bisimilarity $\sim \subseteq S \times S$ let

$$TS/\sim \; = \; (S', Act, \longrightarrow', I_0', AP, L'), \quad \text{the } quotient \text{ of } TS \text{ under } \sim$$

where

represent every equivalence class under $\sim$ by a single state

Quotient LTS under $\sim$

Quotient transition system

For $TS = (S, Act, \longrightarrow, I_0, AP, L)$ and strong bisimilarity $\sim \subseteq S \times S$ let

$$TS/\sim \; = \; (S', Act, \longrightarrow', I'_0, AP, L'), \qquad \text{the } \textit{quotient} \text{ of } TS \text{ under } \sim$$

where

- $S' = S/\sim \; = \; \{ [s]_\sim \mid s \in S \}$ with $[s]_\sim \; = \; \{ s' \in S \mid s \sim s' \}$
- $\longrightarrow'$ is defined by: $\dfrac{s \xrightarrow{\alpha} s'}{[s]_\sim \xrightarrow{\alpha}{}' [s']_\sim}$
- $I'_0 = \{ [s_0]_\sim \mid s_0 \in I_0 \}$, the equivalence class of the initial states in TS
- $L'([s]_\sim) = L(s)$.

$[s]_\sim$ L(s)

$L'([s]_\sim)$

$[s]_\sim$

Quotient LTS under $\sim$

Quotient transition system

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- $I_0' = \{ [s_0]_\sim \mid s_0 \in I_0 \}$, the equivalence class of the initial states in TS
- $L'([s]_\sim) = L(s)$.

Remarks

L' is well-defined as all states in $[s]_\sim$ are equally labeled. Note that if $s \xrightarrow{\alpha} s'$, then for all $t \sim s$ we have $t \xrightarrow{\alpha} t'$ with $s' \sim t'$.

Quotient transition system

For any transition system TS it holds: $TS \;\sim\; TS/\!\sim$.

Proof:

The binary relation:

$$R \;=\; \{\, (s, [s]_{\sim}) \mid s \in S \,\}$$

is a strong bisimulation on the disjoint union $TS \uplus TS/\!\sim$.

$$\begin{tikzcd} s \arrow[r, "\alpha"] \arrow[d, dash, dotted] & t \\ {[s]_\sim} & \end{tikzcd}$$

then

$$\begin{tikzcd} s \arrow[r, "\alpha"] \arrow[d, dash, dotted] & t \arrow[d, dash, dotted] \\ {[s]_\sim} \arrow[r, "\alpha"] & {[t]_\sim} \end{tikzcd}$$

by def.

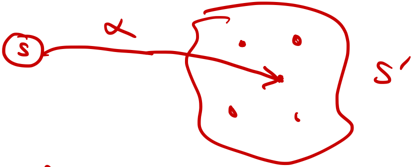
Strong bisimulation revisited

Auxiliary predicate

Let $P : S \times Act \times 2^S \to \{0, 1\}$ be a predicate such that for $S' \subseteq S$:

$$P(s, \alpha, S') = \begin{cases} 1 & \text{if } \exists s' \in S'.\ s \xrightarrow{\alpha} s' \\ 0 & \text{otherwise.} \end{cases}$$

true if



$P(s, \alpha, S') = 1$

Strong bisimulation revisited

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$$P(s, \alpha, S') = \begin{cases} 1 & \text{if } \exists s' \in S'.\ s \xrightarrow{\alpha} s' \\ 0 & \text{otherwise.} \end{cases}$$

Alternative definition of strong bisimulation

Let $TS = (S, Act, \longrightarrow, I_0, AP, L)$ and R an *equivalence relation* on S.
Then: R is a *strong bisimulation* on S if for $(s, t) \in R$:

1. $L(s) = L(t)$, and
2. $P(s, \alpha, C) \ = \ P(t, \alpha, C)$ for all C in S/R and $\alpha \in Act$.

$s \sim' t$, if there *exists* a strong bisimulation R such that $(s, t) \in R$.

It can be easily proven that $\sim$ coincides with $\sim'$. Proof is omitted.

Overview

Probabilistic bisimulation: intuition

Intuition

- Strong bisimulation is used to compare labeled transition systems.
- Strongly bisimilar states exhibit the same step-wise behaviour.
- Our aim: adapt bisimulation to discrete-time Markov chains.
- This yields a probabilistic variant of strong bisimulation.

- When do two DTMC states exhibit the same step-wise behaviour?
- Key: if their transition probability for each equivalence class coincides.

1989: Kim G. Larsen and Arne Skou

Kim G. Larsen

Arne Skou

Probabilistic bisimulation

$P(s, \alpha, C) = P(t, \alpha, C)$

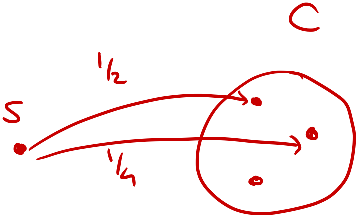
Probabilistic bisimulation [Larsen & Skou, 1989]

Let $\mathcal{D} = (S, \mathbf{P}, \iota_{\mathrm{init}}, AP, L)$ be a DTMC and $R \subseteq S \times S$ an equivalence.
Then: R is a *probabilistic bisimulation* on S if for any $(s, t) \in R$:

1. $L(s) = L(t)$, and
2. $\mathbf{P}(s, C) = \mathbf{P}(t, C)$ for all equivalence classes $C \in S/R$

where $\mathbf{P}(s, C) = \sum_{s' \in C} \mathbf{P}(s, s')$.

includes $[s]_R$



$P(s, C) = \frac{3}{4}$

Probabilistic bisimulation

Probabilistic bisimulation [Larsen & Skou, 1989]

Let $\mathcal{D} = (S, \mathbf{P}, \iota_{\text{init}}, AP, L)$ be a DTMC and $R \subseteq S \times S$ an equivalence. Then: R is a *probabilistic bisimulation* on S if for any $(s, t) \in R$:

1. $L(s) = L(t)$, and
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where $\mathbf{P}(s, C) = \sum_{s' \in C} \mathbf{P}(s, s')$.

For states in R, the probability of moving to some equivalence class is equal.

Probabilistic bisimilarity

Let $\mathcal{D}$ be a DTMC and s, t states in $\mathcal{D}$.

Probabilistic bisimulation

Probabilistic bisimulation [Larsen & Skou, 1989]

Let $\mathcal{D} = (S, \mathbf{P}, \iota_{\text{init}}, AP, L)$ be a DTMC and $R \subseteq S \times S$ an equivalence.
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For states in R, the probability of moving to some equivalence class is equal.

Probabilistic bisimilarity

Let $\mathcal{D}$ be a DTMC and s, t states in $\mathcal{D}$. Then: s is *probabilistic bisimilar* to t, denoted $s \sim_p t$, if there exists a probabilistic bisimulation R with $(s, t) \in R$.

Probabilistic bisimulation

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$\{a_1, \ldots, a_{10}\}$ $\{a_1, \ldots, a_{10}\}$

s t

Probabilistic bisimulation

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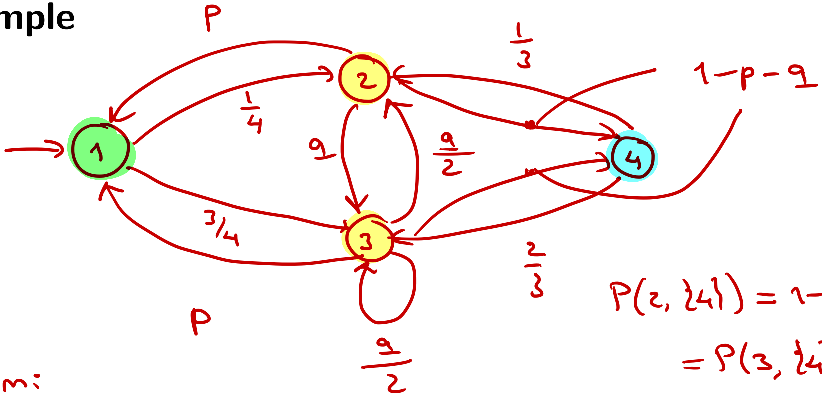
Remarks

As opposed to bisimulation on states in transition systems, any probabilistic bisimulation is an equivalence.

But: not every strong bisimulation is an equivalence

Example



$P(2, \{4\}) = 1-p-q$
$= P(3, \{4\})$

Claim:

$R = \{ (1,1), \ldots, (4,4), (2,3), (3,2) \}$ is a prob. bisim.

$P(2, \{1\}) = p = P(3, \{1\})$

$P(2, \{2,3\}) = q = \frac{q}{2} + \frac{q}{2} = P(3, \{2,3\})$

Bisimilar DTMCs

Bisimilar DTMCs

Let $\mathcal{D}_1$, $\mathcal{D}_2$ be DTMCs over the same set of atomic propositions with initial distributions ι^1_{init} and ι^2_{init}, respectively.

Consider the DTMC $\mathcal{D} = \mathcal{D}_1 \uplus \mathcal{D}_2$ that results from the disjoint union of $\mathcal{D}_1$ and $\mathcal{D}_2$. Consider $\sim_p$ on $\mathcal{D} = \mathcal{D}_1 \uplus \mathcal{D}_2$.

Then $\mathcal{D}_1$ and $\mathcal{D}_2$ are bisimilar, denoted $\mathcal{D}_1 \sim_p \mathcal{D}_2$ whenever

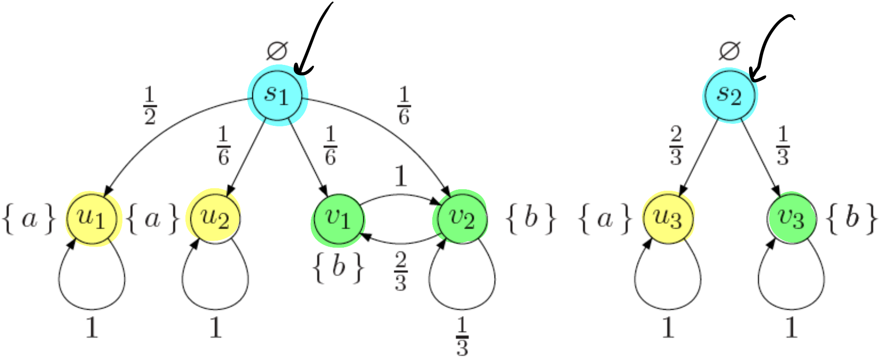
$$\iota^1_{\mathrm{init}}(C) = \iota^2_{\mathrm{init}}(C)$$

for each bisimulation equivalence class C of $\mathcal{D} = \mathcal{D}_1 \uplus \mathcal{D}_2$ under $\sim_p$.

Here, $\iota_{\mathrm{init}}(C)$ denotes $\sum\limits_{s \in C} \iota_{\mathrm{init}}(s)$.

Example

$$P(s_1, \quad) = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} = P(s_2, \quad)$$



Quotient under $\sim_p$

Quotient DTMC under $\sim_p$

For $\mathcal{D} = (S, \mathbf{P}, \iota_{\text{init}}, AP, L)$ and probabilistic bisimilarity $\sim_p \subseteq S \times S$ let

$$\mathcal{D}/\sim_p \;=\; (S', \mathbf{P}', \iota'_{\text{init}}, AP, L'), \qquad \text{the } \textit{quotient} \text{ of } \mathcal{D} \text{ under } \sim_p$$

where

- $S' = S/\sim_p = \{ [s]_{\sim_p} \mid s \in S \}$ with $[s]_{\sim_p} \;=\; \{ s' \in S \mid s \sim_p s' \}$
- $\mathbf{P}'([s]_{\sim_p}, [s']_{\sim_p}) \;=\; \mathbf{P}(s, [s']_{\sim_p})$

↳ total prob. mass of going from s to

$[s']_{\sim_p}$

Quotient under $\sim_p$

Quotient DTMC under $\sim_p$

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where

- $S' = S/\sim_p = \{\, [s]_{\sim_p} \mid s \in S \,\}$ with $[s]_{\sim_p} \;=\; \{\, s' \in S \mid s \sim_p s' \,\}$
- $\mathbf{P}'([s]_{\sim_p}, [s']_{\sim_p}) \;=\; \mathbf{P}(s, [s']_{\sim_p})$
- $\iota'_{\text{init}}([s]_{\sim_p}) = \sum_{s' \in [s]_{\sim_p}} \iota_{\text{init}}(s)$
- $L'([s]_{\sim_p}) = L(s)$.

Remarks

The transition probability from $[s]_{\sim_p}$ to $[t]_{\sim_p}$ is $\mathbf{P}(s, [t]_{\sim_p})$.

Quotient under $\sim_p$

Quotient DTMC under $\sim_p$

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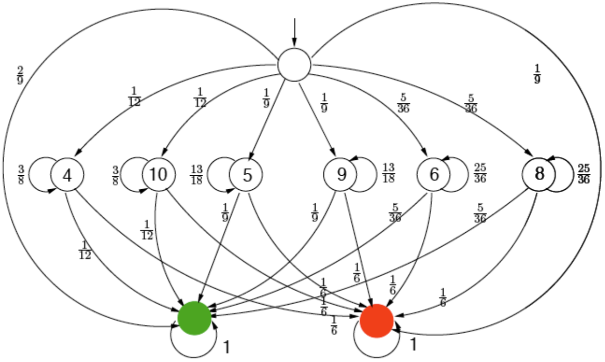
- $\mathbf{P}'([s]_{\sim_p}, [s']_{\sim_p}) \; = \; \mathbf{P}(s, [s']_{\sim_p})$
- $\iota'_{\text{init}}([s]_{\sim_p}) = \sum_{s' \in [s]_{\sim_p}} \iota_{\text{init}}(s)$
- $L'([s]_{\sim_p}) = L(s)$.

Remarks

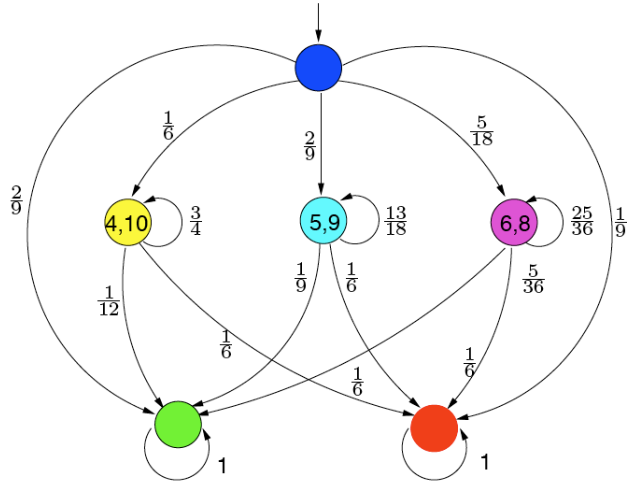
The transition probability from $[s]_{\sim_p}$ to $[t]_{\sim_p}$ is $\mathbf{P}(s, [t]_{\sim_p})$. This is well-defined as $\mathbf{P}(s, C) = \mathbf{P}(s', C)$ for all $s \sim_p s'$ and all bisimulation equivalence classes C.

A DTMC model of Craps

- Come-out roll:
  - 7 or 11: win
  - 2, 3, or 12: lose
  - else: roll again
- Next roll(s):
  - 7: lose
  - point: win
  - else: roll again



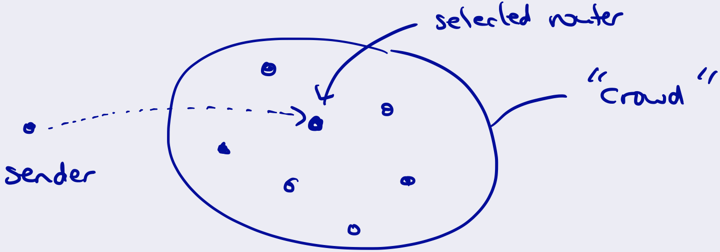
Quotient DTMC of Craps under $\sim_p$



Example: Crowds protocol

Security: Crowds protocol [Reiter & Rubin, 1998]

- A protocol for anonymous web browsing (variants: mCrowds, BT-Crowds)
- Hide user's communication by random routing within a crowd
  - sender selects a crowd member randomly using a uniform distribution



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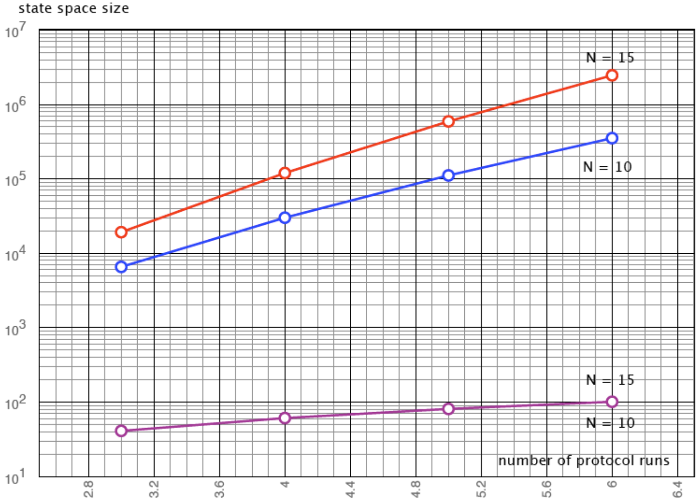
- A protocol for anonymous web browsing (variants: mCrowds, BT-Crowds)
- Hide user's communication by random routing within a crowd
  - sender selects a crowd member randomly using a uniform distribution
  - selected router flips a biased coin:
    - with probability $1 - p$: direct delivery to final destination
    - otherwise: select a next router randomly (uniformly)
  - once a routing path has been established, use it until crowd changes
- Rebuild routing paths on crowd changes
- Property: Crowds protocol ensures "probable innocence":
  - probability real sender is discovered $< \frac{1}{2}$ if $N \geqslant \frac{p}{p-\frac{1}{2}} \cdot (c+1)$
  - where N is crowd's size and c is number of corrupt crowd members

State space reduction under $\sim_p$



IEEE 802.11 group communication protocol

	original DTMC			quotient DTMC		red. factor	
OD	states	transitions	ver. time	blocks	total time	states	time
4	1125	5369	122	71	13	**15.9**	**9.00**
12	37349	236313	7180	1821	642	**20.5**	**11.2**
20	231525	1590329	50133	10627	5431	**21.8**	**9.2**
28	804837	5750873	195086	35961	24716	**22.4**	**7.9**
36	2076773	15187833	5103900	91391	77694	**22.7**	**6.6**
40	3101445	22871849	7725041	135752	127489	**22.9**	**6.1**

all times in milliseconds

$Pr(\Diamond G)$

$22.9 \approx \frac{3101445}{135752}$

Overview

PCTL syntax

Probabilistic Computation Tree Logic: Syntax

PCTL consists of state- and path-formulas.

- PCTL *state formulas* over the set AP obey the grammar:

$$\Phi \ ::= \ \text{true} \ \Big| \ a \ \Big| \ \Phi_1 \wedge \Phi_2 \ \Big| \ \neg\Phi \ \Big| \ \mathbb{P}_J(\varphi)$$

  where $a \in AP$, φ is a path formula and interval $J \subseteq [0,1]$.

- PCTL *path formulae* are formed according to the following grammar:

$$\varphi \ ::= \ \bigcirc \Phi \ \Big| \ \Phi_1 \, \mathsf{U} \, \Phi_2 \ \Big| \ \Phi_1 \, \mathsf{U}^{\leqslant n} \, \Phi_2$$

  where Φ, Φ_1, and Φ_2 are state formulae and $n \in \mathbb{N}$.

Preservation of PCTL-formulas

Preservation of PCTL-formulas

Bisimulation preserves PCTL

Let $\mathcal{D}$ be a DTMC and s, t states in $\mathcal{D}$. Then:

$$s \sim_p t \quad \text{if and only if} \quad s \text{ and } t \text{ are PCTL-equivalent.}$$

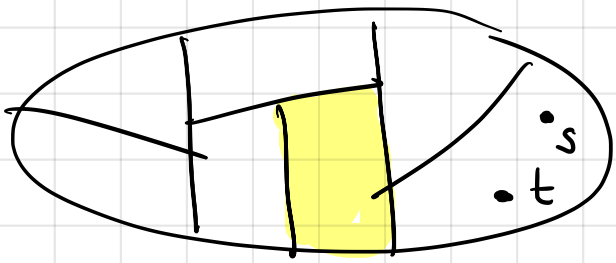
$$s \sim_p t \implies \left(\forall \Phi \in PCTL.\ s \models \Phi \text{ iff } t \models \Phi \right)$$

$$s \not\sim_p t \impliedby \left(\exists \Phi \in PCTL.\ s \not\models \Phi \text{ and } t \models \Phi \right)$$

Lemma DTMC $\mathcal{D}$, $s,t \in \mathcal{D}$, $G \in S/_{\sim_p}$

Then:

$$s \sim_p t \implies \Pr(s \models \Diamond G) = \Pr(t \models \Diamond G)$$



Proof (sketch). If $[s]_{\sim_p} = G = [t]_{\sim_p}$ easy,

$\Pr(s \models \Diamond G) = 1 = \Pr(t \models \Diamond G)$.

Assume $[s]_{\sim_p} \cap G = \emptyset$

Let $S' = \{ s \in S \mid \Pr(s \models \Diamond G) > G \} \setminus G$

Then S' is a disjoint union of eq. classes under $\sim_p$. For $s \in S'$, the vector

$\left(\Pr(s \models \Diamond G)\right)_{s \in S'}$ is the unique

solution of the lin. equation system

$$x_s = P(s,G) + \sum_{s' \in S'} P(s,s') \cdot x_{s'} \qquad (*)$$

Now consider the unique solution

$$(x_B)_{B \in S/_{\sim_p},\, B \subseteq S'} \quad \text{of}$$

(**)

$$x_B = \underbrace{P(s_B, G)}_{\substack{\text{1-step prob.} \\ \text{to go to } G}} + \sum_{\substack{C \in S/_{\sim_p} \\ C \subseteq S'}} \underbrace{P(s_B, C)}_{\downarrow} \cdot x_C \quad \text{for } s_B \in B$$

$$\sum_{t \in C} P(s_B, t)$$

<u>Claim</u> $x_s = x_B$ for all states $s \in B$.

<u>Proof:</u> prove that $(y_s)_{s \in S'}$ is a solution of (*) where $y_s = x_B$ if $s \in B$, $B \in S'/_{\sim_p}$

Derive: $$y_s = P(s, G) + \sum_{s' \in S'} P(s, s') \cdot y_{s'}$$

$=$ (* partition S' into eq. classes *)

$$P(s, G) + \sum_{\substack{s' \in C \\ C \in S'/_{\sim_p}}} P(s, s') \cdot y_{s'}$$

$$P(s,G) + \sum_{\substack{s' \in C \\ C \in S'/_{\sim_P}}} P(s,s') \cdot y_{s'}$$

$$= \quad (* \; y_{s'} = x_B \text{ if } s' \in B \text{ and } B \in S/_{\sim_P} *)$$

$$P(s,G) + \sum_{\substack{s' \in C \\ C \in S'/_{\sim_P}}} P(s,s') \cdot x_C$$

$$= P(s,G) + \sum_{C \in S'/_{\sim_P}} \underbrace{\sum_{s' \in C} P(s,s')}_{P(s,C)} \cdot x_C$$

$$= \quad (* \; s \sim_P t \text{ , so } P(s,G) = P(t,G) \text{ and } P(s,C) = P(t,C)$$

$$P(t,G) + \sum_{C \in S'/_{\sim_P}} P(t,C) \cdot x_C$$

$$= x_B$$

Preservation of PCTL-formulas

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Let $\mathcal{D}$ be a DTMC and s, t states in $\mathcal{D}$. Then:

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Remarks

$s \sim_p t$ implies that

1. transient probabilities, reachability probabilities,

$\Diamond^{=n}$

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Remarks

$s \sim_p t$ implies that

1. transient probabilities, reachability probabilities,
2. repeated reachability, persistence probabilities

$$Pr(s \models \Box\Diamond G) = Pr(t \models \Box\Diamond G)$$

$$\Diamond\Box \qquad\qquad \Diamond\Box$$

Preservation of PCTL-formulas

Bisimulation preserves PCTL

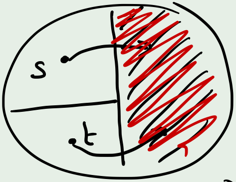
Let $\mathcal{D}$ be a DTMC and s, t states in $\mathcal{D}$. Then:

$$s \sim_p t \quad \text{if and only if} \quad s \text{ and } t \text{ are PCTL-equivalent.}$$

Remarks

$s \sim_p t$ implies that

1. transient probabilities, reachability probabilities,
2. repeated reachability, persistence probabilities
3. all qualitative PCTL formulas



for s and t are equal.

$s \models \mathbb{P}_{>\frac{1}{2}}(\Diamond \, red)$

$t \not\models \mathbb{P}_{>\frac{1}{2}}(\Diamond \, red)$

If for PCTL-formula Φ we have $s \models \Phi$ but $t \not\models \Phi$, then it follows $s \not\sim_p t$.

Preservation of PCTL-formulas

Bisimulation preserves PCTL

Let $\mathcal{D}$ be a DTMC and s, t states in $\mathcal{D}$. Then:

$$s \sim_p t \quad \text{if and only if} \quad s \text{ and } t \text{ are PCTL-equivalent.}$$

Remarks

$s \sim_p t$ implies that

1. transient probabilities, reachability probabilities,
2. repeated reachability, persistence probabilities
3. all qualitative PCTL formulas

for s and t are equal.

If for PCTL-formula Φ we have $s \models \Phi$ but $t \not\models \Phi$, then it follows $s \not\sim_p t$.
A single PCTL-formula suffices!

PCTL* syntax

PCTL: $\bigcirc \Phi \mid \Phi \cup \Phi \mid \Phi \cup^{\leq n} \Phi$

Probabilistic Computation Tree Logic: Syntax

PCTL* consists of state- and path-formulas.

- PCTL* *state formulas* over the set AP obey the grammar:

$$\Phi ::= \text{true} \;\big|\; a \;\big|\; \Phi_1 \wedge \Phi_2 \;\big|\; \neg\Phi \;\big|\; \mathbb{P}_J(\varphi)$$

  where $a \in AP$, φ is a path formula and $J \subseteq [0,1]$, $J \neq \varnothing$ is a non-empty interval.

  $\bigcirc\bigcirc a$

- PCTL* *path formulae* are formed according to the following grammar:

LTL
$$\varphi ::= \Phi \;\big|\; \neg\varphi \;\big|\; \varphi_1 \wedge \varphi_2 \;\big|\; \bigcirc \varphi \;\big|\; \varphi_1 \,\mathsf{U}\, \varphi_2$$

  where Φ is a state formula and φ, φ_1, and φ_2 are path formulae.

$(a \cup b) \cup c$

PCTL* semantics (1)

Notation

$\mathcal{D}, s \models \Phi$ if and only if state-formula Φ holds in state s of (possibly infinite) DTMC $\mathcal{D}$. As $\mathcal{D}$ is known from the context we simply write $s \models \Phi$.

Satisfaction relation for state formulas

The satisfaction relation $\models$ is defined for PCTL* state formulas by:

$$
\begin{array}{lll}
s \models a & \text{iff} & a \in L(s) \\
s \models \neg\,\Phi & \text{iff} & \text{not } (s \models \Phi) \\
s \models \Phi \wedge \Psi & \text{iff} & (s \models \Phi) \text{ and } (s \models \Psi) \\
s \models \mathbb{P}_J(\varphi) & \text{iff} & Pr(s \models \varphi) \in J
\end{array}
$$

where $Pr(s \models \varphi) = Pr_s\{\, \pi \in Paths(s) \mid \pi \models \varphi \,\}$

PCTL* semantics (2)

Satisfaction relation for path formulas

Let $\pi = s_0\, s_1\, s_2 \ldots$ be an infinite path in (possibly infinite) DTMC $\mathcal{D}$. Let $\pi^i = s_i\, s_{i+1}\, s_{i+2} \ldots$ denotes the i-th suffix of π.

The satisfaction relation $\models$ is defined for state formulas by:

$$\begin{array}{lll}
\pi \models \Phi & \text{iff} & \pi[0] \models \Phi \\
\pi \models \neg\varphi & \text{iff} & \text{not } \pi \models \varphi \\
\pi \models \varphi_1 \wedge \varphi_2 & \text{iff} & \pi \models \varphi_1 \text{ and } \pi \models \varphi_2 \\
\pi \models \bigcirc\varphi & \text{iff} & \pi^1 \models \varphi \\
\pi \models \varphi_1 \,\mathsf{U}\, \varphi_2 & \text{iff} & \exists k \geqslant 0. (\pi^k \models \varphi_2 \wedge \forall 0 \leqslant i < k.\, \pi^i \models \varphi_1)
\end{array}$$

Measurability

PCTL* measurability

For any PCTL* path formula φ and state s of DTMC $\mathcal{D}$,
the set $\{\, \pi \in \mathit{Paths}(s) \mid \pi \models \varphi \,\}$ is measurable.

Proof:

Left as an exercise, using the result for PCTL measurability and the measurability of ω-regular properties.

Bounded until in PCTL*

Bounded until

Bounded until can be defined using the other operators:

$$\underbrace{\varphi_1 \, \mathsf{U}^{\leqslant n} \varphi_2}_{\text{in PCTL}} \;=\; \bigvee_{0 \leqslant i \leqslant n} \psi_i \quad \text{where } \psi_0 = \varphi_2 \text{ and } \psi_{i+1} = \varphi_1 \wedge \bigcirc \psi_i \text{ for } i \geqslant 0.$$

$\Phi_1 \, \mathsf{U}^{\leq n} \, \Phi_2$

state formulas

$k = 3$

$\psi_3 = \varphi_1 \wedge \bigcirc \psi_2$

$\psi_2 = \varphi_1 \wedge \bigcirc \psi_1$

$\psi_1 = \varphi_1 \wedge \bigcirc \psi_0$

$\psi_0 = \varphi_2$

Bounded until in PCTL*

Bounded until

Bounded until can be defined using the other operators:

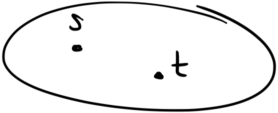
$$\varphi_1 \, \mathsf{U}^{\leqslant n} \varphi_2 \;=\; \bigvee_{0 \leqslant i \leqslant n} \psi_i \quad \text{where } \psi_0 = \varphi_2 \text{ and } \psi_{i+1} = \varphi_1 \wedge \bigcirc \psi_i \text{ for } i \geqslant 0.$$

Examples in PCTL* but not in PCTL

$\mathbb{P}_{>\frac{1}{4}}(\bigcirc a \, \mathsf{U} \, \bigcirc b)$ and $\mathbb{P}_{=1}(\mathbb{P}_{>\frac{1}{2}}(\Box\Diamond a \vee \Diamond\Box b))$.

$\mathbb{P}_{>\frac{1}{2}}(\Box\Diamond a) \vee \mathbb{P}_{>\frac{1}{2}}(\Diamond\Box b) \leftarrow$ PCTL

Preservation of PCTL*-formulas



Bisimulation preserves PCTL*

Let $\mathcal{D}$ be a DTMC and s, t states in $\mathcal{D}$. Then:

$$s \sim_p t \quad \text{if and only if} \quad s \text{ and } t \text{ are PCTL}^*\text{-equivalent.}$$

if and only if s and t are PCTL- equivalet

Remarks

1. Bisimulation thus preserves not only all PCTL but also all PCTL* formulas.
2. By the last two results it follows that PCTL- and PCTL*-equivalence coincide.

Preservation of PCTL*-formulas

Bisimulation preserves PCTL*

Let $\mathcal{D}$ be a DTMC and s, t states in $\mathcal{D}$. Then:

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Remarks

1. Bisimulation thus preserves not only all PCTL but also all PCTL* formulas.
2. By the last two results it follows that PCTL- and PCTL*-equivalence coincide. Thus any two states that satisfy the same PCTL formulas, satisfy the same PCTL* formulas.

Bisimilar paths

$$\pi_1 = s_{0,1}\, s_{1,1}\, s_{2,1}\, s_{3,1} \dots$$
$$\pi_2 = s_{0,2}\, s_{1,2}\, s_{2,2}\, s_{3,2} \dots$$

$\pi_1 \sim_p \pi_2$ iff $\forall i \geq 0.\ s_{i,1} \sim_p s_{i,2}$

Bisimulation-closed σ-algebra

let $C_0, \dots, C_n \in S/_{\sim_p}$. let $\mathbb{C}_{\sim_p}$ be the σ-algebra generated by

$$Cyl(C_0 \dots C_n) = \{ \sigma \in Paths^D \mid \sigma = t_0 \dots t_n t_{n+1} t_{n+2} \dots \text{ with } \forall 0 \leq i \leq n.\ t_i \in C_i \}$$

let $\Pi \in \mathbb{C}_{\sim_p}$ be a measurable set of paths (in D)

Π is bisimulation-closed if

$$\forall \pi_1 \in \Pi.\ \pi_1 \sim_p \pi_2 \Rightarrow \pi_2 \in \Pi.$$

Lemma:

$s_1 \sim_p s_2 \Longrightarrow \forall$ bisimulation-closed Π:

$$Pr_{s_1}(\Pi) = Pr_{s_2}(\Pi).$$

Why is this useful?

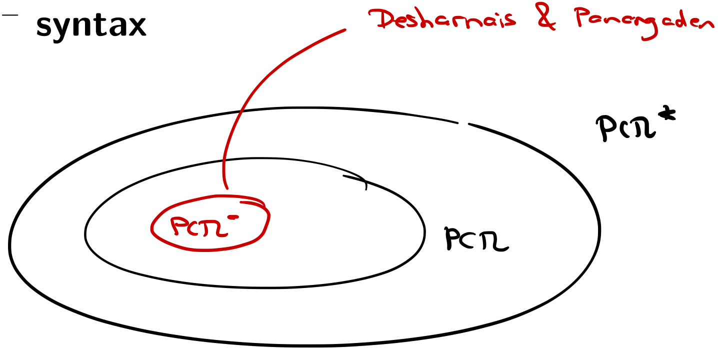
To show $s_1 \sim_p s_2 \implies s_1, s_2$ are PCTL*-equiv.

1. $s_1 \sim_p s_2 \implies s_1 \models \Phi$ iff $s_2 \models \Phi$

(*) 2. $s_1 \sim_p s_2 \implies \underbrace{Pr(s_1 \models \varphi) = Pr(s_2 \models \varphi)}_{\Pi_J(\varphi)}$

3. $\pi_1 \sim_p \pi_2$ implies $\pi_1 \models \varphi$ iff $\pi_2 \models \varphi$

PCTL⁻ syntax

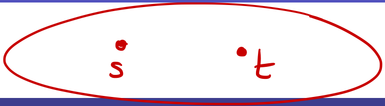


$$s_1 \sim_p s_2 \quad \underline{\underline{\text{iff}}} \quad \forall \Phi \in PCTL^-. \, (s_1 \models \Phi \text{ iff } s_2 \models \Phi)$$

$$\Longrightarrow$$

$$s_1 \not\sim_p s_2 \quad \Longleftarrow \quad \exists \Phi \in PCTL^-. \, (s_1 \not\models \Phi \wedge s_2 \models \Phi)$$

PCTL⁻ syntax



Simple Probabilistic Computation Tree Logic: Syntax

PCTL⁻ only consists of state-formulas. These formulas over the set AP obey the grammar:

$$\Phi \ ::= \ a \ \Big| \ \Phi_1 \wedge \Phi_2 \ \Big| \ \Phi_1 \vee \Phi_2 \ \Big| \ \mathbb{P}_{\leqslant p}(\bigcirc \Phi)$$

where $a \in AP$ and p is a probability in $[0, 1]$.

Remarks

This is a truly simple logic. It does not contain the until-operator. Negation is not present and cannot be expressed. Only upper bounds on probabilities.

It turns out that PCTL-, PCTL*- and PCTL⁻-equivalence coincide.

Preservation of PCTL

PCTL/PCTL* and Bisimulation Equivalence

Let $\mathcal{D}$ be a DTMC and s_1, s_2 states in $\mathcal{D}$. Then, the following statements are equivalent:

finite / countably infinite /

(a) $s_1 \sim_p s_2$.

(b) s_1 and s_2 are PCTL*-equivalent, i.e., fulfill the same PCTL* formulas

(c) s_1 and s_2 are PCTL-equivalent, i.e., fulfill the same PCTL formulas

(d) s_1 and s_2 are PCTL$^-$-equivalent, i.e., fulfill the same PCTL$^-$ formulas

$$\sim_p \;=\; \equiv_{PCTL^*} \;=\; \equiv_{PCTL} \;=\; \equiv_{PCTL^-}$$

Preservation of PCTL

PCTL/PCTL* and Bisimulation Equivalence

Let $\mathcal{D}$ be a DTMC and s_1, s_2 states in $\mathcal{D}$. Then, the following statements are equivalent:

(a) $s_1 \sim_p s_2$.

(b) s_1 and s_2 are PCTL*-equivalent, i.e., fulfill the same PCTL* formulas

(c) s_1 and s_2 are PCTL-equivalent, i.e., fulfill the same PCTL formulas

(d) s_1 and s_2 are PCTL$^-$-equivalent, i.e., fulfill the same PCTL$^-$ formulas

Proof:

1. (a) $\Longrightarrow$ (b): by structural induction on PCTL* formulas.
2. (b) $\Longrightarrow$ (c): trivial as PCTL is a sublogic of PCTL*.
3. (c) $\Longrightarrow$ (d): trivial as PCTL$^-$ is a sublogic of PCTL.
4. (d) $\Longrightarrow$ (a): involved. First finite DTMCs, then for arbitrary DTMCs.

Proof

Finite case D is finite.

$$\underbrace{s_1 \equiv_{PCTL^-} s_2}_{s_1 + s_2 \text{ are PCTL}^-\text{-equivalet}} \implies s_1 \sim_p s_2$$

Proof Assume $s_1 \equiv_{PCTL^-} s_2$. We show that

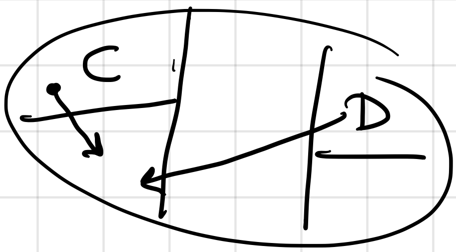
$$R = \{ (s_1, s_2) \mid s_1 \equiv_{PCTL^-} s_2 \} \quad \text{is a}$$

probabilistic bisimulation. Let $(s_1, s_2) \in R$.

(1) $L(s_1) = L(s_2)$. Easy since $s_1 \equiv_{PCTL^-} s_2$ and all atomic propositions are expressible in $PCTL^-$.

(2) Let $C \in S/_R$. Does $P(s_1, C) = P(s_2, C)$?

Let $C, D \in S/_R$; and $\Phi_{C,D}$ be a $PCTL^-$ state formula s.t. $Sat(\Phi_{C,D}) \subseteq C \wedge Sat(\Phi_{C,D}) \cap D = \emptyset$



Let $\Phi_C = \bigwedge_{\substack{D \in S/_{\sim_p} \\ D \neq C}} \Phi_{C,D}$

finitary conjunction since $|S/_{\sim_p}| < \infty$

Then: $Sat(\Phi_C) = C$.

W.l.o.g. assume $P(s_1, C) \leq P(s_2, C)$

Assume $P(s_1, C) \subsetneq P(s_2, C)$

let $s_1 \models \mathbb{P}_{\leq p}(\bigcirc \Phi_C)$ where $p = P(s_1, C)$

Since $s_1 \equiv_{PCTL^-} s_2$ it follows

$$s_2 \models \mathbb{P}_{\leq p}(\bigcirc \Phi_C)$$

But then $\boxed{P(s_1, C) = P(s_2, C)}$ $\boxtimes$

In the infinite case, this proof principle can be applied since

$$\underbrace{\begin{array}{c}\text{"negation"} \\ \text{"countable conjunction"}\end{array}}_{\text{logic}} \begin{array}{c} = \\ = \end{array} \underbrace{\begin{array}{c}\text{"complementation"} \\ \text{"countable intersections"}\end{array}}_{\text{measurable spaces.}}$$

Theorem $s_1 \equiv_{PCTL^-} s_2 \implies s_1 \sim_p s_2$

Proof: $R = \{ (s_1, s_2) \mid s_1 \equiv_{PCTL^-} s_2 \}$ is a prob. bisimulation.

(1) $L(s_1) = L(s_2)$ as before

(2) let $(s_1, s_2) \in R$. let $Sat(\Phi)$, $\Phi \in PCTL^-$ be basic events on S and $\mathfrak{C}_S$ is the smallest σ-algebra containing the sets $\{Sat(\Phi) \mid \Phi \in PCTL^-\}$

Since $S_{\sim p}$ is countable, every $PCTL^-$ equivalence class $C \in S/_R$ can be written as a countable intersection of $Sat(\Phi)$ and $C \subseteq Sat(\Phi)$.

Thus: all $PCTL^-$-eq. classes belong to $\mathfrak{C}_S$.

As $PCTL^-$ permits (finitary) conjunction, the set of all sets $Sat(\Phi)$ is closed under finite intersections.

Prop. for every prob. measure $\mu_1, \mu_2 \in \mathfrak{C}_S$

Proposition

for every prob. measure $\mu_1, \mu_2 \in \mathfrak{C}_S$

$$\mu_1(Sat(\Phi)) = \mu_2(Sat(\Phi)) \text{ for } \forall \Phi \in PCTL^-$$

implies

$$\mu_1 = \mu_2 .$$

Overview

1960: Laurie Snell and John Kemeny

Laurie Snell

John Kemeny

Lumpability

Ignore the initial distribution and state-labelling of a Markov chain.

Lumpability **[Kemeny & Snell, 1960]**

Let $\mathcal{D}$ be a (possibly countably infinite) DTMC with state space S and $\mathcal{B} = \{ B_1, \ldots, B_n \}$ be a partitioning of S (where B_j may be countably infinite). $\mathcal{D}$ is lumpable with respect to $\mathcal{B}$ iff for any B_i and B_j in $\mathcal{B}$ and any $s, s' \in B_i$:

$$\sum_{u \in B_j} \mathbf{P}(s, u) \; = \; \sum_{u \in B_j} \mathbf{P}(s', u) \quad \text{that is} \quad \mathbf{P}(s, B_j) = \mathbf{P}(s', B_j).$$

If $\mathcal{D}$ is lumpable with respect to $\mathcal{B}$, $\mathcal{B}$ is called a lumpable partition

It is easy to show that $S/\sim_p$ is a lumpable partition of the state space S. In fact, it is the coarsest possible lumpable partition.

Lumping equivalence

Lumping equivalence **[Kemeny & Snell, 1960]**

The DTMCs $\mathcal{D}$ and $\mathcal{D}'$ are lumping equivalent if there are lumpable partitions $\mathcal{B}$ of $\mathcal{D}$ and $\mathcal{B}'$ of $\mathcal{D}'$ such that there is an injective function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that:

$$\mathbf{P}(B_i, B_j) \;=\; \mathbf{P}'(B'_{f(i)}, B'_{f(j)}).$$

Corollary

$\mathcal{D} \sim_p \mathcal{D}'$ if and only if $\mathcal{D}$ and $\mathcal{D}'$ are lumping equivalent (with respect to the coarsest possible lumpable partition on their union).

Lumping equivalence

Remark

For finite Markov chains, the correspondence between lumping equivalence and $\sim_p$ allows to obtain the coarsest possible lumpable partition in an algorithmic, i.e., automated manner.

This can be considered as a breakthrough in Markov chain theory.

Overview

Summary

- Bisimilar states have equal transition probabilities for every equivalence class.
- $\sim_p$ is the coarsest probabilistic bisimulation.
- All states in a quotient DTMC are equivalence classes under $\sim_p$.
- $\sim_p$ and PCTL-equivalence coincide.
- PCTL, PCTL*, and PCTL$^-$-equivalence coincide.
- To show $s \not\sim_p t$, show $s \models \Phi$ and $t \not\models \Phi$ for $\Phi \in$ PCTL$^-$.
- Bisimulation may yield up to exponential savings in state space.

Take-home message

Probabilistic bisimulation on Markov chains coincides with a notion from the sixties, named (ordinary) lumpability.