

Concurrency Theory

Lecture 20: McMillan Prefixes

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Overview

- 1 Introduction
- 2 Branching processes
- 3 The true concurrency semantics of a net
- 4 McMillan's finite prefix
- 5 Summary

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Introduction

- ▶ **Interleaving semantics** of Petri nets = set of **sequential** runs
 - ▶ a sequential run is a **total ordering** of transition occurrences
- ▶ The set of all sequential runs can be represented by a marking graph
- ▶ **Partial-order semantics** of Petri nets = set of **distributed** runs
 - ▶ a distributed run is an acyclic (**causal**) net which contains no choices
 - ▶ a distributed run is a **partial ordering** of transition occurrences
- ▶ Today: the set of all distributed runs can be represented by a **finite prefix** of the unfolding of the net.

Branching process: preamble

- ▶ A **branching process** represents a set of distributed runs
- ▶ It explicitly represents **each possible** resolution of each choice¹
- ▶ It is an acyclic (**occurrence**) net containing choices.
- ▶ It is a **partial ordering with conflicts** of transition occurrences.
- ▶ The true concurrency semantics of a net is a specific branching process, called **unfolding**.
- ▶ A net unfolding is the true concurrency counterpart of a marking graph.
- ▶ It is the **unique maximal** branching process in a **complete lattice**.
- ▶ The reachable markings of a 1-bounded net are covered by a **finite prefix** of this maximal branching process.

¹In net jargon, a choice is called a **conflict**.

Conflicts

A distributed run is based on a **causal** net. A branching process on an **occurrence** net. Main difference: the presence of conflicts (choices).

Conflict

Let $N = (P, T, F, M)$ be a net. Nodes x_1 and x_2 are in **conflict**, denoted $x_1 \# x_2$, if there exist distinct transitions $t_1, t_2 \in T$ such that $\bullet t_1 \cap \bullet t_2 \neq \emptyset$ and $(t_1, x_1) \in F^*$ and $(t_2, x_2) \in F^*$.

Node x is in **self-conflict** whenever $x \# x$.

Note that in a causal net $\# = \emptyset$ as $\bullet t_1 \cap \bullet t_2 = \emptyset$ for any two distinct transitions t_1 and t_2 .

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Occurrence net

Occurrence net

A net $K = (Q, V, G, M)$ is an **occurrence** net iff:

1. for each $q \in Q$, $|\bullet q| \leq 1$
2. the transitive closure G^+ of G is irreflexive
3. for each node $x \in Q \cup V$ we have $\{y \mid (y, x) \in G^+\}$ is finite
4. no transition $v \in V$ is in self-conflict
5. $M_0 = \circ K = \{q \in Q \mid \bullet q = \emptyset\}$.

Remark

Since $\# = \emptyset$ in a causal net, and each causal net fulfils the remaining conditions, every causal net is an occurrence net.

Branching process

Branching process

[Engelfriet 1991]

A **branching process** of net N is a pair (K, h) where $K = (Q, V, G, M)$ is an occurrence net and h a net homomorphism from K to N such that:

$$\forall v, v' \in Q. (\bullet v = \bullet v' \text{ and } h(v) = h(v') \text{ implies } v = v').$$

Every distributed run is a branching process. The reverse is not true.

Examples

On the black board.

Relating branching processes

Homomorphisms and isomorphisms between branching processes

Let $B_1 = (K_1, h_1)$ and $B_2 = (K_2, h_2)$ be two branching processes of net N . A **homomorphism** from B_1 to B_2 is a homomorphism h from K_1 to K_2 such that $h_2 \circ h = h_1$.²

An **isomorphism** is a **bijective** homomorphism. B_1 and B_2 are **isomorphic** if there is an isomorphism from B_1 to B_2 .

Being isomorphic is an equivalence relation. Its equivalence classes are called isomorphism classes.

²The composition of two net homomorphisms is a net homomorphism.

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Approximation of branching processes

Approximation

Let B_1 and B_2 be two branching processes of net N . B_1 **approximates** B_2 , denoted $B_1 \sqsubseteq B_2$, if there exists an **injective** homomorphism from B_1 to B_2 .

Intuition

B_1 approximates B_2 whenever every (partial) distributed run in B_1 is also contained in B_2 . In other words, B_1 is isomorphic to an initial part of B_2 . Being an approximation on branching processes is the analogue of being a prefix on sequences.

Approximation of branching processes

Approximation

Let B_1 and B_2 be two branching processes of net N . B_1 **approximates** B_2 , denoted $B_1 \sqsubseteq B_2$, if there exists an injective homomorphism from B_1 to B_2 .

Lemma

Approximation is preserved by isomorphism: if B'_i is isomorphic to B_i (for $i = 1, 2$), then $B_1 \sqsubseteq B_2$ implies $B'_1 \sqsubseteq B'_2$. Thus, \sqsubseteq can be extended to a partial order on isomorphism classes (of branching processes).

Proof.

Home exercise. Basically juggling with homomorphisms. \square

The true concurrency semantics of a net

Corollary: the unfolding of a net

Every one-bounded net has a unique maximal (with respect to \sqsubseteq) branching process up to isomorphism. This is called the **unfolding** or **true concurrency semantics** of net N .

We denote by $B_{\max} = ((P_{\max}, T_{\max}, F_{\max}), h_{\max})$ a representative of the isomorphism class of the maximal branching process of N .

Example

On the black board.

Engelfriet's theorem

Engelfriet's branching process theorem

The set of isomorphism classes of branching processes of net N is a **complete lattice** with respect to the approximation relation \sqsubseteq . Formally, $(\mathbb{B}, \sqsubseteq)$ is a complete partial order, where \mathbb{B} is the set of isomorphism classes of branching processes.

Complete lattice

Recall that a complete lattice is a partial order $(\mathbb{B}, \sqsubseteq)$ such that all subsets of \mathbb{B} have LUBs and GLBs.

The true concurrency semantics of Petri nets

The **true concurrency** semantics of a Petri net is given by its **unfolding**.

Recall: The **interleaving** semantics of a Petri net is given by its **marking graph**.

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Configurations

Configurations

Let $K = (Q, V, G, M_0)$ be an occurrence net, $\prec = G^+$ and $\preceq = G^*$.

The set $C \subseteq V$ is a **configuration** of K whenever:

1. $x \in C$ implies $y \in C$, for all $y \preceq x$ (downward-closed wrt. \preceq)
2. $\forall x, y \in C. \neg(x\#y)$ (conflict-free)

Intuition and examples

A configuration can be seen as the set of transitions that have occurred so far in a distributed run. Examples on the black board.

Fact

For configuration C of B_{\max} (of net N), and $x_1 \dots x_n$ a **linearisation** of the transitions in C (respecting \preceq), the sequence $h_{\max}(x_1) \dots h_{\max}(x_n)$ is a **sequential run** of the original net N .

Finite prefix

The maximal branching process under \sqsubseteq may be infinite.

Prefix of maximal branching process

Branching process $B = (P, T, F, M_0)$ is a prefix of B_{\max} if $B \sqsubseteq B_{\max}$ and $P \subseteq P_{\max}$ and $T \subseteq T_{\max}$. B is finite whenever P and T are finite.

Finite prefix existence theorem

[McMillan, 1992]

For every finite one-bounded net N , there exists a finite prefix B_{fin} of B_{\max} that covers all reachable markings of N . The size of the finite prefix can maximally be exponential in the size of N .

Proof.

Follows directly from two facts:

1. Every reachable marking is represented by some **cut** of B_{\max} , and
2. The set of reachable markings of a finite one-bounded net is finite.

Cuts

Cuts

Let C be a finite configuration of a branching process $B = (K, h)$. Then:

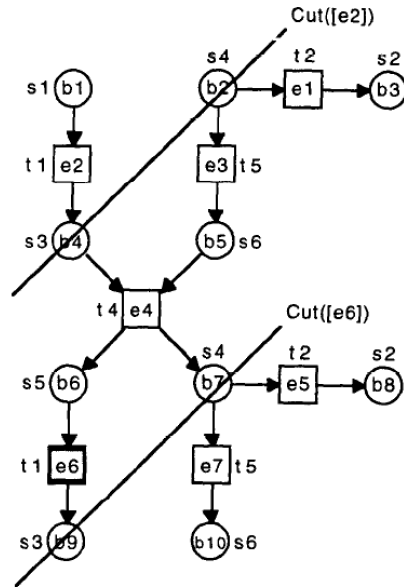
$$\text{Cut}(C) = ({}^\circ K \cup C) \setminus \bullet C.$$

If B is a branching process of N , then $h(\text{Cut}(C))$ is a **reachable marking** of net N . We denote $h(\text{Cut}(C))$ by $M(C)$, the marking of configuration C .

Intuition

Cuts correspond to markings reached by firing all transitions in a given finite configuration.

Example



Cut-off event

Cut-off event

Let $B_{\max} = ((P_{\max}, T_{\max}, G_{\max}), h_{\max})$. Transition $t \in T_{\max}$ is a **cut-off** transition if there exists a transition $t' \in T_{\max} \cup \{\perp\}$ such that:

$$|[t']| < |[t]| \text{ and } M([t]) = M([t']).$$

Dummy transition

Remark: \perp is a dummy transition having no input places and ${}^{\circ}B_{\max}$ as output places, for which we let $[\perp] = \emptyset$. This yields that if $M([t]) = M_0$, then t is a cut-off transition.

Fact

If $|[t']| < |[t]|$ and $M([t]) = M([t'])$, then the "continuations" of B_{\max} from $Cut([t])$ and $Cut([t'])$ are isomorphic.

Transition causes

Transition causes

Let $K = (Q, V, G)$ be an occurrence net and $v \in V$. The set $[v]$ of **causes** of v is defined by:

$$[v] = \{v' \in V \mid v' \preceq v\}.$$

(Recall that \preceq denotes G^* , the reflexive and transitive closure of G .)

Example

On the black board

Facts

1. For each v , $[v]$ is a finite configuration.
2. For every configuration C of K , either $v \notin C$ or $[v] \subseteq C$.

McMillan prefix

McMillan prefix

The **McMillan prefix** of one-bounded net N is the branching process B_{fin} , the unique prefix of B_{\max} having T_{fin} as set of transitions satisfying for each $t \in T_{\max}$:

$$t \in T_{\text{fin}} \text{ iff no transition } t' \prec t \text{ is a cut-off transition.}$$

Computing the McMillan prefix

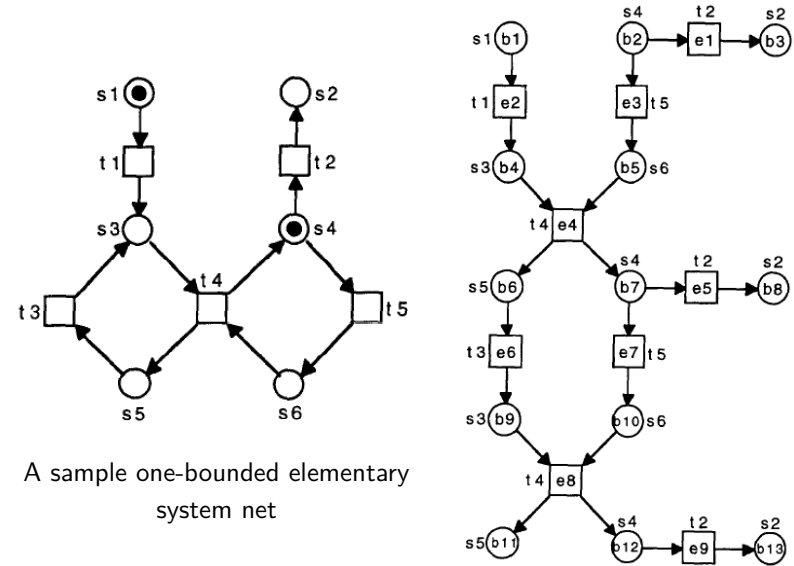
Algorithm

1. Start with the empty branching process
2. Add transitions one at the time, in order of increasing size of their sets of causes
3. On adding t , compare $M([t])$ with $M([t'])$ for each t' that was added before t
4. If $M([t]) = M([t'])$, then t is a cut-off transition, and its successors are not explored
5. Terminate when no further transitions can be added.

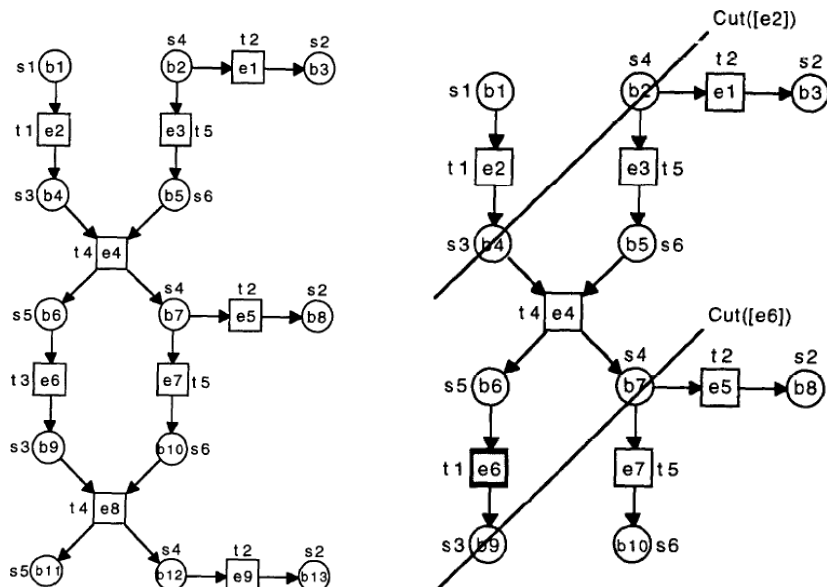
Remark

Termination is ensured by the finiteness of the number of reachable markings on N , as N is one-bounded.

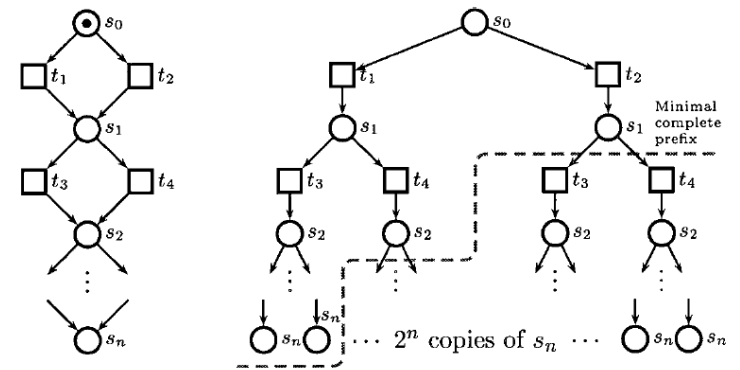
Example net and one of its branching processes



Its McMillan prefix



An exponentially-sized McMillan prefix



For every marking M all the configurations $[t]$ satisfying $M([t]) = M$ have the same size, and therefore there exist no cut-off events [Kishinevsky and Taubin]

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Summary

- ▶ A branching process captures several distributed runs of N
- ▶ It is represented by a relaxed notion of causal net, the occurrence net
- ▶ Branching processes are mapped to N via homomorphisms
- ▶ A homomorphism is a structure-preserving mapping between two nets
- ▶ Approximation (denoted \sqsubseteq) is a partial-order on branching processes
- ▶ Isomorphic branching process with \sqsubseteq are a complete lattice
- ▶ True concurrency semantics of $N =$ the maximal element (under \sqsubseteq)
- ▶ For 1-bounded nets, the McMillan prefix covers all reachable markings