Concurrency Theory

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Lecture 8: The π-Calculus

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Recap: Modelling Mutual Exclusion Algorithms

Modelling the Processes in CCS

Assumption: P_i cannot fail or terminate within critical section

Peterson's algorithm

while true do
  "non-critical section";
  $b_i := \text{true}$;
  $k := j$;
  while $b_j \wedge k = j$ do skip;
  "critical section";
  $b_i := \text{false}$;
end

CCS representation

$$P_1 = \overline{b1wt}.\overline{kw2}.P_{11}$$
$$P_{11} = b2rf.P_{12} + b2rt.(kr1.P_{12} + kr2.P_{11})$$
$$P_{12} = enter_1.exit_1.\overline{b1wf}.P_1$$
$$P_2 = \overline{b2wt}.\overline{kw1}.P_{21}$$
$$P_{21} = b1rf.P_{22} + b1rt.(kr1.P_{21} + kr2.P_{22})$$
$$P_{22} = enter_2.exit_2.\overline{b2wf}.P_2$$
$$Peterson = (P_1 \parallel P_2 \parallel B_{1f} \parallel B_{2f} \parallel K_1) \setminus L$$
$$\text{for } L = \{b1rf, b1rt, b1wf, b1wt, b2rf, b2rt, b2wf, b2wt, kr1, kr2, kw1, kw2\}$$

Recap: Modelling Mutual Exclusion Algorithms

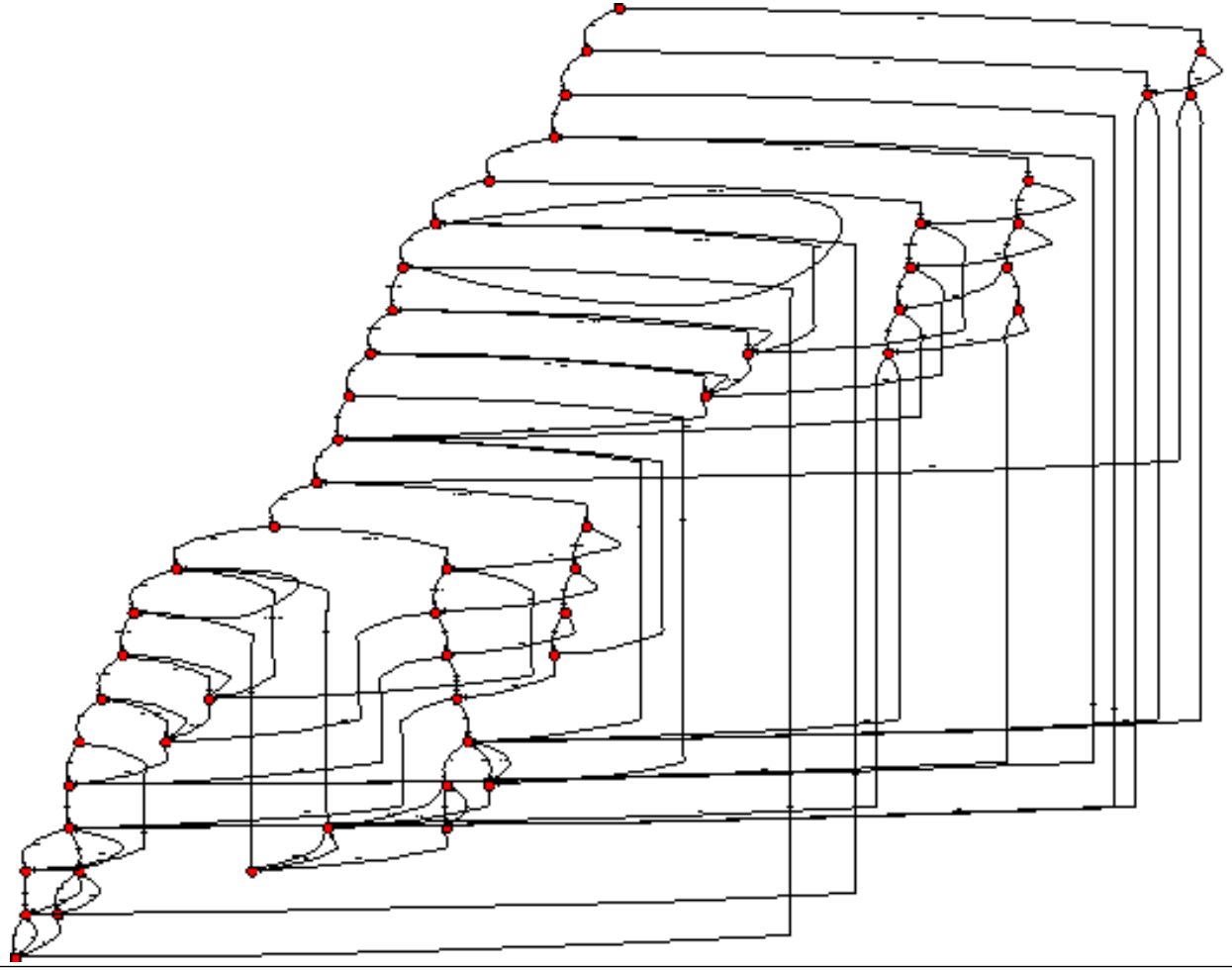
Obtaining the LTS I

Alternatives:

- By hand (really painful)
- By tools:
  - CAAL (Concurrency Workbench, Aalborg Edition): http://caal.cs.aau.dk
    - smart editor
    - visualisation of generated LTS
    - equivalence checking w.r.t. several bisimulation, simulation and trace equivalences
    - generation of distinguishing formulae for nonequivalent processes
    - model checking of recursive HML formulae
    - (bi)simulation and model checking games.
    - see exercises
  - TAPAs (Tool for the Analysis of Process Algebras): http://rap.dsi.unifi.it/tapas/
    - CCS specification of Peterson's algorithm available as example
    - yields LTS with 54 states (see next slide)
  - CWB (Edinburgh Concurrency Workbench): http://homepages.inf.ed.ac.uk/perdita/cwb/
    - somewhat outdated

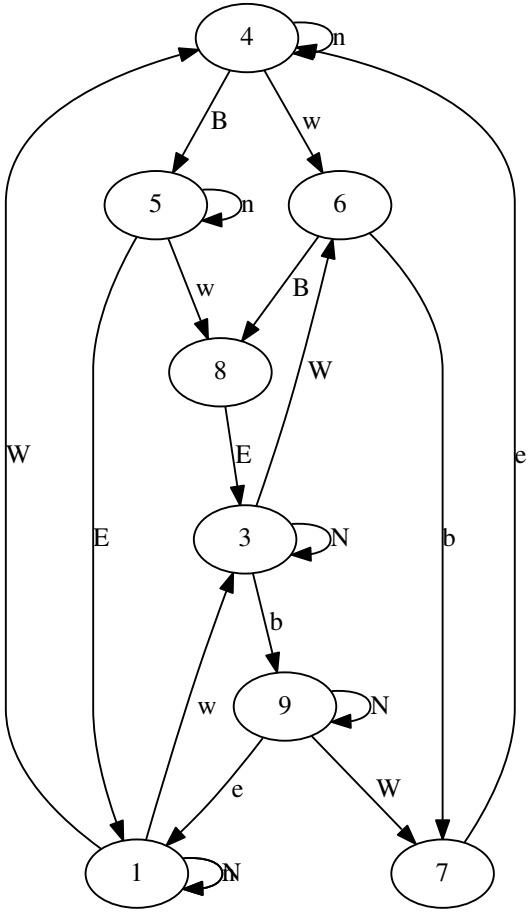
Recap: Modelling Mutual Exclusion Algorithms

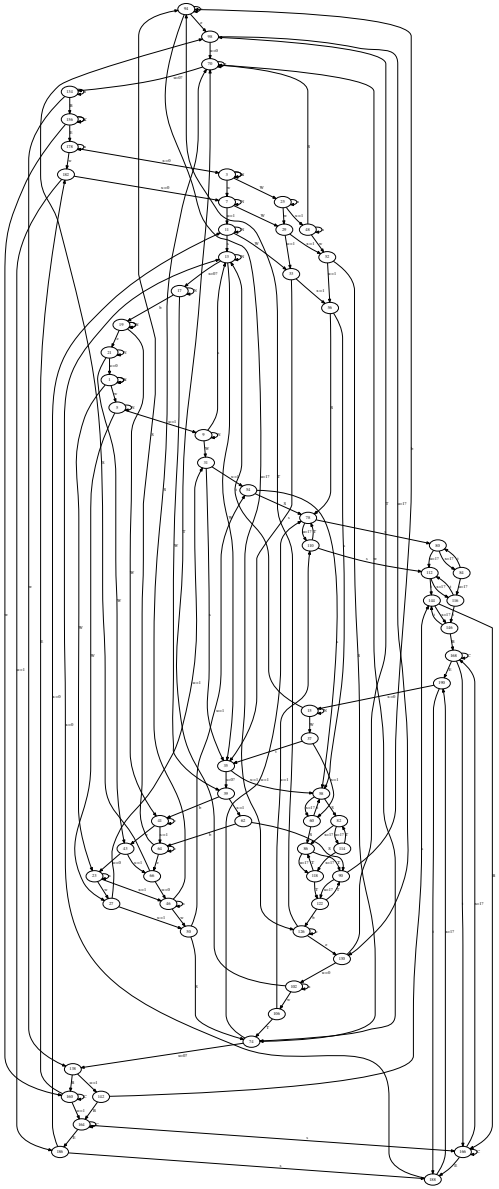
Obtaining the LTS II



Was passiert, wenn P_0 und P_1 laufen?

Betrachte $(P_0 \sqcup\!\sqcup P_1) \circ B_u \circ B_x \circ B_s$.





Syntax of Value-Passing CCS

Definition (Syntax of value-passing CCS)

- Let A, $\overline{A}$, Pid (ranked) as in Definition 2.1.
- Let e and b be integer and Boolean expressions, resp., built from integer variables $x, y, \ldots$
- The set Prc^+ of value-passing process expressions is defined by:

$$
\begin{array}{lll}
P ::= & \mathsf{nil} & \text{(inaction)} \\
\mid & a(x).P & \text{(input prefixing)} \\
\mid & \overline{a}(e).P & \text{(output prefixing)} \\
\mid & \tau.P & (\tau \text{ prefixing}) \\
\mid & P_1 + P_2 & \text{(choice)} \\
\mid & P_1 \parallel P_2 & \text{(parallel composition)} \\
\mid & P \setminus L & \text{(restriction)} \\
\mid & P[f] & \text{(relabelling)} \\
\mid & \mathsf{if}\ b\ \mathsf{then}\ P & \text{(conditional)} \\
\mid & C(e_1, \ldots, e_n) & \text{(process call)}
\end{array}
$$

where $a \in A$, $L \subseteq A$, $C \in Pid$ (of rank $n \in \mathbb{N}$), and $f : A \to A$.

Semantics of Value-Passing CCS I

Definition (Semantics of value-passing CCS)

A value-passing process definition $\left(C_i(x_1, \ldots, x_{n_i}) = P_i \mid 1 \leq i \leq k\right)$ determines the LTS $\left(Prc^+, Act, \longrightarrow\right)$ with $Act := \left(A \cup \overline{A}\right) \times \mathbb{Z} \cup \{\tau\}$ whose transitions can be inferred from the following rules ($P, P', Q, Q' \in Prc^+$, $a \in A$, x_i integer variables, e_i/b integer/Boolean expressions, $z \in \mathbb{Z}$, $\alpha \in Act$, $\lambda \in \left(A \cup \overline{A}\right) \times \mathbb{Z}$):

$$\text{(In)}\ \frac{}{a(x).P \xrightarrow{a(z)} P[z/x]} \qquad \text{(Out)}\ \frac{(z \text{ value of } e)}{\overline{a}(e).P \xrightarrow{\overline{a}(z)} P} \qquad \text{(Tau)}\ \frac{}{\tau.P \xrightarrow{\tau} P}$$

$$\text{(Sum}_1\text{)}\ \frac{P \xrightarrow{\alpha} P'}{P + Q \xrightarrow{\alpha} P'} \qquad \text{(Sum}_2\text{)}\ \frac{Q \xrightarrow{\alpha} Q'}{P + Q \xrightarrow{\alpha} Q'}$$

$$\text{(Par}_1\text{)}\ \frac{P \xrightarrow{\alpha} P'}{P \parallel Q \xrightarrow{\alpha} P' \parallel Q} \qquad \text{(Par}_2\text{)}\ \frac{Q \xrightarrow{\alpha} Q'}{P \parallel Q \xrightarrow{\alpha} P \parallel Q'} \qquad \text{(Com)}\ \frac{P \xrightarrow{\lambda} P' \quad Q \xrightarrow{\overline{\lambda}} Q'}{P \parallel Q \xrightarrow{\tau} P' \parallel Q'}$$

Semantics of Value-Passing CCS II

Definition (Semantics of value-passing CCS; continued)

$$\text{(Rel)}\ \frac{P \xrightarrow{\alpha} P'}{P[f] \xrightarrow{f(\alpha)} P'[f]} \qquad \text{(Res)}\ \frac{P \xrightarrow{\alpha} P' \ (\alpha \notin (L \cup \overline{L}) \times \mathbb{Z})}{P \setminus L \xrightarrow{\alpha} P' \setminus L}$$

$$\text{(If)}\ \frac{P \xrightarrow{\alpha} P' \ (b \text{ true})}{\text{if } b \text{ then } P \xrightarrow{\alpha} P'} \qquad \text{(Call)}\ \frac{P[z_1/x_1, \ldots, z_n/x_n] \xrightarrow{\alpha} P' \quad (C(x_1, \ldots, x_n) = P, z_i \text{ value of } e_i)}{C(e_1, \ldots, e_n) \xrightarrow{\alpha} P'}$$

Remarks:

- $P[z_1/x_1, \ldots, z_n/x_n]$ denotes the substitution of each free (i.e., unbound) occurrence of x_i by z_i $(1 \leq i \leq n)$
- Relabelling functions are extended to actions by letting

$$f(a(z)) := f(a)(z) \quad \text{and} \quad f(\overline{a}(z)) := \overline{f(a)}(z) \quad (\text{and } f(\tau) := \tau)$$

Translation of Value-Passing into Pure CCS

Definition (Translation of value-passing into pure CCS)

For each $P \in Prc^+$ without free variables, its translated form $\widehat{P} \in Prc$ is given by

$$\widehat{\mathsf{nil}} := \mathsf{nil} \qquad \widehat{\tau.P} := \tau.\widehat{P}$$

$$\widehat{a(x).P} := \sum_{z \in \mathbb{Z}} a_z.\widehat{P[z/x]} \qquad \widehat{\overline{a}(e).P} := \overline{a_z}.\widehat{P} \quad (z \text{ value of } e)$$

$$\widehat{P_1 + P_2} := \widehat{P}_1 + \widehat{P}_2 \qquad \widehat{P_1 \parallel P_2} := \widehat{P}_1 \parallel \widehat{P}_2$$

$$\widehat{P \setminus L} := \widehat{P} \setminus \{a_z \mid a \in L, z \in \mathbb{Z}\} \qquad \widehat{P[f]} := \widehat{P}[\widehat{f}] \quad (\widehat{f}(a_z) := f(a)_z)$$

$$\widehat{\text{if } b \text{ then } P} := \begin{cases} \widehat{P} & \text{if } b \text{ true} \\ \mathsf{nil} & \text{otherwise} \end{cases} \qquad \widehat{C(e_1, \ldots, e_n)} := C_{z_1,\ldots,z_n} \quad (z_i \text{ value of } e_i)$$

Moreover, each defining equation $C(x_1, \ldots, x_n) = P$ of a process identifier is translated into the indexed collection of process definitions

$$\left(C_{z_1,\ldots,z_n} = \widehat{P[z_1/x_1, \ldots, z_n/x_n]} \mid v_1, \ldots, v_n \in \mathbb{Z} \right)$$

Mobility in Concurrent Systems I

Observation: CCS imposes static communication structures: if $P, Q \in Prc$ want to communicate, then both must syntactically refer to the same action name

$\Rightarrow$ every potential communication partner known beforehand,
no dynamic passing of communication links

$\Rightarrow$ lack of modelling capabilities for mobility

Goal: develop calculus in the spirit of CCS which supports mobility

$\Rightarrow$ π-Calculus

Mobility in Concurrent Systems II

Example 8.1 (Dynamic access to resources)

- Server S controls access to printer P
- Client C wishes to use P
- In CCS: P and C must share some action name a
  $\Rightarrow$ C could access P without being granted it by S
- In π-Calculus:
  - initially only S has access to P (using link a)
  - using another link b, C can request access to P
- Formally:

$$\underbrace{\overline{b}\langle a\rangle.S'}_{S} \parallel \underbrace{b(c).\overline{c}\langle d\rangle.C'}_{C} \parallel \underbrace{a(e).P'}_{P}$$
$$\xrightarrow{\tau} S' \parallel \overline{a}\langle d\rangle.C'[a/c] \parallel a(e).P'$$
$$\xrightarrow{\tau} S' \parallel C'[a/c] \parallel P'[d/e]$$

- a: link to P
- b: link between S and C
- c: "placeholder" for a
- d: data to be printed
- e: "placeholder" for d

Mobility in Concurrent Systems III

Example 8.1 (Dynamic access to resources; continued)

- Different rôles of action name a:
  - in interaction between S and C: object transferred from S to C
  - in interaction between C and P: name of communication link
- Intuitively, names represent access rights:
  - a: to P
  - b: to S
  - d: to data to be printed
- If a is only way to access P
  $\Rightarrow$ P "moves" from S to C

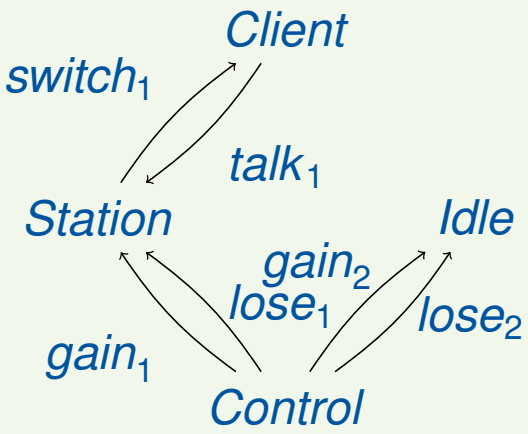
Mobile Clients I

Example 8.2 (Hand-over protocol)

Scenario:

- client devices moving around (phones, PCs, sensors, ...)
- each radio-connected to some base station
- stations wired to central control
- some event (e.g., signal fading) may cause a client to be switched to another station
- essential: specification of switching process (“hand-over protocol”)

Simplest configuration:
two stations, one client



Mobile Clients II

Example 8.2 (Hand-over protocol; continued)

- Every station is in one of two modes: *Station* (active; four links) or *Idle* (inactive; two links)
- *Client* can talk via *Station*, and at any time *Control* can request *Station*/*Idle* to lose/gain *Client*:

$$\begin{aligned} \mathit{Station}(\mathit{talk}, \mathit{switch}, \mathit{gain}, \mathit{lose}) &= \mathit{talk}.\mathit{Station}(\mathit{talk}, \mathit{switch}, \mathit{gain}, \mathit{lose}) + \\ &\quad \mathit{lose}(t, s).\overline{\mathit{switch}}\langle t, s\rangle.\mathit{Idle}(\mathit{gain}, \mathit{lose}) \\ \mathit{Idle}(\mathit{gain}, \mathit{lose}) &= \mathit{gain}(t, s).\mathit{Station}(t, s, \mathit{gain}, \mathit{lose}) \end{aligned}$$

- If *Control* decides *Station* to lose *Client*, it issues a new pair of channels to be shared by *Client* and *Idle*:

$$\begin{aligned} \mathit{Control}_1 &= \overline{\mathit{lose}_1}\langle \mathit{talk}_2, \mathit{switch}_2\rangle.\overline{\mathit{gain}_2}\langle \mathit{talk}_2, \mathit{switch}_2\rangle.\mathit{Control}_2 \\ \mathit{Control}_2 &= \overline{\mathit{lose}_2}\langle \mathit{talk}_1, \mathit{switch}_1\rangle.\overline{\mathit{gain}_1}\langle \mathit{talk}_1, \mathit{switch}_1\rangle.\mathit{Control}_1 \end{aligned}$$

- *Client* can either talk or, if requested, switch to a new pair of channels:

$$\mathit{Client}(\mathit{talk}, \mathit{switch}) = \overline{\mathit{talk}}.\mathit{Client}(\mathit{talk}, \mathit{switch}) + \mathit{switch}(t, s).\mathit{Client}(t, s)$$

Mobile Clients III

Example 8.2 (Hand-over protocol; continued)

- As usual, the whole system is a restricted composition of processes:

$$System_1 = \text{new } L\,(Client_1 \parallel Station_1 \parallel Idle_2 \parallel Control_1)$$

where

$$\begin{aligned} Client_i &:= Client(talk_i, switch_i) \\ Station_i &:= Station(talk_i, switch_i, gain_i, lose_i) \\ Idle_i &:= Idle(gain_i, lose_i) \\ L &:= (talk_i, switch_i, gain_i, lose_i \mid i \in \{1, 2\}) \end{aligned}$$

- After having formally defined the π-Calculus we will see that this protocol is correct, i.e., that the hand-over does indeed occur:

$$System_1 \longrightarrow^* System_2$$

where

$$System_2 = \text{new } L\,(Idle_1 \parallel Client_2 \parallel Station_2 \parallel Control_2)$$

Syntax of the Monadic π-Calculus

Introduction

Literature on π-Calculus:

- Initial research paper:
  R. Milner, J. Parrow, D. Walker: *A calculus of mobile processes*, Part I/II. Journal of Inf. & Comp., 100:1–77, 1992
- Overview article:
  J. Parrow: *An introduction to the π-Calculus*. Chapter 8 of *Handbook of Process Algebra*, 479–543, Elsevier, 2001
- Textbook:
  R. Milner: *Communicating and mobile systems: the π-Calculus*. Cambridge University Press, 1999

To simplify the presentation (as in Milner's book):

1. Monadic π-Calculus with replication (message = one name, no process identifiers)
2. Extension to polyadic calculus
3. Extension by process equations

Syntax of the Monadic π-Calculus

Definition 8.3 (Syntax of monadic π-Calculus)

- Let $A = \{a, b, c \ldots, x, y, z, \ldots\}$ be a set of names.
- The set of action prefixes is given by

$$\begin{array}{rll} \pi ::= & x(y) & \text{(receive } y \text{ along } x\text{)} \\ \mid & \overline{x}\langle y \rangle & \text{(send } y \text{ along } x\text{)} \\ \mid & \tau & \text{(unobservable action)} \end{array}$$

- The set Prc^{π} of π-Calculus process expressions is defined by the following syntax:

$$\begin{array}{rll} P ::= & \sum_{i \in I} \pi_i . P_i & \text{(guarded sum)} \\ \mid & P_1 \parallel P_2 & \text{(parallel composition)} \\ \mid & \text{new } x\ P & \text{(restriction)} \\ \mid & !P & \text{(replication)} \end{array}$$

(where I finite index set, $x \in A$)

Conventions: $\text{nil} := \sum_{i \in \emptyset} \pi_i . P_i$, $\text{new } x_1, \ldots, x_n\ P := \text{new } x_1 \left(\ldots \text{new } x_n\ P \right)$

Free and Bound Names

Definition 8.4 (Free and bound names)

- The input prefix $x(y)$ and the restriction new y P both bind y.
- Every other occurrence of a name (i.e., x in $x(y)$ and x, y in $\overline{x}\langle y \rangle$) is free.
- The set of bound/free names of a process expressions $P \in Prc^{\pi}$ is respectively denoted by $bn(P)/fn(P)$.

Remark: $bn(P) \cap fn(P) \neq \emptyset$ is possible

Example 8.5

For $P = \text{new}\, x\, (x(y).\text{nil} \parallel \overline{z}\langle y \rangle.\text{nil})$:

$$bn(P) = \{x, y\}, fn(P) = \{y, z\}$$

Structural Congruence

Goal: simplify definition of operational semantics by ignoring "purely syntactic" differences between processes

Definition 8.6 (Structural congruence)

$P, Q \in Prc^{\pi}$ are structurally congruent, written $P \equiv Q$, if one can be transformed into the other by applying the following operations and equations:

1. renaming of bound names (α-conversion)
2. reordering of terms in a summation (commutativity of $+$)
3. $P \parallel Q \equiv Q \parallel P$, $P \parallel (Q \parallel R) \equiv (P \parallel Q) \parallel R$, $P \parallel \text{nil} \equiv P$ (Abelian monoid laws for $\parallel$)
4. $\text{new}\, x\, \text{nil} \equiv \text{nil}$, $\text{new}\, x, y\, P \equiv \text{new}\, y, x\, P$, $P \parallel \text{new}\, x\, Q \equiv \text{new}\, x\, (P \parallel Q)$ if $x \notin fn(P)$ (scope extension)
5. $!P \equiv P \parallel !P$ (unfolding)

A Standard Form

Theorem 8.7 (Standard form)

Every process expression is structurally congruent to a process of the standard form

$$\text{new } x_1, \ldots, x_k \, (P_1 \parallel \ldots \parallel P_m \parallel !Q_1 \parallel \ldots \parallel !Q_n)$$

where each P_i is a non-empty sum, and each Q_j is in standard form.

(If $m = n = 0$: nil; if $k = 0$: restriction absent)

Proof.

by induction on the structure of $R \in Prc^{\pi}$ (on the board) □

The Reaction Relation

Thanks to Theorem 8.7, only processes in standard form need to be considered for defining the operational semantics:

Definition 8.8

The reaction relation $\longrightarrow \subseteq Prc^{\pi} \times Prc^{\pi}$ is generated by the rules:

$$\text{(Tau)}\ \frac{}{\tau.P + Q \longrightarrow P}$$

$$\text{(React)}\ \frac{}{(x(y).P + R) \parallel (\overline{x}\langle z\rangle.Q + S) \longrightarrow P[z/y] \parallel Q}$$

$$\text{(Par)}\ \frac{P \longrightarrow P'}{P \parallel Q \longrightarrow P' \parallel Q} \qquad \text{(Res)}\ \frac{P \longrightarrow P'}{\text{new } x\ P \longrightarrow \text{new } x\ P'}$$

$$\text{(Struct)}\ \frac{P \longrightarrow P'}{Q \longrightarrow Q'} \quad \text{if } P \equiv Q \text{ and } P' \equiv Q'$$

- $P[z/y]$ replaces every free occurrence of y in P by z.
- In (React), the pair $(x(y), \overline{x}\langle z\rangle)$ is called a redex.

Example: Printer Server

Example 8.9

1. Printer server (cf. Example 8.1):

$$\underbrace{\overline{b}\langle a\rangle.S'}_{S} \parallel \underbrace{a(e).P'}_{P} \parallel \underbrace{b(c).\overline{c}\langle d\rangle.C'}_{C} \longrightarrow S' \parallel a(e).P' \parallel \overline{a}\langle d\rangle.C'[a/c]$$

$$S' \parallel a(e).P' \parallel \overline{a}\langle d\rangle.C'[a/c] \longrightarrow S' \parallel P'[d/e] \parallel C'[a/c]$$

   (on the board)

2. With scope extension ($P \parallel \text{new } x\, Q \equiv \text{new } x\, (P \parallel Q)$ if $x \notin fn(P)$):

$$\begin{aligned} &\text{new } b\,(\text{new } a\,(\overline{b}\langle a\rangle.S' \parallel a(e).P') \parallel b(c).\overline{c}\langle d\rangle.C') \\ \longrightarrow\ &\text{new } a, b\,(S' \parallel a(e).P' \parallel \overline{a}\langle d\rangle.C'[a/c]) \end{aligned}$$

   (on the board)