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Modeling and Verification of Probabilistic Systems

— Exercise Sheet 5 —

Notes:

- The exercise sheets need to be solved in groups of 2 students.
- Hand in the solution until 21.11.2018 before the exercise class.
- We do not accept solutions via L2p or email.
- Write your names and matriculation numbers on the front page and staple all pages.

Exercise 1 (Definition Strong Bisimulation): (24 Points)

In the lecture two definitions of strong bisimulation were given.

Definition 1 (Lecture 9, Slide 5)
Let $TS = (S, Act, \rightarrow, I_0, AP, L)$ be a transition system and $R \subset S \times S$. Then R is a strong bisimulation on TS whenever for all $(s, t) \in R$:

1. $L(s) = L(t)$
2. if $s \xrightarrow{\alpha} s'$ then there exists $t' \in S$ such that $t \xrightarrow{\alpha} t'$ and $(s', t') \in R$
3. if $t \xrightarrow{\alpha} t'$ then there exists $s' \in S$ such that $s \xrightarrow{\alpha} s'$ and $(s', t') \in R$

Definition 2 (Lecture 9, Slide 13)
Let $TS = (S, Act, \rightarrow, I_0, AP, L)$ and R an equivalence relation on S. Then: R is a strong bisimulation on S if for $(s, t) \in R$:

1. $L(s) = L(t)$, and
2. $P(s, \alpha, C) = P(t, \alpha, C)$ for all $C \in S/R$ and $\alpha \in Act$.

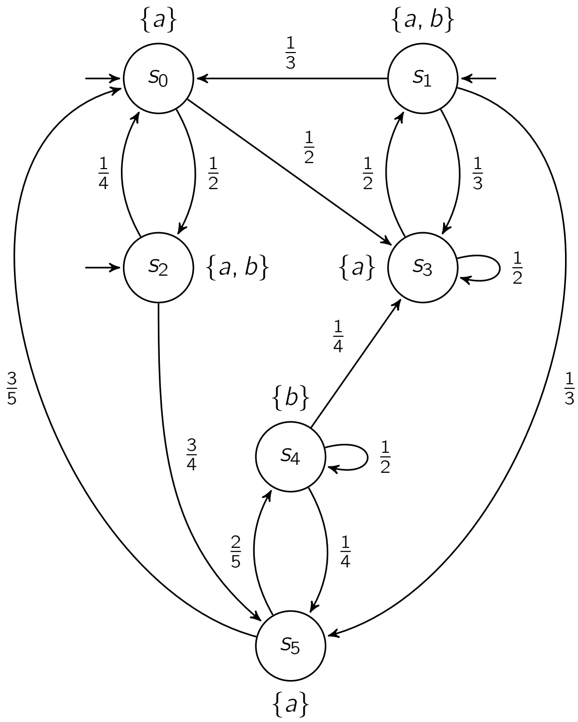
where

$$P(s, \alpha, S') = \begin{cases} 1 & \text{if } \exists s' \in S'.\ s \xrightarrow{\alpha} s' \\ 0 & \text{otherwise} \end{cases}$$

Prove that for a given equivalence relation R the two definitions are equivalent.

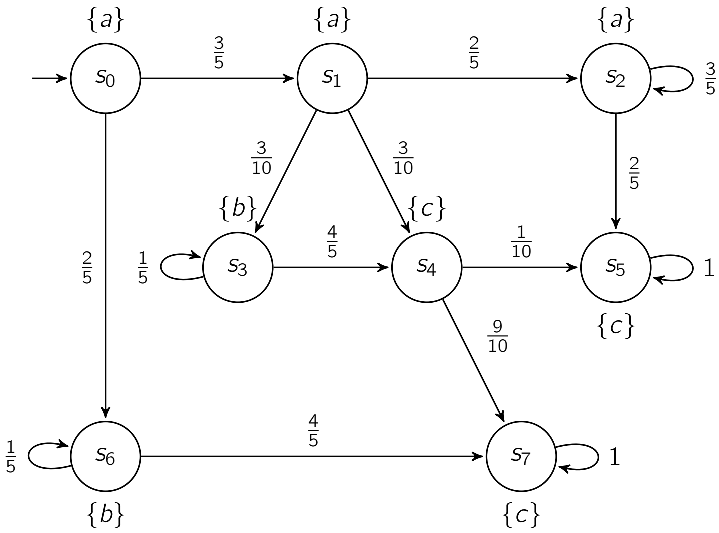
Exercise 2 (Bisimulation): (24 + 16 = 40 Points)

a) Consider the following DTMC:



Show that:
(i) $s_0 \not\sim_p s_1$ (ii) $s_0 \not\sim_p s_5$ (iii) $s_0 \not\sim_p s_3$ (iv) $s_1 \not\sim_p s_2$
by provide distinguishing PCTL* formulas for each pair of states. Denote which of the two states satisfy the formula.

b) Give $\mathcal{D}/\sim_p$ for the following DTMC $\mathcal{D}$



Exercise 3 (Policies): (12 + 12 + 12 = 36 Points)

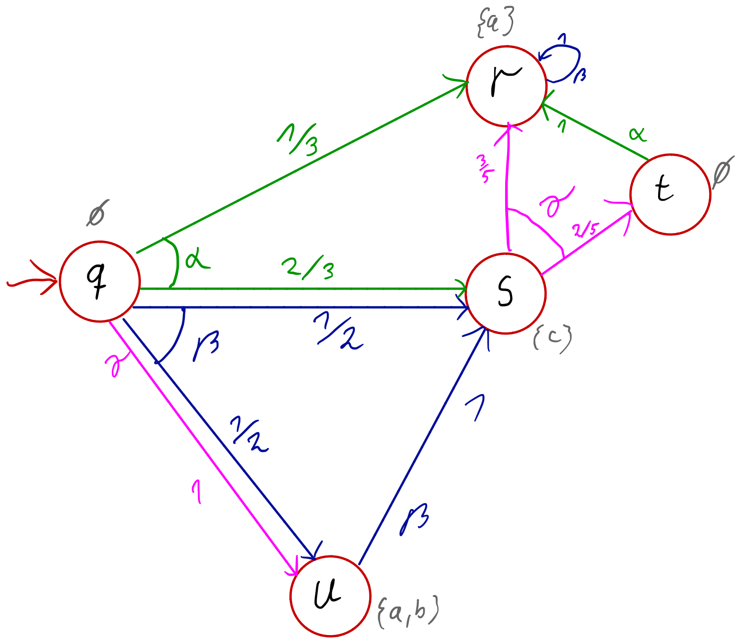
In this exercise, we consider randomized policies. For a countable set X, let $Dist(X)$ be the set of probability distributions over X, i.e., $Dist(X) = \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1\}$.
Let $\mathcal{M} = (S, Act, \mathbf{P}, \iota_{init}, AP, L)$ be an MDP. A *randomized policy* for $\mathcal{M}$ is a function $\mathfrak{S}: S^+ \to Dist(Act)$ such that $\mathfrak{S}(s_0 s_1 \ldots s_n) \in Dist(Act(s_n))$. Let $Scheds(\mathcal{M})$ denote the set of all randomized schedulers for $\mathcal{M}$. The class of *positional randomized schedulers* is defined straightforwardly.
The *induced DTMC* of $\mathcal{M}$ by a randomized policy $\mathfrak{S}$ is given by $\mathcal{M}_{\mathfrak{S}} = (S^+, \mathbf{P}_{\mathfrak{S}}, \iota_{init}, AP, L')$ where for $\sigma = s_0 s_1 \ldots s_n \in S^+$ we set $L'(\sigma) = L(s_n)$ and

$$\mathbf{P}_{\mathfrak{S}}(\sigma, \sigma s_{n+1}) = \sum_{\alpha \in Act(s_n)} \mathfrak{S}(\sigma)(\alpha) \cdot \mathbf{P}(s_n, \alpha, s_{n+1})$$

a) Depict the induced DTMC of the following MDP by the randomized policy $\mathfrak{S}$ with

$$\mathfrak{S}(s_0 s_1 \ldots s_n) = unif(Act(s_n)), \quad \text{where for } \alpha \in Act(s): \; unif(Act(s))(\alpha) = \frac{1}{|Act(s)|}$$

It suffices to depict the fragment reachable from the (only) initial state q.



b) A *multi-objective PCTL formula* is a vector $(\Phi_1, \ldots, \Phi_m)$ of $m > 0$ PCTL state formulas. For an MDP $\mathcal{M}$ and state s of $\mathcal{M}$ we write

$$\mathcal{M}, s \models_\exists (\Phi_1, \ldots, \Phi_m) \quad \text{iff } \exists \mathfrak{S} \in Scheds(\mathcal{M}): \forall i \in \{1, \ldots, m\}: \mathcal{M}_{\mathfrak{S}}, s \models \Phi_i.$$

Show that (non-randomized) schedulers as defined in the lecture are insufficient for multi-objective PCTL formulas, i.e., provide an MDP $\mathcal{M}$ and a multi-objective PCTL formula $(\Phi_1, \ldots, \Phi_m)$ such that $\mathcal{M}, s \models_\exists (\Phi_1, \ldots, \Phi_m)$ only holds when allowing *randomized schedulers*.

c) Show that positional randomized schedulers are insufficient for multi-objective PCTL formulas, i.e., provide an MDP $\mathcal{M}$ and a multi-objective PCTL formula $(\Phi_1, \ldots, \Phi_m)$ such that $\mathcal{M}, s \models_\exists (\Phi_1, \ldots, \Phi_m)$ only holds when allowing non-positional schedulers.