



Semantics and Verification of Software

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Lecture 4: Operational Semantics of WHILE III
(Summary & Application to Compiler Correctness)

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Recap: Execution of Statements

Execution of Statements

Remember:

$c ::= \text{skip} \mid x := a \mid c_1; c_2 \mid \text{if } b \text{ then } c_1 \text{ else } c_2 \text{ end} \mid \text{while } b \text{ do } c \text{ end} \in \text{Cmd}$

Definition (Execution relation for statements)

For $c \in \text{Cmd}$ and $\sigma, \sigma' \in \Sigma$, the **execution relation** $\langle c, \sigma \rangle \rightarrow \sigma'$ is defined by:

$$\begin{array}{c} \frac{}{\langle \text{skip}, \sigma \rangle \rightarrow \sigma} \text{(skip)} \qquad \frac{}{\langle a, \sigma \rangle \rightarrow z} \text{(asgn)} \\ \frac{\langle c_1, \sigma \rangle \rightarrow \sigma' \quad \langle c_2, \sigma' \rangle \rightarrow \sigma''}{\langle c_1; c_2, \sigma \rangle \rightarrow \sigma''} \text{(seq)} \qquad \frac{\langle b, \sigma \rangle \rightarrow \text{true} \quad \langle c_1, \sigma \rangle \rightarrow \sigma'}{\langle \text{if } b \text{ then } c_1 \text{ else } c_2 \text{ end}, \sigma \rangle \rightarrow \sigma'} \text{(if-t)} \\ \frac{\langle b, \sigma \rangle \rightarrow \text{false} \quad \langle c_2, \sigma \rangle \rightarrow \sigma'}{\langle \text{if } b \text{ then } c_1 \text{ else } c_2 \text{ end}, \sigma \rangle \rightarrow \sigma'} \text{(if-f)} \qquad \frac{\langle b, \sigma \rangle \rightarrow \text{false}}{\langle \text{while } b \text{ do } c \text{ end}, \sigma \rangle \rightarrow \sigma} \text{(wh-f)} \\ \frac{\langle b, \sigma \rangle \rightarrow \text{true} \quad \langle c, \sigma \rangle \rightarrow \sigma' \quad \langle \text{while } b \text{ do } c \text{ end}, \sigma' \rangle \rightarrow \sigma''}{\langle \text{while } b \text{ do } c \text{ end}, \sigma \rangle \rightarrow \sigma''} \text{(wh-t)} \end{array}$$

Recap: Execution of Statements

Determinism of Execution Relation

This operational semantics is well defined in the following sense:

Theorem

*The execution relation for statements is **deterministic**, i.e., whenever $c \in \text{Cmd}$ and $\sigma, \sigma', \sigma'' \in \Sigma$ such that $\langle c, \sigma \rangle \rightarrow \sigma'$ and $\langle c, \sigma \rangle \rightarrow \sigma''$, then $\sigma' = \sigma''$.*

- How to prove this theorem?
- Idea:
 - employ corresponding result for **expressions** (Lemma 3.6)
 - use **induction on the syntactic structure** of c ⚡
- Instead: **structural induction on derivation trees**

Functional of the Operational Semantics

Functional of the Operational Semantics

The determinism of the execution relation (Theorem 3.5) justifies the following definition:

Definition 4.1 (Operational functional)

The **functional of the operational semantics**,

$$\mathcal{D}[\cdot] : \mathit{Cmd} \rightarrow (\Sigma \dashrightarrow \Sigma),$$

assigns to every statement $c \in \mathit{Cmd}$ a **partial state transformation** $\mathcal{D}[c] : \Sigma \dashrightarrow \Sigma$, which is defined as follows:

$$\mathcal{D}[c]\sigma := \begin{cases} \sigma' & \text{if } \langle c, \sigma \rangle \rightarrow \sigma' \text{ for some } \sigma' \in \Sigma \\ \text{undefined} & \text{otherwise} \end{cases}$$

Remark: $\mathcal{D}[c]\sigma$ can indeed be undefined (consider e.g. $c = \text{while true do skip end}$; see Corollary 3.4)

Functional of the Operational Semantics

Equivalence of Statements

Underlying principle: two (syntactic) objects are considered (semantically) **equivalent** if they have the same “meaning”

- finite automata: $A_1 \sim A_2$ iff $L(A_1) = L(A_2)$
- context-free grammars: $G_1 \sim G_2$ iff $L(G_1) = L(G_2)$
- Turing machines: $T_1 \sim T_2$ iff both compute same function

Definition 4.2 (Operational equivalence)

Two statements $c_1, c_2 \in \mathit{Cmd}$ are called **(operationally) equivalent** (notation: $c_1 \sim c_2$) iff

$$\mathcal{D}[[c_1]] = \mathcal{D}[[c_2]].$$

Thus:

- $c_1 \sim c_2$ iff $\mathcal{D}[[c_1]]\sigma = \mathcal{D}[[c_2]]\sigma$ for every $\sigma \in \Sigma$
- In particular, $\mathcal{D}[[c_1]]\sigma$ is undefined iff $\mathcal{D}[[c_2]]\sigma$ is undefined

Functional of the Operational Semantics

“Unwinding” of Loops

Simple application of statement equivalence: test of execution condition in a `while` loop can be represented by an `if` statement

Lemma 4.3

For every $b \in BExp$ and $c \in Cmd$,

`while b do c end` \sim `if b then c; while b do c end else skip end.`

Proof.

on the board



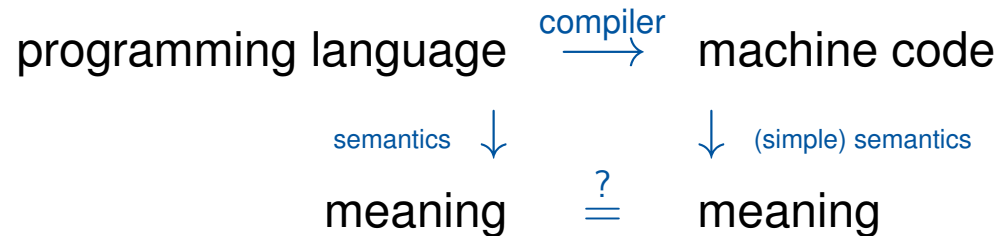
Summary: Operational Semantics

Summary: Operational Semantics

- Formalized by **evaluation/execution relations**
- Inductively defined by **derivation trees** using **structural operational rules**
- Enables proofs about operational behaviour of programs using **structural induction** on derivation trees
- **Semantic functional** characterizes complete input/output behavior of programs

Application: Compiler Correctness

Compiler Correctness



To do:

1. Definition of **abstract machine**
2. Definition of (operational) **semantics of machine instructions**
3. Definition of **translation** WHILE \rightarrow machine code (“compiler”)
4. **Proof:** semantics of generated machine code = semantics of original source code

The Abstract Machine

The Abstract Machine

Definition 4.4 (Abstract machine)

The **abstract machine (AM)** is given by

- **programs** $P \in \text{Code}$ and **instructions** p :

$$P ::= p^*$$

$$p ::= \text{PUSH}(z) \mid \text{PUSH}(t) \mid \text{ADD} \mid \text{SUB} \mid \text{MULT} \mid \text{EQ} \mid \text{GT} \mid \text{NOT} \mid \text{AND} \mid \text{OR} \mid \\ \text{LOAD}(x) \mid \text{STO}(x) \mid \text{JMP}(k) \mid \text{JMPF}(k)$$

(where $z, k \in \mathbb{Z}$, $t \in \mathbb{B}$, and $x \in \text{Var}$)

- **configurations** of the form $\langle pc, e, \sigma \rangle \in \text{Cnf}$ where
 - $pc \in \mathbb{Z}$ is the **program counter** (i.e., address of next instruction to be executed)
 - $e \in \text{Stk} := (\mathbb{Z} \cup \mathbb{B})^*$ is the **evaluation stack** (top right)
 - $\sigma \in \Sigma := (\text{Var} \rightarrow \mathbb{Z})$ is the **(storage) state**(thus $\text{Cnf} = \mathbb{Z} \times \text{Stk} \times \Sigma$)
- **initial configurations** of the form $\langle 0, \varepsilon, \sigma \rangle$
- **final configurations** of the form $\langle |P|, e, \sigma \rangle$

The Abstract Machine

Semantics of AM-Code I

Definition 4.5 (Transition relation of AM)

For $P = p_0; \dots; p_{n-1} \in \text{Code}$ and $0 \leq pc < n$, the **transition relation** $\triangleright \subseteq \text{Cnf} \times \text{Cnf}$ is given by

$P \vdash \langle pc, e, \sigma \rangle \triangleright \langle pc + 1, e : z, \sigma \rangle$	if $p_{pc} = \text{PUSH}(z)$
$P \vdash \langle pc, e, \sigma \rangle \triangleright \langle pc + 1, e : t, \sigma \rangle$	if $p_{pc} = \text{PUSH}(t)$
$P \vdash \langle pc, e : z_1 : z_2, \sigma \rangle \triangleright \langle pc + 1, e : (z_1 + z_2), \sigma \rangle$	if $p_{pc} = \text{ADD}$
$P \vdash \langle pc, e : z_1 : z_2, \sigma \rangle \triangleright \langle pc + 1, e : (z_1 - z_2), \sigma \rangle$	if $p_{pc} = \text{SUB}$
$P \vdash \langle pc, e : z_1 : z_2, \sigma \rangle \triangleright \langle pc + 1, e : (z_1 \cdot z_2), \sigma \rangle$	if $p_{pc} = \text{MULT}$
$P \vdash \langle pc, e : z_1 : z_2, \sigma \rangle \triangleright \langle pc + 1, e : (z_1 = z_2), \sigma \rangle$	if $p_{pc} = \text{EQ}$
$P \vdash \langle pc, e : z_1 : z_2, \sigma \rangle \triangleright \langle pc + 1, e : (z_1 > z_2), \sigma \rangle$	if $p_{pc} = \text{GT}$
$P \vdash \langle pc, e : t, \sigma \rangle \triangleright \langle pc + 1, e : (\neg t), \sigma \rangle$	if $p_{pc} = \text{NOT}$
$P \vdash \langle pc, e : t_1 : t_2, \sigma \rangle \triangleright \langle pc + 1, e : (t_1 \wedge t_2), \sigma \rangle$	if $p_{pc} = \text{AND}$
$P \vdash \langle pc, e : t_1 : t_2, \sigma \rangle \triangleright \langle pc + 1, e : (t_1 \vee t_2), \sigma \rangle$	if $p_{pc} = \text{OR}$
$P \vdash \langle pc, e, \sigma \rangle \triangleright \langle pc + 1, e : \sigma(x), \sigma \rangle$	if $p_{pc} = \text{LOAD}(x)$
$P \vdash \langle pc, e : z, \sigma \rangle \triangleright \langle pc + 1, e, \sigma[x \mapsto z] \rangle$	if $p_{pc} = \text{STO}(x)$
$P \vdash \langle pc, e, \sigma \rangle \triangleright \langle pc + k, e, \sigma \rangle$	if $p_{pc} = \text{JMP}(k)$
$P \vdash \langle pc, e : \text{true}, \sigma \rangle \triangleright \langle pc + 1, e, \sigma \rangle$	if $p_{pc} = \text{JMPF}(k)$
$P \vdash \langle pc, e : \text{false}, \sigma \rangle \triangleright \langle pc + k, e, \sigma \rangle$	if $p_{pc} = \text{JMPF}(k)$

The Abstract Machine

Semantics of AM-Code II

Corollary 4.6

▷ *is not total*, i.e., there exists $\gamma \in \text{Cnf}$ such that

$$\gamma \not\triangleright \gamma'$$

for all $\gamma' \in \text{Cnf}$

Proof.

Possible cases:

- γ **final** (that is, $\gamma = \langle |P|, e, \sigma \rangle$)
- γ **stuck**
 - e.g., $\gamma = \langle pc, 1, \sigma \rangle$ with $p_{pc} = \text{ADD}$ or $p_{pc} = \text{JMPF}(k)$
 - or $\gamma = \langle pc, e, \sigma \rangle$ with $pc \notin \{0, \dots, |P|\}$



The Abstract Machine

Alternative Choices

Remark: more realistic machine architectures

- **Variables referenced by address** (and not by name)
 - configurations $\langle pc, e, \mu \rangle$ with **memory** $\mu \in (\mathbb{N} \rightarrow \mathbb{Z})$
 - $LOAD(x)/STO(x)$ replaced by $LOAD(m)/STO(m)$ (where $m \in \mathbb{N}$)(requires symbol table for translation)
- **Registers** for storing intermediate values
(in place of evaluation stack e ; involves register allocation)

Terminating and Looping Computations I

Definition 4.7 (AM computations)

- A **finite computation** is a finite configuration sequence of the form $\gamma_0, \gamma_1, \dots, \gamma_k$ where $k \in \mathbb{N}$ and $\gamma_{i-1} \triangleright \gamma_i$ for each $i \in \{1, \dots, k\}$
- If, in addition, there is no γ such that $\gamma_k \triangleright \gamma$, then $\gamma_0, \gamma_1, \dots, \gamma_k$ is called **terminating**
- A **looping computation** is an infinite configuration sequence of the form $\gamma_0, \gamma_1, \gamma_2, \dots$ where $\gamma_i \triangleright \gamma_{i+1}$ for each $i \in \mathbb{N}$

Note: according to (the proof of) Corollary 4.6, a terminating computation may end in a final or in a stuck configuration

Terminating and Looping Computations II

Example 4.8

1. For $P := 0:\text{LOAD}(x); 1:\text{PUSH}(1); 2:\text{ADD}; 3:\text{STO}(x)$ and $\sigma(x) = 3$, we obtain the following terminating computation:

$$\langle 0, \varepsilon, \sigma \rangle \triangleright \langle 1, 3, \sigma \rangle \triangleright \langle 2, 3 : 1, \sigma \rangle \triangleright \langle 3, 4, \sigma \rangle \triangleright \langle 4, \varepsilon, \sigma[x \mapsto 4] \rangle$$

Remark: implements statement $x := x + 1$

2. For $P := 0:\text{PUSH}(\text{true}); 1:\text{JMPF}(2); 2:\text{JMP}(-2)$, the following computation loops:

$$\langle 0, \varepsilon, \sigma \rangle \triangleright \langle 1, \text{true}, \sigma \rangle \triangleright \langle 2, \varepsilon, \sigma \rangle \triangleright \langle 0, \varepsilon, \sigma \rangle \triangleright \dots$$

Remark: implements statement `while true do skip end`

A New Inductive Principle

Application: Finite computations (Def. 4.7)

Definition: a finite computation $\gamma_0, \gamma_1, \dots, \gamma_k$ has length k

Induction base: property holds for all computations of length 0

Induction hypothesis: property holds for all computations of length $\leq k$

Induction step: property holds for all computations of length $k + 1$

Properties of AM

Application: Extension of Code and Stack

Lemma 4.9

If $P \vdash \langle pc, e, \sigma \rangle \triangleright^* \langle pc', e', \sigma' \rangle$, then

$$P_1; P; P_2 \vdash \langle |P_1| + pc, e_0 : e, \sigma \rangle \triangleright^* \langle |P_1| + pc', e_0 : e', \sigma' \rangle$$

for all $P_1, P_2 \in \text{Code}$ and $e_0 \in \text{Stk}$.

Interpretation: both the code and the stack component can be extended without actually changing the behaviour of the machine

Proof.

by induction on the length of the computation (on the board) □

Properties of AM

Another Property: Determinism

Lemma 4.10

The semantics of AM is **deterministic**: for all $\gamma, \gamma', \gamma'' \in \text{Cnf}$,

$$\gamma \triangleright \gamma' \text{ and } \gamma \triangleright \gamma'' \text{ imply } \gamma' = \gamma''.$$

Proof (Idea).

- Instruction to be executed is unambiguously given by program counter
- Topmost stack entries and storage state then yield unique successor configuration □

Thus the following function is well defined:

Definition 4.11 (Semantics of AM)

The **semantics of an AM program** is given by $\mathfrak{M}[\cdot] : \text{Code} \rightarrow (\Sigma \dashrightarrow \Sigma)$ as follows:

$$\mathfrak{M}[P]\sigma := \begin{cases} \sigma' & \text{if } P \vdash \langle 0, \varepsilon, \sigma \rangle \triangleright^* \langle |P|, e, \sigma' \rangle \text{ for some } e \in \text{Stk} \\ \text{undefined} & \text{otherwise} \end{cases}$$