Prof. Dr. Ir. Dr. h. c. Joost-Pieter Katoen

Jip Spel, Tim Quatmann

Modeling and Verification of Probabilistic Systems
— Exercise Sheet 6 —

Notes:

- The exercise sheets need to be solved in groups of 2 students.
- Hand in the solution until 28.11.2018 before the exercise class.
- We do not accept solutions via L2p or email.
- Write your names and matriculation numbers on the front page and staple all pages.

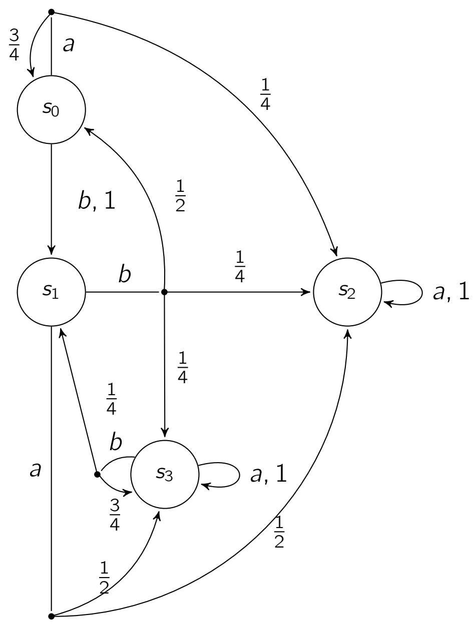
Exercise 1 (Positional Policy): (15 Points)

Let $\mathcal{M}$ be a finite MDP with state space S and initial state s_0.
Prove or disprove: For $T \subseteq S$ and for any $k \in \mathbb{N}$ there exists a positional policy such that

$$\Pr^{\mathcal{M}}_{\sigma}(s_0 \models \lozenge^{\leq k} T) = \Pr^{\mathcal{M}}_{\min}(s_0 \models \lozenge^{\leq k} T)$$

Exercise 2 (Value Iteration): (30 Points)

Consider the MDP depicted below.



Use value iteration to compute the probabilities $\Pr^{max}(s \models \lozenge s_2)$ for all $s \in S$. You may abort the process after 5 iterations.

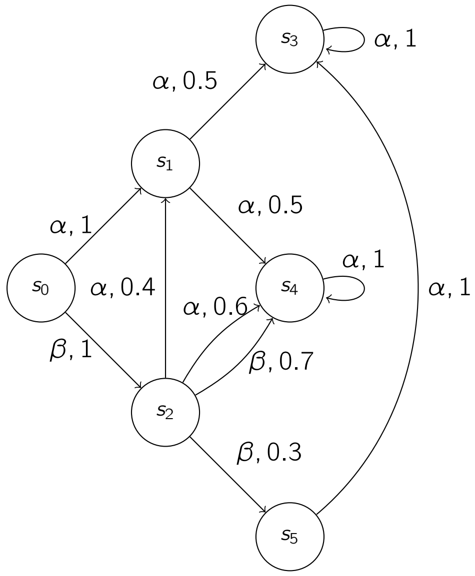
Exercise 3 (Acyclic MDPs): (12 + 18 = 30 Points)

We call an MDP $\mathcal{M} = (S, Act, \mathbf{P}, \iota_{init}, AP, L)$ *acyclic* if all SCCs of the underlying graph

$$G = (S, E) \text{ with } E = \{(s, t) \in S \times S \mid \exists \alpha \in Act \colon \mathbf{P}(s, \alpha, t) > 0\}$$

are trivial bottom SCCs (i.e., singleton sets that can not be left). Put differently, if a state s lies on a cycle, then $\mathbf{P}(s, \alpha, s) = 1$ for all $\alpha \in Act(s)$.

a) Compute $\Pr^{\max}(s \models \Diamond\{s4\})$ for each state s of the following acyclic MDP.



Hints:

- Solve the equation system for max-reach probabilities from the lecture by computing the values x_s in a specific order.

b) Describe an Algorithm (in pseudo code) that gets an acyclic MDP and a set of goal states $T \subseteq S$ as input and computes $\Pr^{\max}(s \models \Diamond T)$ for all $s \in S$ in *linear time*. More precisely, the worst-case runtime of your algorithm should be in $\mathcal{O}(|\mathbf{P}|)$, where $|\mathbf{P}| = |\{(s, \alpha, t) \in S \times Act \times S \mid \mathbf{P}(s, \alpha, t) > 0\}|$. Briefly justify the runtime of your algorithm (e.g. by giving the runtime of the individual steps).

Hints:

- Make use of dynamic programing as well as the equation system for max-reach probabilities.
- You may assume that non-zero transition probabilities are stored in adjacency lists. This means, for example, that the sets $Post(s, \alpha) = \{t \in S \mid \mathbf{P}(s, \alpha, t) > 0\}$ can be computed in $\mathcal{O}(|Post(s, \alpha)|)$ time.

Exercise 4 (Alternative Value Iteration): (15 + 10 = 25 Points)

Let $\mathcal{M} = (S, Act, \mathbf{P}, \iota_{init}, AP, L)$ be an MDP with a set of goal states $G \subseteq S$. We consider a slight variant of the value iteration algorithm from the lecture: For $s \in G$ we set $x_s = 1$, for $s \not\models \exists\Diamond G$ we set $x_s = 0$, and for the remaining states we set

$$x_s^{(0)} = 1 \text{ and } x_s^{(n+1)} = \max\left\{\sum_{t \in S} \mathbf{P}(s, \alpha, t) \cdot x_t^{(n)} \mid \alpha \in Act(s)\right\}.$$

Notice that the only difference is that we now start the iterations at $x_s^{(0)} = 1$.

a) Show that, in general, the sequence $x_s^{(0)}, x_s^{(1)}, x_s^{(2)}, \ldots$ no longer converges to $x_s = \Pr^{\max}(s \models \Diamond G)$.

b) Argue why this problem does not occur when computing *minimal* reachability probabilities. A formal proof is not required.