Prof. Dr. Ir. Dr. h. c. Joost-Pieter Katoen
Jip Spel, Tim Quatmann

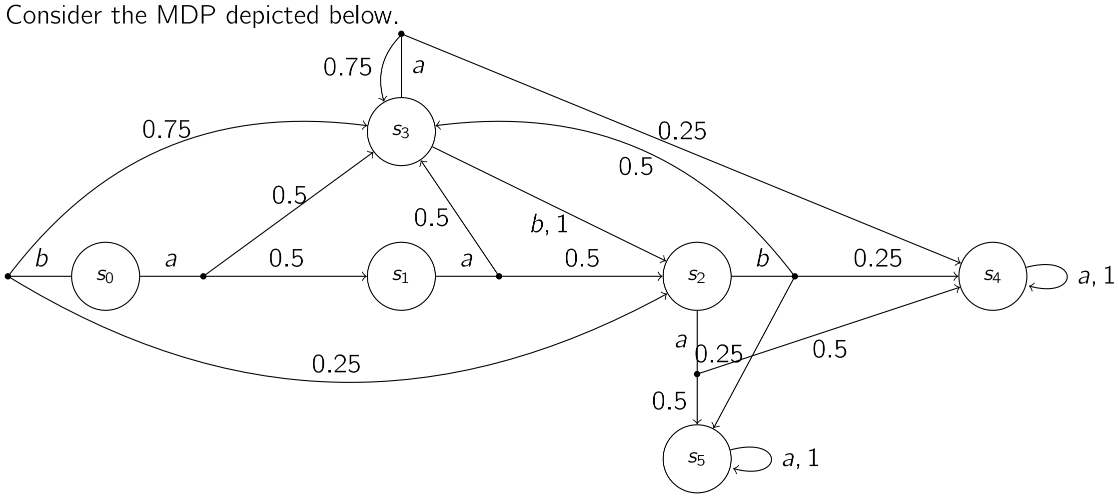
Modeling and Verification of Probabilistic Systems
— Exercise Sheet 7 —

Notes:

- The exercise sheets need to be solved in groups of 2 students.
- Hand in the solution until 5.12.2018 before the exercise class.
- We do not accept solutions via L2p or email.
- Write your names and matriculation numbers on the front page and staple all pages.

Exercise 1 (Policy Iteration): (10 + 10 = 20 Points)

Consider the MDP depicted below.



Consider the probabilities for the following property: $\mathrm{Pr}^{\mathcal{M}}_{max}(s \models \Diamond s_4)$ for all $s \in S$.

a) Execute policy iteration, start with taking a in all states. If you need an ordering on the states, use the ordering of the state numbers.

b) Give the LP formulation to characterize the property above.

Exercise 2 (PCTL Equivalence): (15 + 7 + 7 + 11 = 40 Points)

a) Prove or disproof:

$$\mathbb{P}_{\geq 1}(\Diamond a) \wedge \mathbb{P}_{\geq 1}(\Diamond b) \equiv_{MDP} \mathbb{P}_{\geq 1}\Big(\Diamond\big((a \wedge \mathbb{P}_{\geq 1}(\Diamond b)) \vee (b \wedge \mathbb{P}_{\geq 1}(\Diamond a))\big)\Big)$$

b) Let $\mathcal{M}$ be an MDP. We consider the existential satisfaction relation $\models_\exists$ for PCTL formulas, where

$$\mathcal{M}, s \models_\exists \mathbb{P}_J(\varphi) \text{ iff there **exists** a policy } \mathfrak{S} \text{ on } \mathcal{M}.\mathrm{Pr}^{\mathfrak{S}}(s \models^\exists \varphi) \in J.$$

For other PCTL state and path formulas, $\models_\exists$ is defined similar to the satisfaction relation $\models$ from the lecture.

Prove or disprove for all MDP $\mathcal{M}$:

$$\mathcal{M}, s \models \mathbb{P}_J(\varphi) \implies \mathcal{M}, s \models_\exists \mathbb{P}_J(\varphi)$$

c) Prove or disprove for all MDP $\mathcal{M}$:

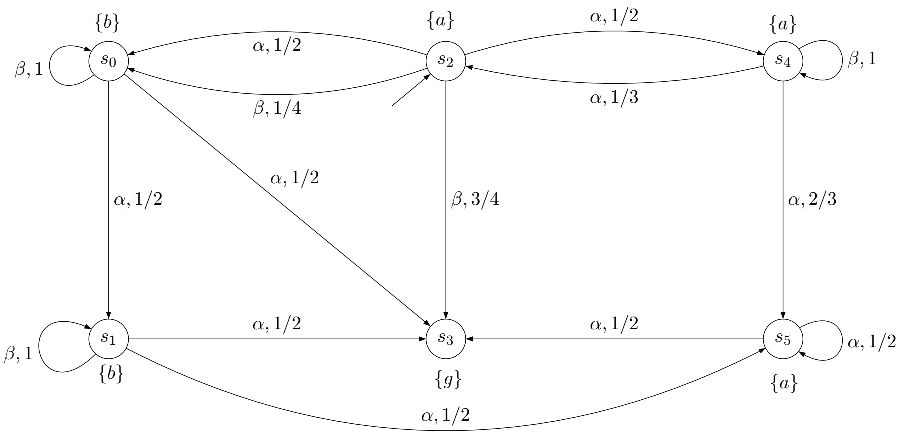
$$\mathcal{M}, s \models_\exists \mathbb{P}_J(\varphi) \implies \mathcal{M}, s \models \mathbb{P}_J(\varphi)$$

d) For two PCTL formulas Φ and Ψ, we write $\Phi \equiv^\exists_{MDP} \Psi$ if and only if for all MDPs $\mathcal{M}$, it holds: $Sat^\exists_{\mathcal{M}}(\Phi) = Sat^\exists_{\mathcal{M}}(\Psi)$, where $Sat^\exists_{\mathcal{M}}(\Phi) = \{s \mid \mathcal{M}, s \models_\exists \Phi\}$.

Prove or disprove for all PCTL formulas Φ and Ψ:

$$\Phi \equiv_{MDP} \Psi \text{ if and only if } \Phi \equiv^\exists_{MDP} \Psi$$

Exercise 3 (PCTL and Policies): (25 Points)



Consider the MDP M as shown above. Find out the satisfaction sets of the following properties or list the probabilities corresponding to a maximizing scheduler for all states, give the used policies, and describe their properties:

1. $\mathbb{P}_{\geq 0.5}(\bigcirc a)$
2. $\mathbb{P}_{\geq 0.6}(\bigcirc b)$
3. $\mathbb{P}_{\leq 0.3}(\lozenge g)$
4. $\Pr^M(s \models (\square \lozenge a))$
5. $\Pr^M(s \models (\square \lozenge a \land \neg \square \lozenge b))$

Exercise 4 (CTMC): (15 Points)

Model the following server system as a CTMC.

- The server system consists of two servers and a queue of capacity one.
- Jobs arrive with a rate λ, and are scheduled to any available server.
- If no server is available, they're put in the queue.
- The servers handle the jobs with rate μ.
- If done, a job from the queue is taken, if there is one. Otherwise, the server is idle.