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Modeling and Verification of Probabilistic Systems
— Exercise Sheet 2 —

Notes:

- The exercise sheets need to be solved in groups of 2 students.
- Hand in the solution until 31.10.2018 before the exercise class.
- We do not accept solutions via L2p or email.
- Write your names and matriculation numbers on the front page and staple all pages.

Exercise 1 (Cylindersets): (6 + 6 + 6 + 7 = 25 Points)

a) Prove or disprove the following claims:

(i) For all DTMC $\mathcal{D}$ and paths $\hat{\pi}_1, \hat{\pi}_2 \in \text{Paths}^*(\mathcal{D})$: $\hat{\pi}_2$ is a prefix of $\hat{\pi}_1 \implies Cyl(\hat{\pi}_1) \subseteq Cyl(\hat{\pi}_2)$

(ii) For all DTMC $\mathcal{D}$ and paths $\hat{\pi}_1, \hat{\pi}_2 \in \text{Paths}^*(\mathcal{D})$: $Cyl(\hat{\pi}_1) \subseteq Cyl(\hat{\pi}_2) \implies \hat{\pi}_2$ is a prefix of $\hat{\pi}_1$

(iii) For all DTMC $\mathcal{D}$ and paths $\hat{\pi}_1, \hat{\pi}_2 \in \text{Paths}^*(\mathcal{D})$: $Cyl(\hat{\pi}_1) \cap Cyl(\hat{\pi}_2) \in (\{\emptyset\} \cup \{Cyl(\hat{\pi}) | \hat{\pi} \in \text{Paths}^*(D)\})$, i.e., the intersection of two cylinder sets is either empty or again a cylinder set.

b) Consider the DTMC $\mathcal{D} = (S, \mathbf{P}, \iota_{init}, AP, L)$, with

- $S = \{0, 1, \ldots, 9\}$,
- $\mathbf{P}(s, s') = \frac{1}{10}$ for all $s, s' \in S$,
- $\iota_{init}(s) = \frac{1}{10}$ for all $s \in S$,
- $AP = \emptyset$, and $L(s) = \emptyset$ for all $s \in S$.

Furthermore, let $f\colon \text{Paths}(\mathcal{D}) \to [0, 1]$ be a function that maps an infinite path $s_0 s_1 \ldots$ of $\mathcal{D}$ to the real number $f(s_0 s_1 \ldots) = 0.s_0 s_1 \cdots \in \mathbb{R}$. For example, $f(345(0^\omega)) = 0.345$ or $f(3^\omega) = 0.\overline{3} = \frac{1}{3}$.

Show that the set $\Pi = \{\pi \in \text{Paths}(\mathcal{D}) \mid f(\pi) \in \mathbb{Q}\}$ is measurable with respect to the probability space of the DTMC $\mathcal{D}$.

Hints:

- Express the set Π in terms of countable unions and/or complements of cylinder sets and proof that your expression indeed characterizes the set Π.
- A number $x \in [0.1]$ is rational ($x \in \mathbb{Q}$) iff it can be represented by a repeating decimal of the form $0.\alpha\beta^\omega$ with $\alpha, \beta \in \{0, 1, \ldots, 9\}^*$.

Exercise 2 (Probabilities vs. Qualitative Properties): (8 + 9 + 8 = 25 Points)

Let $D = (S, \mathbf{P}, s_{init}, AP, L)$ be a finite DTMC, $s \in S$, $a, b \in AP$. For each of the following statements, explain informally whether it is correct or not. If it is not correct, give a counterexample and indicate which of the implications (if any) hold.

Hints:

- $s \models \forall\phi$ iff for each path $\pi \in \text{Paths}(s)$: $\pi \models \phi$
- $s \models \exists\phi$ iff there exists a path $\pi \in \text{Paths}(s)$ such that $\pi \models \phi$

- For more information look at the slides on *Linear temporal logic* and *Computation tree logic* of the course *Introduction to model checking*. `https://moves.rwth-aachen.de/teaching/ss-18/introduction-to-model-checking/`.

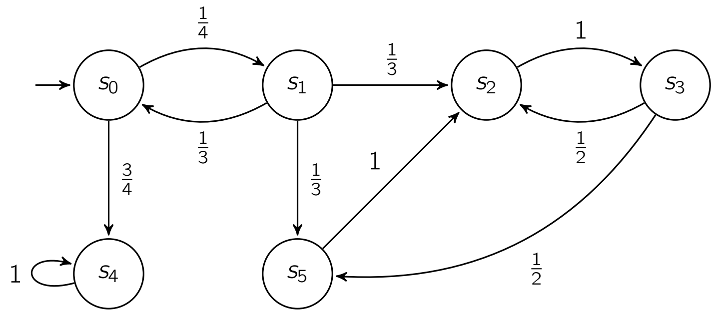
a) $Pr(s \models \Box a) = 1$ if and only if $s \models \forall \Box a$

b) $Pr(s \models \Diamond a) < 1$ if and only if $s \not\models \forall \Diamond a$

c) $Pr(s \models \Box a) > 0$ if and only if $s \models \exists \Box a$

Exercise 3 (Fairness): (5 + 10 + 10 = 25 Points)

Consider the following DTMC:



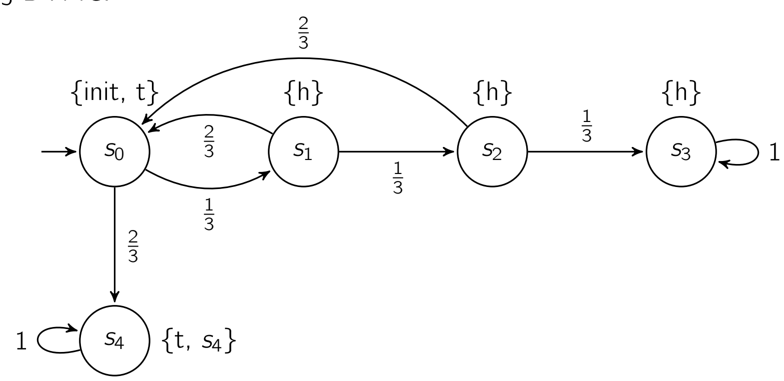
a) Denote the SCCs and BSCCs of this DTMC

b) Provide a set $G \subseteq S$, with $|G| = 2$, such that for all $s \in S$ we have $\Pr(s \models \Box\Diamond G) = 1$.

c) Provide a set $G \subseteq S$, with $|G| = 4$, such that for all $s \in S$ we have $\Pr(s \models \Diamond\Box G) = 1$.

Exercise 4 (Product Markov chain): (10 + 15 = 25 Points)

Consider the following DTMC:



Note that with probability $\frac{1}{3}$ heads (h) is thrown and with probability $\frac{2}{3}$ tails (t) is thrown.

a) Depict a deterministic finite-state automaton(DFA) with the following property:
   - After initially heads, reach s_4 but with at most two times heads in total.

b) Give the product Markov chain of the given DTMC and the automaton of exercise a).