Concurrency Theory

Lecture 19: Branching Processes

Joost-Pieter Katoen and Thomas Noll

Lehrstuhl für Informatik 2
Software Modeling and Verification Group

http://moves.rwth-aachen.de/teaching/ws-1718/ct

December 18, 2017

RWTH AACHEN UNIVERSITY

Overview

Overview

Introduction

- Interleaving semantics of Petri nets = set of sequential runs
  - a sequential run is a total ordering of transition occurrences
- The set of all sequential runs can be represented by a marking graph
- Partial-order semantics of Petri nets = set of distributed runs
  - a distributed run is an acyclic (causal) net which contains no choices
  - a distributed run is a partial ordering of transition occurrences
- Today: the set of all distributed runs can be represented by a specific branching process, the unfolding

Branching process: preamble

- A branching process represents a set of distributed runs
- It explicitly represents each possible resolution of each choice[1]
- It is an acyclic (occurrence) net containing choices.
- It is a partial ordering with conflicts of transition occurrences.
- The true concurrency semantics of a net is a specific branching process, called unfolding.
- A net unfolding is the true concurrency counterpart of a marking graph.
- It is the unique maximal branching process in a complete lattice.
- The reachable markings of a 1-bounded net are covered by a finite prefix of this maximal branching process.

[1] In net jargon, a choice is called a conflict.

Overview

Elementary system nets

Net

An elementary net system N is a tuple (P, T, F, M_0) where:

- P is a countable set of places
- T is a countable set of transitions with $P \cap T = \varnothing$
- $F \subseteq (P \times T) \cup (T \times P)$ are the arcs satisfying:

$$\forall t \in T.\ {}^\bullet t \text{ and } t^\bullet \text{ are finite and non-empty}$$

- $M_0 : P \to \mathbb{N}$ is the initial marking.

Places and transitions are generically called nodes.

Assumption: (possibly) infinite elementary nets are 1-bounded. Thus any marking can be viewed as a subset of places.

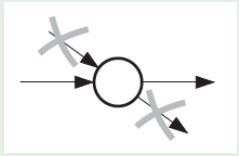
Causal nets

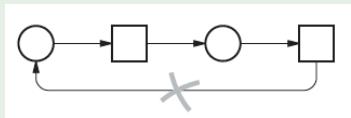
A causal net constitutes the basis for a "distributed" run.
It is a (possibly infinite) net which satisfies:

1. It has no place branches: at most one arc ends or starts in a place
2. It is acyclic
3. Each sequences of arcs (flows) has a unique first element
4. The initial marking contains all places without incoming arcs.

Intuition

No place branches, no sequence of arcs forms a loop, and each sequence of arcs has a first node.





Causal nets

A causal net constitutes the basis for a distributed run.
It is a possibly infinite net which satisfies:

1. Has no place branches: at most one arc ends or starts in a place
2. Is acyclic
3. Each sequences of arcs (flows) has a first element
4. The initial marking contains all places without incoming arcs

Causal net

A (possibly infinite) net $K = (Q, V, G, M_0)$ is called a causal net iff:

1. for each $q \in Q$, $|{}^\bullet q| \leqslant 1$ and $|q^\bullet| \leqslant 1$
2. the transitive closure (called causal order) G^+ of G is irreflexive
3. for each node $x \in Q \cup V$, the set $\{\, y \mid (y, x) \in G^+ \,\}$ is finite
4. M_0 equals the minimal set of places in K under G^+, i.e.,

$$M_0 \;=\; {}^\circ K \;=\; \{\, q \in Q \mid {}^\bullet q = \varnothing \,\}.$$

Properties of causal nets

Boundedness of causal nets

Every causal net is one-bounded, i.e., in every marking every place will hold at most one token.

Absence of superfluous places and transitions

Every causal net has a step sequence that visits all places and fires every transition.

What is a distributed run?

Distributed run

A distributed run of a one-bounded elementary net system N is:

1. a labeled causal net K
2. in which each transition t (with ${}^\bullet t$ and $t^\bullet$) is an action of N.

A distributed run K of N is complete iff (the marking) ${}^\circ K$ represents the initial marking of N and (the marking) K° does not enable any transition.

Examples on the black board.

Today: a characterization of distributed runs using homomorphisms.

Net homomorphisms

Homomorphism

A homomorphism from $N_1 = (P_1, T_1, F_1, M_{0,1})$ to $N_2 = (P_2, T_2, F_2, M_{0,2})$ is a mapping $h : P_1 \cup T_1 \to P_2 \cup T_2$ such that: [2]

1. $h(P_1) \subseteq P_2$ and $h(T_1) \subseteq T_2$, and
2. $\forall t \in T_1$, the restriction of h to ${}^\bullet t$ is a bijection between ${}^\bullet t$ (in N_1) and ${}^\bullet h(t)$ (in N_2), and similarly for $t^\bullet$ and $h(t)^\bullet$, and
3. the restriction of h to $M_{0,1}$ is a bijection between $M_{0,1}$ and $M_{0,2}$.[3]

Intuition

A homomorphism is a mapping between nets that preserves the nature of nodes and the environment of nodes. A homomorphism from N_1 to N_2 means that N_1 can be folded onto a part of N_2, or in other words, that N_1 can be obtained by partially unfolding a part of N_2.

[2] Here $h(X)$ for set X of nodes is defined by $h(X) = \bigcup_{x \in X} h(x)$.
[3] Due to the 1-boundedness, a marking M is a subset of the set P of places.

Distributed run using homomorphisms

Distributed run [Best and Fernandez, 1988]

A distributed run of an elementary net system N is a pair (K, h) where K is a causal net and h is a homomorphism from K to N.[4]

Intuition

A distributed run (K, h) of N may be viewed as a net K of which the places and transitions are labeled by places and transitions of N such that the labeling h forms a net homomorphism from K to N.[5]

[4] Best and Fernandez called this a process of a net.
[5] In the previous lecture, the labeling h was explicitly given as ℓ.

Examples

Overview

Conflicts

A distributed run is based on a causal net. A branching process on an occurrence net. Main difference: the presence of conflicts (choices).

Conflict

Let $N = (P, T, F, M)$ be a net. Nodes x_1 and x_2 are in conflict, denoted $x_1 \# x_2$, if there exist distinct transitions $t_1, t_2 \in T$ such that ${}^\bullet t_1 \cap {}^\bullet t_2 \neq \varnothing$ and $(t_1, x_1) \in F^*$ and $(t_2, x_2) \in F^*$.

Node x is in self-conflict whenever $x \# x$.

Examples

On the black board.

Note that in a causal net $\# = \varnothing$ as ${}^\bullet t_1 \cap {}^\bullet t_2 = \varnothing$ for any two distinct transitions t_1 and t_2.

Occurrence net

Occurrence net

A net $K = (Q, V, G, M)$ is an occurrence net iff:

1. for each $q \in Q$, $|{}^\bullet q| \leqslant 1$
2. the transitive closure G^+ of G is irreflexive
3. for each node $x \in Q \cup V$ we have $\{ y \mid (y, x) \in G^+ \}$ is finite
4. no transition $v \in V$ is in self-conflict
5. $M_0 = {}^\circ K = \{ q \in Q \mid {}^\bullet q = \varnothing \}$.

Remark

Since $\# = \varnothing$ in a causal net, and each causal net fulfils the remaining conditions, every causal net is an occurrence net.

Example

Branching process

Branching process **[Engelfriet 1991]**

A branching process of net N is a pair (K, h) where $K = (Q, V, G, M)$ is an occurrence net and h a net homomorphism from K to N such that:

$$\forall v, v' \in Q.\ ({}^\bullet v = {}^\bullet v' \text{ and } h(v) = h(v') \quad \text{implies} \quad v = v')\ .$$

Every distributed run is a branching process. The reverse is not true.

Examples

On the black board.

Overview

Relating branching processes

Homomorphisms and isomorphisms between branching processes

Let $B_1 = (K_1, h_1)$ and $B_2 = (K_2, h_2)$ be two branching processes of net N. A homomorphism from B_1 to B_2 is a homomorphism h from K_1 to K_2 such that $h_2 \circ h = h_1$.[6]

An isomorphism is a bijective homomorphism. B_1 and B_2 are isomorphic if there is an isomorphism from B_1 to B_2.

Being isomorphic is an equivalence relation. Its equivalence classes are called isomorphism classes.

[6] The composition of two net homomorphisms is a net homomorphism.

Approximation of branching processes

Approximation

Let B_1 and B_2 be two branching processes of net N. B_1 approximates B_2, denoted $B_1 \sqsubseteq B_2$, if there exists an injective homomorphism from B_1 to B_2.

Intuition

B_1 approximates B_2 whenever every (partial) distributed run in B_1 is also contained in B_2. In other words, B_1 is isomorphic to an initial part of B_2. Being an approximation on branching processes is the analogue of being a prefix on sequences.

Examples

On the black board. Obviously, $\sqsubseteq$ is a partial order on branching processes.

Approximation of branching processes

Approximation

Let B_1 and B_2 be two branching processes of net N. B_1 approximates B_2, denoted $B_1 \sqsubseteq B_2$, if there exists an injective homomorphism from B_1 to B_2.

Lemma

Approximation is preserved by isomorphism: if B_i' is isomorphic to B_i (for $i = 1, 2$), then $B_1 \sqsubseteq B_2$ implies $B_1' \sqsubseteq B_2'$. Thus, $\sqsubseteq$ can be extended to a partial order on isomorphism classes (of branching processes).

Proof.

Home exercise. Basically juggling with homomorphisms. □

Engelfriet's theorem

Engelfriet's branching process theorem

The set of isomorphism classes of branching processes of net N is a complete lattice with respect to the approximation relation $\sqsubseteq$. Formally, $(\mathbb{B}, \sqsubseteq)$ is a complete partial order, where $\mathbb{B}$ is the set of isomorphism classes of branching processes.

Complete lattice

Recall that a complete lattice is a partial order $(\mathbb{B}, \sqsubseteq)$ such that all subsets of $\mathbb{B}$ have LUBs and GLBs.

The true concurrency semantics of a net

Corollary: the unfolding of a net

Every one-bounded net has a unique maximal (with respect to $\sqsubseteq$) branching process up to isomorphism. This is called the unfolding or true concurrency semantics of net N.

We denote by $B_{\max} = ((P_{\max}, T_{\max}, F_{\max}), h_{\max})$ a representative of the isomorphism class of the maximal branching process of N.

Example

On the black board.

The true concurrency semantics of Petri nets

The true concurrency semantics of a Petri net is given by its unfolding.

Recall: The interleaving semantics of a Petri net is given by its marking graph.