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Modeling and Verification of Probabilistic Systems
— Exercise Sheet 1 —

Notes:

- The exercise sheets need to be solved in groups of 2 students.
- Make sure that you have access to the L2P room of this course since we use it to manage your exercise points.
- Hand in the solution until 24.10.2018 before the exercise class.
- We do not accept solutions via L2p or email.
- Write your names and matriculation numbers on the front page and staple all pages.

Exercise 1 (Sigma Algebras): (6 + 7 + 7 + 7 = 27 Points)

a) Formally define a probability space $\mathcal{P} = (\Omega, \mathcal{F}, \Pr)$ that models two consecutive tosses of a fair coin. Formally specify the following events (i.e., provide the corresponding element of $\mathcal{F}$) and compute their probabilities:

i) Both tosses yield `Heads`.

ii) The first coin toss yields `Heads`.

iii) At least one of the two tosses yields `Heads`.

b) Let Ω' be a countably infinite set and define $\mathcal{F}_{\Omega'}$ as the smallest class of subsets of Ω' such that for all $A \subseteq \Omega'$

(i) if A is finite, then $A \in \mathcal{F}_{\Omega'}$, and

(ii) if $A \in \mathcal{F}_{\Omega'}$, then $A^c \in \mathcal{F}_{\Omega'}$ for $A^c := (\Omega' \setminus A)$.

Show that the definition is non-trivial, i.e., in general $\mathcal{F}_{\Omega'} \neq 2^{\Omega'}$.
(Hint: find a set Ω' and a subset $A \subseteq \Omega'$ which cannot be in $\mathcal{F}_{\Omega'}$ according to the above definition.)

c) Prove that the definition of task b) becomes trivial (i.e., $\mathcal{F}_{\Omega'} = 2^{\Omega'}$) when we replace condition (i) by

(i) if A is *infinite*, then $A \in \mathcal{F}_{\Omega'}$.

d) Prove or disprove that $(\Omega', \mathcal{F}_{\Omega'})$ as defined in task b) is a σ-algebra for any countably infinite set Ω'.

Exercise 2 (Geometric Distributions): (15 + 15 = 30 Points)

Recall the definition of a *geometric distribution* as given in the lecture:

Definition 1. *Let X be a discrete random variable, $k \in \mathbb{N}_{>0}$ and $0 < p \leq 1$. The mass function of a* geometric distribution *is given by:*

$$\Pr\{X = k\} = (1-p)^{k-1} \cdot p$$

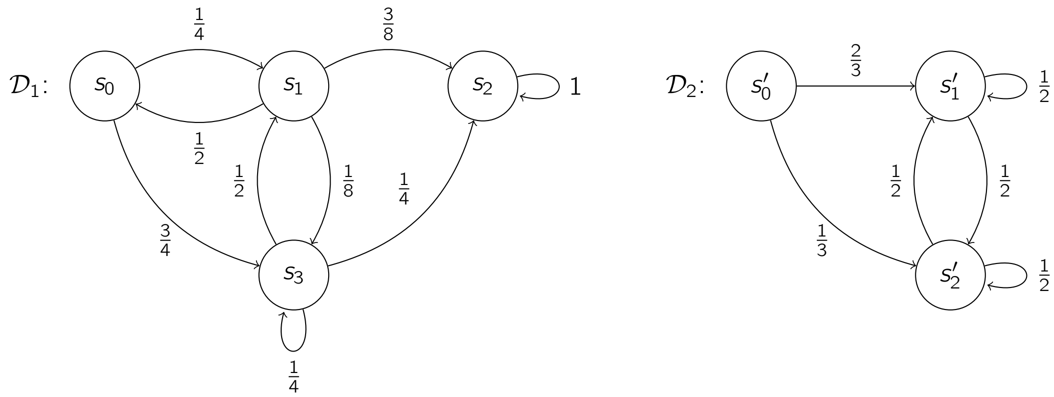
Let X now be be geometrically distributed with parameter p.

a) Show that $E[X] = \frac{1}{p}$.

b) Show that $\mathrm{Var}[X] = \frac{1-p}{p^2}$.

Exercise 3 (Probabilities in DTMCs): (7 + 7 + 7 = 21 Points)

Consider the following two DTMCs $\mathcal{D}_1$ and $\mathcal{D}_2$ as shown below..



a) Compute the transient state probability $\Theta_3^{\mathcal{D}_1}(s_3)$ assuming a uniform initial distribution (over all states).

b) For $\mathcal{D}_1$, compute the probability that after *exactly* 3 steps we are *not* in s_3, where the initial distribution is

$$\iota_{\text{init}}(s_i) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

c) Show that the transition probability matrix of $\mathcal{D}_2$ is ergodic and compute the limiting probability of being in state s_2'.

Exercise 4 (Absent minded professor): (2 + 20 = 22 Points)

We consider the following scenario.
An absent minded professor has 2 umbrellas that he/she uses when commuting from home to office and back.

- Suppose that it rains with probability q each time if he/she commutes.
- If it rains, and an umbrella is available at his/her location, he/she takes it.
- If it is not raining, and he/she has two umbrellas at his/her current location, he/she takes one with him with probability p.
- In all other cases, he/she only changes location.

a) For certain values of p and q, this scenario is a stochastic process. For which values?

b) Depict a DTMC for this process, in which you denote the number of umbrellas at each location and the location of the professor.