Concurrency Theory

Winter Semester 2017/18

Lecture 5: Fixed-Point Theory

Joost-Pieter Katoen and Thomas Noll
Software Modeling and Verification Group
RWTH Aachen University

http://moves.rwth-aachen.de/teaching/ws-1718/ct/

Software Modeling and Verification Chair | RWTH AACHEN UNIVERSITY

Introducing Recursion

Solution: employ recursion!

- $Inv(\langle a\rangle tt) \equiv \langle a\rangle tt \wedge [a]\, Inv(\langle a\rangle tt)$
- $Pos([a]ff) \equiv [a]ff \vee \langle a\rangle\, Pos([a]ff)$

Interpretation: the sets of states $X, Y \subseteq S$ satisfying the respective formula should solve the corresponding equation, i.e.,

- $X = \langle\cdot a\cdot\rangle(S) \cap [\cdot a\cdot](X)$
- $Y = [\cdot a\cdot](\emptyset) \cup \langle\cdot a\cdot\rangle(Y)$

Open questions

- Do such recursive equations (always) have solutions?
- If so, are they unique?
- How can we compute whether a process satisfies a recursive formula?

Syntax of HML with One Recursive Variable

Initially: only one variable
Later: mutual recursion

Definition (Syntax of HML with one variable)

The set HMF_X of Hennessy-Milner formulae with one variable X over a set of actions Act is defined by the following syntax:

$$
\begin{array}{rcll}
F & ::= & X & \text{(variable)} \\
 & \mid & \mathrm{tt} & \text{(true)} \\
 & \mid & \mathrm{ff} & \text{(false)} \\
 & \mid & F_1 \wedge F_2 & \text{(conjunction)} \\
 & \mid & F_1 \vee F_2 & \text{(disjunction)} \\
 & \mid & \langle\alpha\rangle F & \text{(diamond)} \\
 & \mid & [\alpha] F & \text{(box)}
\end{array}
$$

where $\alpha \in Act$.

Semantics of HML with One Recursive Variable I

So far: $[\![F]\!] \subseteq S$ for $F \in HMF$ and LTS $(S, Act, \longrightarrow)$

Now: semantics of formula depends on states that (are assumed to) satisfy X

Definition (Semantics of HML with one variable)

Let $(S, Act, \longrightarrow)$ be an LTS and $F \in HMF_X$. The semantics of F,

$$[\![F]\!] : 2^S \to 2^S,$$

is defined by

$$\begin{aligned}
[\![X]\!](T) &:= T \\
[\![\mathrm{tt}]\!](T) &:= S \\
[\![\mathrm{ff}]\!](T) &:= \emptyset \\
[\![F_1 \wedge F_2]\!](T) &:= [\![F_1]\!](T) \cap [\![F_2]\!](T) \\
[\![F_1 \vee F_2]\!](T) &:= [\![F_1]\!](T) \cup [\![F_2]\!](T) \\
[\![\langle\alpha\rangle F]\!](T) &:= \langle\cdot\alpha\cdot\rangle([\![F]\!](T)) \\
[\![[\alpha]F]\!](T) &:= [\cdot\alpha\cdot]([\![F]\!](T))
\end{aligned}$$

Semantics of HML with One Recursive Variable III

- Idea underlying the definition of

$$[\![.]\!] : HMF_X \to (2^S \to 2^S) :$$

  if $T \subseteq S$ gives the set of states that satisfy X, then $[\![F]\!](T)$ will be the set of states that satisfy F
- How to determine this T?
- According to previous discussion: as solution of recursive equation of the form $X = F_X$ where $F_X \in HMF_X$
- But: solution not unique; therefore write:

$$X \stackrel{min}{=} F_X \qquad \text{or} \qquad X \stackrel{max}{=} F_X$$

- In the following we will see:
  1. Equation $X = F_X$ always solvable
  2. Least and greatest solutions are unique and can be obtained by fixed-point iteration

Complete Lattices

Definition (Complete lattice)

A complete lattice is a partial order $(D, \sqsubseteq)$ such that all subsets of D have LUBs and GLBs. In this case,

$$\bot := \bigsqcup \emptyset \ (= \bigsqcap D) \qquad \text{and} \qquad \top := \bigsqcap \emptyset \ (= \bigsqcup D)$$

respectively denote the least and greatest element of D.

Application to HML with Recursion

Lemma

Let $(S, Act, \longrightarrow)$ be an LTS. Then $(2^S, \subseteq)$ is a complete lattice with

- $\bigsqcup \mathcal{T} = \bigcup \mathcal{T} = \bigcup_{T \in \mathcal{T}} T$ *for all* $\mathcal{T} \subseteq 2^S$
- $\bigsqcap \mathcal{T} = \bigcap \mathcal{T} = \bigcap_{T \in \mathcal{T}} T$ *for all* $\mathcal{T} \subseteq 2^S$
- $\bot = \bigsqcup \emptyset = \bigsqcap 2^S = \emptyset$
- $\top = \bigsqcap \emptyset = \bigsqcup 2^S = S$

Proof.

omitted □

Fixed Points

Definition 5.1 (Fixed point)

Let D be some domain, $d \in D$, and $f : D \to D$. If

$$f(d) = d$$

then d is called a fixed point of f.

Example 5.2

1. The (only) fixed points of $f_1 : \mathbb{N} \to \mathbb{N} : n \mapsto n^2$ are 0 and 1
2. A subset $T \subseteq \mathbb{N}$ is a fixed point of $f_2 : 2^{\mathbb{N}} \to 2^{\mathbb{N}} : T \mapsto T \cup \{1, 2\}$ iff $\{1, 2\} \subseteq T$

Monotonicity of Functions

Definition 5.3 (Monotonicity)

Let $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$ be partial orders. A function $f : D \to D'$ is called monotonic (w.r.t. $(D, \sqsubseteq)$ and $(D', \sqsubseteq')$) if, for every $d_1, d_2 \in D$,

$$d_1 \sqsubseteq d_2 \implies f(d_1) \sqsubseteq' f(d_2).$$

Example 5.4

1. $f_1 : \mathbb{N} \to \mathbb{N} : n \mapsto n^2$ is monotonic w.r.t. $(\mathbb{N}, \leq)$
2. $f_2 : 2^{\mathbb{N}} \to 2^{\mathbb{N}} : T \mapsto T \cup \{1, 2\}$ is monotonic w.r.t. $(2^{\mathbb{N}}, \subseteq)$
3. Let $\mathcal{T} := \{T \subseteq \mathbb{N} \mid T \text{ finite}\}$.
   Then $f_3 : \mathcal{T} \to \mathbb{N} : T \mapsto \sum_{n \in T} n$ is monotonic w.r.t. $(2^{\mathbb{N}}, \subseteq)$ and $(\mathbb{N}, \leq)$.
4. $f_4 : 2^{\mathbb{N}} \to 2^{\mathbb{N}} : T \mapsto \mathbb{N} \setminus T$ is not monotonic w.r.t. $(2^{\mathbb{N}}, \subseteq)$
   (since, e.g., $\emptyset \subseteq \mathbb{N}$ but $f_4(\emptyset) = \mathbb{N} \not\subseteq f_4(\mathbb{N}) = \emptyset$).

The Fixed-Point Theorem I

Alfred Tarski (1901–1983)

Theorem 5.5 (Tarski's fixed-point theorem)

Let $(D, \sqsubseteq)$ *be a complete lattice and* $f : D \to D$ *monotonic. Then* f *has a least fixed point* $\text{fix}(f)$ *and a greatest fixed point* $\text{FIX}(f)$ *given by*

$$\text{fix}(f) = \bigsqcap \{d \in D \mid f(d) \sqsubseteq d\} \qquad \textit{(GLB of all pre-fixed points of } f\textit{)}$$

$$\text{FIX}(f) = \bigsqcup \{d \in D \mid d \sqsubseteq f(d)\} \qquad \textit{(LUB of all post-fixed points of } f\textit{)}$$

Proof.

on the board □

The Fixed-Point Theorem II

Example 5.6 (cf. Example 5.2)

- Let $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}} : T \mapsto T \cup \{1,2\}$
- As seen before: $f(T) = T$ iff $\{1,2\} \subseteq T$
- Theorem 5.5 for fix:
$$\begin{aligned}\text{fix}(f) &= \textstyle\bigsqcap\{d \in D \mid f(d) \sqsubseteq d\} \\ &= \textstyle\bigcap\{T \subseteq \mathbb{N} \mid f(T) \subseteq T\} \\ &= \textstyle\bigcap\{T \subseteq \mathbb{N} \mid T \cup \{1,2\} \subseteq T\} \\ &= \textstyle\bigcap\{T \subseteq \mathbb{N} \mid \{1,2\} \subseteq T\} \\ &= \{1,2\}\end{aligned}$$
- Theorem 5.5 for FIX:
$$\begin{aligned}\text{FIX}(f) &= \textstyle\bigsqcup\{d \in D \mid d \sqsubseteq f(d)\} \\ &= \textstyle\bigcup\{T \subseteq \mathbb{N} \mid T \subseteq f(T)\} \\ &= \textstyle\bigcup\{T \subseteq \mathbb{N} \mid T \subseteq T \cup \{1,2\}\} \\ &= \textstyle\bigcup 2^{\mathbb{N}} \\ &= \mathbb{N}\end{aligned}$$

The Fixed-Point Theorem for Finite Lattices

Theorem 5.7 (Fixed-point theorem for finite lattices)

Let $(D, \sqsubseteq)$ *be a finite complete lattice and* $f : D \to D$ *monotonic. Then*

$$\mathrm{fix}(f) = f^m(\bot) \quad \textit{and} \quad \mathrm{FIX}(f) = f^M(\top)$$

for some $m, M \in \mathbb{N}$ *where* $f^0(d) := d \quad \textit{and} \quad f^{k+1}(d) := f(f^k(d))$.

Proof.

on the board □

Example 5.8

- Let $f : 2^{\{0,1\}} \to 2^{\{0,1\}} : T \mapsto T \cup \{0\}$
- $f^0(\bot) = \emptyset$, $f^1(\bot) = \{0\}$, $f^2(\bot) = \{0\} = f^1(\bot)$
  $\implies \mathrm{fix}(f) = \{0\}$ for $m = 2$
- $f^0(\top) = \{0, 1\}$, $f^1(\top) = \{0, 1\} = f^0(\top)$
  $\implies \mathrm{FIX}(f) = \{0, 1\}$ for $M = 1$

The Fixed-Point Theorem for Finite Lattices

Application to HML with Recursion

Lemma 5.9

Let $(S, Act, \longrightarrow)$ *be an LTS and* $F \in HMF_X$. *Then*

1. $[\![F]\!] : 2^S \to 2^S$ *is monotonic w.r.t.* $(2^S, \subseteq)$
2. $\text{fix}([\![F]\!]) = \bigcap\{T \subseteq S \mid [\![F]\!](T) \subseteq T\}$
3. $\text{FIX}([\![F]\!]) = \bigcup\{T \subseteq S \mid T \subseteq [\![F]\!](T)\}$

If, in addition, S *is finite, then*

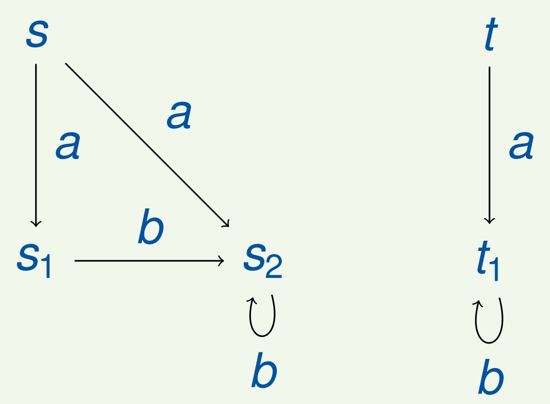
4. $\text{fix}([\![F]\!]) = [\![F]\!]^m(\emptyset)$ *for some* $m \in \mathbb{N}$
5. $\text{FIX}([\![F]\!]) = [\![F]\!]^M(S)$ *for some* $M \in \mathbb{N}$

Proof.

1. by induction on the structure of F (details omitted)
2. by Lemma 5.4 and Theorem 5.5
3. by Lemma 5.4 and Theorem 5.5
4. by Lemma 5.4 and Theorem 5.7
5. by Lemma 5.4 and Theorem 5.7 □

Applying the Fixed-Point Theorem for Finite Lattices

Example 5.10



Let $S := \{s, s_1, s_2, t, t_1\}$.

1. Solution of $X \overset{max}{=} \langle b \rangle tt \wedge [b]X$: on the board
2. Solution of $Y \overset{min}{=} \langle b \rangle tt \vee \langle \{a, b\} \rangle Y$: on the board

Largest Fixed Points and Invariants

- Remember (Example 4.5):
  - Invariant: $Inv(F) \equiv X$ for $F \in HMF$ and $X \overset{max}{=} F \wedge [Act]X$
  - $s \models Inv(F)$ if all states reachable from s satisfy F
- Now: formalize argument and prove its correctness (for arbitrary LTSs)
- Let $inv : 2^S \to 2^S : T \mapsto [\![F]\!] \cap [\cdot Act \cdot] T$ be the corresponding semantic function
- By Theorem 5.5, $\text{FIX}(inv) = \bigcup \{T \subseteq S \mid T \subseteq inv(T)\}$
- Direct formulation of invariance property:

$$Inv = \{s \in S \mid \forall w \in Act^*, s' \in S : s \overset{w}{\longrightarrow} s' \implies s' \in [\![F]\!]\}$$

Theorem 5.11

For every LTS $(S, Act, \longrightarrow)$, $Inv = \text{FIX}(inv)$ holds.

Proof.

on the board □